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On the L-fuzzy generalization of Chu correspondences

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Abstract

In this position paper, we focus on the framework of Chu correspondences extending Mori's approach [16] to formal concept analysis by proposing suitable definitions of the required concepts in an L-fuzzy environment.

1 Introduction

Since its introduction in the eighties, formal concept analysis [9] has become an important and appealing research topic both from the theoretical perspective [18, 11, 21, 3] and from the applicative one. Regarding applications, we can find papers ranging from ontology merging [7, 17], to applications to the Semantic Web by using the notion of concept similarity [8], and from processing of medical records in the clinical domain [10] to the development of recommender systems [5].

Soon after the introduction of "classical" formal concept analysis, a number of different approaches for its generalization were introduced and, nowadays, there are works which extend the theory with ideas from fuzzy set theory [2, 14] or fuzzy logic reasoning [6, 1] or from rough set theory [19, 13, 22] or some integrated approaches such as fuzzy and rough [20], or rough and domain theory [12].

In this paper we concentrate on the categorical approach to formal concept analysis developed in [16], in which the notion of Chu correspondences between formal contexts is introduced. In that paper, the construction of formal concepts associated to a crisp relation between objects and attributes is shown to induce a functor from the category of Chu correspondences to the category of sup-preserving maps between complete lattices. It turns out that the category of Chu correspondences has a *-autonomous category structure which is preserved by the induced functor. Our main contribution here is the development of the initial notions in order to extend the theory of Chu correspondences to an L-fuzzy framework.

2 Preliminaries

We will assume that the reader is familiar with the standard notions of classical formal concept analysis [9], such as context, formal concept lattice, or Galois connection. For the benefit of the reader not acquainted with the basics of the fuzzy extensions of the theory of formal concept analysis, we provide the preliminary notions below.

To begin with, the usual set of boolean values of classical logics (containing true and false), is generalized to the algebraic structure of complete residuated lattice, which allows to provide suitable extensions in a more abstract environment.

Definition 1 A **complete residuated lattice** is an algebra $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ where

1. $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with least element 0 and greatest element 1,
2. $\langle L, \otimes, 1\rangle$ is a commutative monoid,
3. $\otimes$ and $\rightarrow$ are adjoint operators, i.e. $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$, for all $a, b, c \in L$, where $\leq$ is the lattice ordering generated from $\wedge$ and $\vee$.

Now, the natural extension of the notion of context is given below.

Definition 2 Let L be a complete residuated lattice, an **L-fuzzy formal context** is a triple $\langle B, A, r\rangle$ consisting of a set of objects B, a set of attributes A and an L-fuzzy binary relation r, i.e. a mapping from $r\colon B \times A$ to L, which can be alternatively understood as an L-fuzzy subset of $B \times A$.

We finalize the presentation of the preliminary definitions by introducing the L-fuzzy extension provided by Bělohlávek in [2], where we will use the notation Y^X to refer to the set of mappings from X to Y.

Definition 3 Consider an L-fuzzy context $\langle B, A, r\rangle$. A pair of mappings $\uparrow\colon L^B \rightarrow L^A$ and $\downarrow\colon L^A \rightarrow L^B$ is defined as follows:

$$\begin{aligned}
\uparrow f(a) &= \bigwedge_{o \in B} (f(o) \rightarrow r(o, a)) \\
\downarrow g(o) &= \bigwedge_{a \in A} \big(g(a) \rightarrow r(o, a)\big).
\end{aligned}$$

for every $f \in L^B$ and $g \in L^A$.

Lemma 1 *Let L be a complete residuated lattice, let $r \in {}^{B\times A}L$ be an L-fuzzy relation between B and A. Then the pair of operators $\uparrow$ and $\downarrow$ forms a Galois connection between $\langle L^B; \subseteq\rangle$ and $\langle L^A; \subseteq\rangle$.*

This pair of mappings is said to be **closed**, in that they satisfy the equalities in Lemma 2 below.

Lemma 2 *Under the conditions of Lemma 1,the following equalities hold for arbitrary $f \in L^B$ and $g \in L^A$:*

$$\uparrow f = \uparrow\downarrow\uparrow f \text{ and } \downarrow g = \downarrow\uparrow\downarrow g.$$

Definition 4 An L-**fuzzy concept** is a pair $\langle f, g\rangle$ such that $\uparrow f = g, \downarrow g = f$. The set of all L-fuzzy concepts associated to a fuzzy context (B, A, r) will be denoted as L-$FCL(B, A, r)$.

