

## **Fuzzy congruence relations on nd-groupoids**

**P. Cordero<sup>1</sup>, I. de las Peñas<sup>1</sup>, G. Gutiérrez<sup>1</sup>, J. Martínez<sup>1</sup> and  
M. Ojeda-Aciego<sup>1</sup>**

<sup>1</sup> *Department of Applied Mathematics, University of Málaga. Spain*

emails: pcordero@uma.es, ipcabrera@uma.es, ggutierrez@uma.es,  
jmartinezd@uma.es, aciego@uma.es

### **Abstract**

In this work we introduce the notion of fuzzy congruence relation on an nd-groupoid and study conditions on the nd-groupoid which guarantee a complete lattice structure on the set of fuzzy congruence relations. The study of these conditions allowed to construct a counterexample to the statement that the set of fuzzy congruences on a hypergroupoid is a complete lattice.

*Key words: Fuzzy congruence relation, nd-groupoid, multisetilattice*

## **1 Introduction**

The systematic generalization of crisp concepts to the fuzzy case has proven to be an important theoretical tool for the development of new methods of reasoning under uncertainty, imprecision and lack of information.

Regarding the generalization level, it is important to note that the definition of fuzzy sets originally presented as mappings with codomain  $[0, 1]$ , was soon replaced by more general structures, for instance a complete lattice, as in the  $L$ -fuzzy sets introduced by Goguen [8]. This paper continues previous work [4, 5] which is aimed at investigating  $L$ -fuzzy sets where  $L$  has the structure of a multilattice, a structure introduced in [2] and later recovered for use in other contexts, both theoretical and applied [10, 13].

Roughly speaking, a multilattice is an algebraic structure in which the restrictions imposed on a lattice, namely, the “existence of least elements in the sets of upper bounds and greatest in the sets of lower bounds” are relaxed to the “existence of minimals and maximals, respectively, in the corresponding sets of bounds”. Attending to this informal description, the main difference that one notices when working with multilattices is that the operators which compute suprema and infima are no longer single-valued, since there may be several multi-suprema or multi-infima, or may be

none. This immediately leads to the theory of hyperstructures, that is, algebras whose operations are set-valued.

If  $A$  is a non-empty set and  $H$  is a family of set-valued operations on  $A$ , the ordered pair  $(A, H)$  is called a hyperalgebra (or multialgebra, or polyalgebra). The study of hyperalgebras originated in 1934 when Marty introduced the so-called hypergroups in [12]. Since then, a number of papers have been published on this topic, focussing essentially on special types of hyperalgebras (such as hypergroups, hyperrings, hyperfields, vector hyperspaces, boolean hyperalgebras, ...) and guided, sometimes by purely theoretical motivations and sometimes because of their applications in other areas.

In this paper, we will focus on the most general hyperstructures, namely hypergroupoids and nd-groupoids. Our interest in these structures arises from the fact that, in a multilattice, the operators which compute the multi-suprema and multi-infima are precisely nd-groupoids or, if we have for granted that at least a multi-supremum always exists, a hypergroupoid. Actually, some of the results will be stated just in terms of multiseuilattices.

Several papers have investigated the structure of the set of fuzzy congruences on different algebraic structures [1, 6, 7, 15, 17]; and in [4, 5] we initiated our research in this direction. Specifically, we focused on the theory of (crisp) congruences on a multilattice and on an nd-groupoid, as a necessary step prior studying the fuzzy congruences on multilattices and the multilattice-based generalization of the concept of  $L$ -fuzzy congruence. In this paper, we study the notion of fuzzy congruence relation on nd-groupoids.

The fact that the structure of nd-groupoid is simpler than that of a multilattice does not necessarily mean that the theory is simpler as well. We will show that, in general, the set of fuzzy congruences on an nd-groupoid is not a lattice unless we assume some extra properties. This problem led us to review some related literature and, as a result, we found one counter-example in the context of congruences on a hypergroupoid.

## 2 Preliminaries

We can find in the literature we find the definition of a hypergroupoid as a nonempty set endowed with a hyperoperation  $* : H \times H \rightarrow 2^H \setminus \{\emptyset\}$ . However, we are interested in a generalization of hypergroupoid that we will call non-deterministic groupoid (nd-groupoid, for short) which also considers the empty set as possible image of the hyperoperation.

**Definition 2.1** *An nd-groupoid  $(A, *)$  is defined by an nd-operation  $* : A \times A \rightarrow 2^A$  on a nonempty set  $A$ . The induced power groupoid is defined as  $(2^A, *)$  where the operation is given by  $X * Y = \{x * y \mid x \in X, y \in Y\}$  for all  $X, Y \subseteq A$ .*

Notice that the definition allows the assignment of the empty set to a pair of elements, that is  $a * b = \emptyset$ , this mere fact, albeit simple, represents an important difference with hypergroupoids, as it will be explained later.