An ordering between L-fuzzy concepts is defined as follows: $\langle f_1, g_1\rangle \le \langle f_2, g_2\rangle$ if and only if $f_1 \subseteq f_2$ if and only if $g_1 \supseteq g_2$.

Theorem 1 *The poset* $(L\text{-}FCL(B, A, r), \le)$ *is a complete lattice where*

$$\bigwedge_{j\in J} \langle f_j, g_j\rangle = \Big\langle \bigwedge_{j\in J} f_j, \uparrow \big(\bigwedge_{j\in J} f_j\big)\Big\rangle,$$

$$\bigvee_{j\in J} \langle f_j, g_j\rangle = \Big\langle \downarrow \big(\bigwedge_{j\in J} g_j\big), \bigwedge_{j\in J} g_j\Big\rangle.$$

We know recall the basic definitions and notations given in [16].

Definition 5 A **correspondence** from X to Y is a mapping $f\colon X \to 2^Y$. Note that correspondences are also called set-valued or multiple-valued functions.

The **transposed** of a correspondence $f\colon X \to 2^Y$ is a correspondence ${}^t f\colon Y \to 2^X$ defined by ${}^t f(y) = \{x \mid y \in f(x)\}$.

The set $Cors(X, Y)$ of all the correspondences from X to Y can be endowed of a poset structure by defining the ordering $f_1 \le f_2$ as $f_1(x) \subseteq f_2(x)$ for all $x \in X$.

Definition 6 An L-**correspondence** from X to Y is a mapping $\varphi\colon X \to L^Y$.

The **transposed** of an L-correspondence $\varphi\colon X \to L^Y$ is an L-correspondence ${}^t\varphi\colon Y \to {}^X L$ defined by ${}^t\varphi(y)(x) = \varphi(x)(y)$.

The set L-$Cors(X, Y)$ of all the L-correspondences from X to Y can be endowed of a poset structure by defining the ordering $\varphi_1 \le \varphi_2$ as $\varphi_1(x)(y) \le \varphi_2(x)(y)$ for all $x \in X$ and $y \in Y$.

3 Chu correspondences

Let us recall the definition of Chu correspondence in the classical framework of crisp relations as contexts.

Definition 7 Let $C_i = \langle B_i, A_i, R_i\rangle$ $(i = 1, 2)$ be crisp formal contexts. A pair $f = (f_l, f_r)$ is called **a correspondence from** C_1 **to** C_2 if f_l and f_r, respectively, are correspondences from B_1 to B_2 and from A_2 to A_1.

A correspondence f from C_1 to C_2 is said to be a **weak Chu correspondence** if the following equality holds for every $o_1 \in B_1$ and $a_2 \in A_2$

$$\bigwedge_{y\in f_r(a_2)} R_1(o_1, y) = \bigwedge_{x\in f_l(o_1)} R_2(x, a_2)$$

A weak Chu correspondence f C_1 to C_2 is called simply a **Chu correspondence** if the pair $f_l(o_1) \subseteq B_2$ and $f_r(a_2) \subseteq A_1$ is closed for every $o_1 \in B_1$ and $a_2 \in A_2$.

In the following we will concentrate in obtaining a suitable generalization of the previous definition to the framework L-fuzzy sets. To begin with, let us note that a given L-fuzzy context $r\colon B \times A \to L$ can be extended to the set L-fuzzy objects and attributes as follows. We will define a new mapping $\hat{r}\colon L^B \times L^A \to L$ such that for $f \in L^B$ and $g \in L^A$ we have

$$\hat{r}(f,g) = \bigwedge_{\substack{o\in B\\ a\in A}} (f(o) \otimes g(a) \to r(o,a)).$$

This definition allows to provide a suitable generalization of Bělohlávek's Galois connection as follows. Given a singleton $\{x\} \subseteq B$, consider the characteristic function of the singleton $\chi_x \in L^B$ as $\chi_x(x) = 1$ and $\chi_x(o) = 0$ for all $o \in B, o \neq x$. Then

$$\begin{aligned}
\hat{r}(\chi_x, g) &= \bigwedge_{\substack{o\in B\\ a\in A}} (\chi_x(o) \otimes g(a) \to r(o,a)) \\
&= \bigwedge_{\substack{o\in B\\ o\neq x\\ a\in A}} (\chi_x(o) \otimes g(a) \to r(o,a)) \wedge \bigwedge_{a\in A} (\chi_x(x) \otimes g(a) \to r(o,a)) \\
&= \bigwedge_{\substack{o\in B\\ o\neq x\\ a\in A}} (0 \otimes g(a) \to r(o,a)) \wedge \bigwedge_{a\in A} (1 \otimes g(a) \to r(x,a)) \\
&= \bigwedge_{a\in A} (g(a) \to r(x,a))
\end{aligned}$$

which coincides with Bělohlávek's definition, in which the element x has been substituted by the characteristic function χ_x. A similar result can be obtained by fixing a singleton in the set of attributes.

Definition 8 Assume we have two formal contexts $C_i = \langle B_i, A_i, r_i\rangle, (i = 1, 2)$, then the pair $\varphi = (\varphi_l, \varphi_r)$ is called **a correspondence** from C_1 to C_2 if φ_l and φ_r are correspondences respectively from B_1 to B_2 and from A_2 to A_1 (i.e. $\varphi_l : B_1 \to L^{B_2}$ and $\varphi_r : A_2 \to L^{A_1}$). The correspondence φ is called **a weak L-Chu correspondence** if for all $o_1 \in B_1$ and $a_2 \in A_2$ holds

$$\begin{aligned}
\hat{r}_1(\chi_{o_1}, \varphi_r(a_2)) &= \bigwedge_{a_1\in A_1} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) \\
&= \bigwedge_{o_2\in B_2} (\varphi_l(o_1)(o_2) \to r_2(o_2, a_2)) = \hat{r}_2(\varphi_l(o_1), \chi_{a_2}).
\end{aligned}$$

A weak Chu correspondence φ is a **L-Chu correspondence** if the pair of mappings $\varphi_l(o_1)$ and $\varphi_r(a_2)$ is closed for all $o_1 \in B_1$ and $a_2 \in A_2$. We will denote the set of all Chu correspondences from C_1 to C_2 by $ChuCors(C_1, C_2)$.

We can check that this definition allows us to provide a suitable generalization of Mori's definition of weak Chu correspondence and Chu correspondence as follows. Let us assume that we are in the classical case, that is, $L = 2$. Then

$$\begin{aligned}
\widehat{r}_1(\chi_{o_1}, \varphi_r(a_2)) &= \bigwedge_{a_1 \in A_1} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) \\
&= \bigwedge_{\substack{a_1 \in A_1 \\ a_1 \notin \varphi_r(a_2)}} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) \wedge \bigwedge_{a_1 \in \varphi_r(a_2)} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) \\
&= 1 \wedge \bigwedge_{a_1 \in \varphi_r(a_2)} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) \\
&= \bigwedge_{a_1 \in \varphi_r(a_2)} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) \\
&= \bigwedge_{a_1 \in \varphi_r(a_2)} (1 \to r_1(o_1, a_1)) = \bigwedge_{a_1 \in \varphi_r(a_2)} r_1(o_1, a_1)
\end{aligned}$$

and similarly for $\widehat{r}_2$.

4 Bonds

Here we will extend the classical definition of bond, as stated in [9] which is recalled below:

Definition 9 A **bond** from a context $C_1 = \langle B_1, A_1, R_1 \rangle$ to a context $C_2 = \langle B_2, A_2, R_2 \rangle$ is a relation $R_b \subseteq B_1 \times A_2$ for which the following holds:

- $\uparrow_b (o_1) = \{a_2 \in A_2 : (o_1, a_2) \in R_b\}$ is an intent of C_2 for every $o_1 \in B_1$
- $\downarrow_b (a_2) = \{o_1 \in B_1 : (o_1, a_2) \in R_b\}$ is an extent of C_1 for every $a_2 \in A_2$.

Now, we introduce our candidate for the L-fuzzy extension of the notion of bond.

Definition 10 An L**-bond** between two formal contexts $C_1 = \langle B_1, A_1, r_1 \rangle$ and $C_2 = \langle B_2, A_2, r_2 \rangle$ is a correspondence $b : B_1 \to L^{A_2}$ satisfying the condition that the pair $b(o_1)$ and ${}^t b(a_2)$ is closed for all $o_1 \in B_1$ and $a_2 \in A_2$. The set of all bonds from C_1 to C_2 is denotes as $Bonds(C_1, C_2)$.

Every correspondence is a relation, thus an L-bond can be seen as a relation between B_1 and A_2. This certainly suggests a possible relationship between L-Chu correspondences and L-bonds.

Definition 11

- *Let $b\colon C_1 \to C_2$ be an L-bond. We can define a correspondence $\varphi_b\colon C_1 \to C_2$ by*

$$\varphi_{bl}(o_1) = \downarrow_2 (b(o_1)) \in L^{B_2} \text{ for } o_1 \in B_1$$
$$\varphi_{br}(a_2) = \uparrow_1 ({}^t b(a_2)) \in L^{A_1} \text{ for } a_2 \in A_2$$

- *Conversely, consider an L-Chu correspondence φ from C_1 to C_2, and define another correspondence $b_\varphi\colon B_1 \to L^{A_2}$ by*

$$b_\varphi(o_1) = \uparrow_2 (\varphi_l(o_1))$$

Proposition 1 *With the definitions given above*

1. *φ_b is an L-Chu correspondence from C_1 to C_2.*
2. *b_φ is an L-bond from C_1 to C_2.*

Proof: Both proofs follow as a result of more or less straightforward chains of computations. We will only include one of them.

1. Let $o_1 \in B_1$ and $a_2 \in A_2$. Then

$$\begin{aligned}
\widehat{r_2}(\varphi_{bl}(o_1), \chi_{a_2}) &= \bigwedge_{o_2 \in B_2} (\varphi_{bl}(o_1)(o_2) \to r_2(o_2, a_2)) \\
&= \bigwedge_{o_2 \in B_2} (\downarrow_2 (b(o_1))(o_2) \to r_2(o_2, a_2)) \\
&= \uparrow_2 (\downarrow_2 (b(o_1)))(a_2) = b(o_1)(a_2) = {}^t b(a_2)(o_1) = \downarrow_1 (\uparrow_1 ({}^t b(a_2)))(o_1) \\
&= \bigwedge_{a_1 \in A_1} (\uparrow_1 ({}^t b(a_2))(a_1) \to r_1(o_1, a_1)) \\
&= \bigwedge_{a_1 \in A_1} (\varphi_{br}(a_2)(a_1) \to r_1(o_1, a_1)) = \widehat{r_1}(\chi_{o_1}, \varphi_{br}(a_2))
\end{aligned}$$

$\square$

5 Relationship between Chu correspondences and Bonds

The previous proposition suggests a close relationship between the notions of L-Chu correspondence and L-bond between two formal contexts.

The existence of a possible isomorphism between these two type of structure have been checked by computer on several examples in a non-classical context, specifically on 3-valued logic. Now we present the results obtained in one specific cases and, for the purposes of this example we will consider a three-valued lattice $L = \langle \{1, 0, \mathrm{p}\}, \leq \rangle$, with order $\leq$ defined by $0 \leq \mathrm{p} \leq 1$.

Conjunction and implication are defined by

$$a \otimes b = \min\{a, b\} \quad \text{and} \quad a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

In Table 1 we show the two 3-valued contexts we will be working with.

Table 1: 3-contexts C_1 and C_2

C_1	a_{11}	a_{12}	a_{13}
o_{11}	0	1	p
o_{12}	1	0	1
o_{13}	p	1	0
o_{14}	p	1	p

C_2	a_{21}	a_{22}	a_{23}
o_{21}	1	1	p
o_{22}	0	p	1
o_{23}	p	p	1
o_{24}	p	p	p

In Table 2 we show all the 3-bonds b between C_1 and C_2 (obtained by computer) together with all its associated 3-Chu correspondences $(\varphi_{bl}, \varphi_{br})$ computed by

$$\varphi_{bl}(o_1) = \uparrow_2 (b(o_1)) \quad \text{and} \quad \varphi_{br}(a_2) = \downarrow_1 ({}^t b(a_2))$$

for all $o_1 \in B_1$ and $a_2 \in A_2$.

Then, in Table 3 all the 3-Chu correspondences (φ_l, φ_r) (obtained by computer) are presented, together with their associated 3-bonds computed by

$$b_{\varphi_l}(o_1) = \uparrow_2 (\varphi_l(o_1)) \quad \text{or} \quad b_{\varphi_r} = {}^t b_{\varphi_l} = \downarrow_1 (\varphi_r(a_2))$$

for all $o_1 \in B_1$ and $a_2 \in A_2$.

The previous relationship between 3-Chu correspondences and 3-bonds holds as well in all the examples of particular 3-valued pairs of formal contexts we have considered. Therefore, it is likely that they are just instantiations of a general result, as in the classical case. Furthermore, we can define an ordering relation on the set of L-Chu correspondences between C_1 and C_2 as follows $\varphi_1 \leq \varphi_2$ if and only if for all $(o_1, o_2) \in B_1 \times B_2$ the following inequality holds $\varphi_{1l}(o_1)(o_2) \leq \varphi_{2l}(o_1)(o_2)$. And a similar suitable ordering can be defined for L-bonds.

We will finalise this section by stating our main conjecture for future work.

Conjecture 1 *The correspondence which assigns to each Chu correspondence φ the bond b_φ is a bijection. Moreover, L-ChuCors(C_1, C_2) and L-Bonds(C_1, C_2) are isomorphic as ordered structures.*

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Table 2: Relationship between bonds and Chu correspondences between C_1 and C_2.

b	φ_{bl}	φ_{br}
(0,1,1) (0,1,1) (0,1,1) (1,1,1)	(p,p,p,p) (p,p,p,p) (p,p,p,p) (p,0,p,p)	(p,1,p) (0,0,0) (0,0,0)
(1,1,1) (1,1,1) (1,1,1) (1,1,1)	(p,0,p,p) (p,0,p,p) (p,0,p,p) (p,0,p,p)	(0,0,0) (0,0,0) (0,0,0)
(0,1,1) (1,1,1) (1,1,1) (1,1,1)	(p,p,p,p) (p,0,p,p) (p,0,p,p) (p,0,p,p)	(p,0,0) (0,0,0) (0,0,0)
(1,1,1) (1,1,1) (0,1,1) (1,1,1)	(p,0,p,p) (p,0,p,p) (p,p,p,p) (p,0,p,p)	(0,0,p) (0,0,0) (0,0,0)
(0,1,1) (0,1,1) (1,1,1) (1,1,1)	(p,p,p,p) (p,p,p,p) (p,0,p,p) (p,0,p,p)	(p,1,0) (0,0,0) (0,0,0)
(0,1,1) (1,1,1) (0,1,1) (1,1,1)	(p,p,p,p) (p,0,p,p) (p,p,p,p) (p,0,p,p)	(p,0,p) (0,0,0) (0,0,0)
(1,1,1) (0,1,1) (0,1,1) (1,1,1)	(p,0,p,p) (p,p,p,p) (p,p,p,p) (p,0,p,p)	(0,1,p) (0,0,0) (0,0,0)
(1,1,1) (0,1,1) (1,1,1) (1,1,1)	(p,0,p,p) (p,p,p,p) (p,0,p,p) (p,0,p,p)	(0,1,0) (0,0,0) (0,0,0)

Table 3: Relationship between Chu correspondences and bonds between C_1 and C_2.

φ_l	φ_r	b_{φ_l}	$b_{\varphi_r}(= {}^t b_{\varphi_l})$
(p,p,p,p) (p,p,p,p) (p,p,p,p) (p,0,p,p)	(p,1,p) (0,0,0) (0,0,0)	(0,1,1) (0,1,1) (0,1,1) (1,1,1)	(0,0,0,1) (1,1,1,1) (1,1,1,1)
(p,0,p,p) (p,0,p,p) (p,0,p,p) (p,0,p,p)	(0,0,0) (0,0,0) (0,0,0)	(1,1,1) (1,1,1) (1,1,1) (1,1,1)	(1,1,1,1) (1,1,1,1) (1,1,1,1)
(p,p,p,p) (p,0,p,p) (p,0,p,p) (p,0,p,p)	(p,0,0) (0,0,0) (0,0,0)	(0,1,1) (1,1,1) (1,1,1) (1,1,1)	(0,1,1,1) (1,1,1,1) (1,1,1,1)
(p,0,p,p) (p,0,p,p) (p,p,p,p) (p,0,p,p)	(0,0,p) (0,0,0) (0,0,0)	(1,1,1) (1,1,1) (0,1,1) (1,1,1)	(1,1,0,1) (1,1,1,1) (1,1,1,1)
(p,p,p,p) (p,p,p,p) (p,0,p,p) (p,0,p,p)	(p,1,0) (0,0,0) (0,0,0)	(0,1,1) (0,1,1) (1,1,1) (1,1,1)	(0,0,1,1) (1,1,1,1) (1,1,1,1)
(p,p,p,p) (p,0,p,p) (p,p,p,p) (p,0,p,p)	(p,0,p) (0,0,0) (0,0,0)	(0,1,1) (1,1,1) (0,1,1) (1,1,1)	(0,1,0,1) (1,1,1,1) (1,1,1,1)
(p,0,p,p) (p,p,p,p) (p,p,p,p) (p,0,p,p)	(0,1,p) (0,0,0) (0,0,0)	(1,1,1) (0,1,1) (0,1,1) (1,1,1)	(1,0,0,1) (1,1,1,1) (1,1,1,1)
(p,0,p,p) (p,p,p,p) (p,0,p,p) (p,0,p,p)	(0,1,0) (0,0,0) (0,0,0)	(1,1,1) (0,1,1) (1,1,1) (1,1,1)	(1,0,1,1) (1,1,1,1) (1,1,1,1)

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