The following notational conventions will be used hereafter:

- We will use multiplicative notation and, thus, the symbol of the nd-operation will be omitted.
- If  $a \in A$  and  $X \subseteq A$ , we will denote  $aX = \{ax \mid x \in X\}$  and  $Xa = \{xa \mid x \in X\}$ . In particular,  $a\emptyset = \emptyset a = \emptyset$ .
- When the result of the nd-operation is a singleton, we will often omit the braces.

As stated in the introduction, our interest in extending the concept of hypergroupoid is justified by the algebraic characterization of multilattices and multisemilattices, since the operators for multi-suprema and multi-infima are both examples of nd-operators.

With this idea in mind, we introduce below the extension to the framework of nd-groupoids of some well-known properties. Assume that  $(A, \cdot)$  is an nd-groupoid:

- **Idempotency:**  $aa = a$  for all  $a \in A$ .
- **Commutativity:**  $ab = ba$  for all  $a, b \in A$ .
- **Left m-associativity:**  $(ab)c \subseteq a(bc)$  when  $ab = b$ , for all  $a, b, c \in A$ .
- **Right m-associativity:**  $a(bc) \subseteq (ab)c$  when  $bc = c$ , for all  $a, b, c \in A$ .
- **m-associativity:** if it is left and right m-associative.

Note that the prefix ‘m-’ has its origin in the concept of multilattice.

We will focus our interest on the binary relation usually named *natural ordering*, which is defined by

$$a \leq b \text{ if and only if } ab = b$$

Although, in general, this relation is not an ordering, the properties above guarantee that the relation just defined is an ordering. Specifically, it is reflexive if the nd-groupoid is idempotent, the relation is antisymmetric if the nd-groupoid is commutative and, finally, it is transitive if the nd-groupoid is m-associative.

The two following properties of nd-groupoids have an important role in multilattice theory:

- $C_1$ :  $c \in ab$  implies that  $a \leq c$  and  $b \leq c$ .
- $C_2$ :  $c, d \in ab$  and  $c \leq d$  imply that  $c = d$ .

These two properties are named **comparability**. Similarly to lattice theory, we can define algebraically the concept of multisemilattice as an nd-groupoid that satisfies idempotency, commutativity, m-associativity and comparability laws. The ordered and the algebraic definitions of multisemilattice can be proved to be equivalent simply by considering  $a \cdot b = \text{multisup}\{a, b\}$  and  $\leq$  being the natural ordering (see [11, Theorem 2.11]).

**Definition 2.2** (Zadeh, [18]) *Let  $A$  be a nonempty set. A fuzzy relation  $\rho$  on  $A$  is a fuzzy subset of  $A \times A$  (i.e.  $\rho$  is a function from  $A \times A$  to  $[0, 1]$ ).  $\rho$  is reflexive in  $A$  if  $\rho(x, x) = 1$  for all  $x \in A$ ,  $\rho$  is symmetric in  $A$  if  $\rho(x, y) = \rho(y, x)$  for all  $x, y \in A$ , finally,  $\rho$  is transitive if*

$$\sup_{z \in A} \min \{ \rho(x, z), \rho(z, y) \} \leq \rho(x, y) \text{ for all } x, y \in A$$

A fuzzy equivalence relation is a reflexive, symmetric and transitive fuzzy relation.

Since a fuzzy relation in a nonempty set  $A$  is a fuzzy subset of  $A \times A$ , we can define the inclusion, intersection and union of fuzzy relations as follows:  $\rho \subseteq \sigma$  if  $\rho(x, y) \leq \sigma(x, y)$  for all  $x, y \in A$ .  $\bigcap_{i \in \Lambda} \rho_i(x, y) = \inf_{i \in \Lambda} \rho_i(x, y)$  and  $\bigcup_{i \in \Lambda} \rho_i(x, y) = \sup_{i \in \Lambda} \rho_i(x, y)$  for all  $x, y \in A$ .

Let  $FEq(A)$  be the set of fuzzy equivalence relations on a non empty set  $A$ . Murali proved in [14] that  $(FEq(A), \subseteq)$  is a complete lattice where the meet is the intersection and the join is the transitive closure of the union.

The following property is used to provide characterizations of some universal properties in terms of elements; similar definitions are used in other works about fuzzy relations.

**Definition 2.3** *Let  $A$  be a nonempty set and  $\rho$  a fuzzy relation on  $A$ . We say that  $\rho$  satisfies the left (resp. right) sup property if for all nonempty  $X \subseteq A$ , there exist  $x_0$  (resp.  $y_0$ )  $\in X$  such that  $\sup_{x \in X} \rho(x, a) = \rho(x_0, a)$  (resp.  $\sup_{y \in X} \rho(a, y) = \rho(a, y_0)$ ).*

**Definition 2.4** *Let  $\rho$  be a fuzzy relation on a groupoid  $(G, \cdot)$ ; we say that  $\rho$  is right compatible with  $\cdot$  if  $\rho(ac, bc) \geq \rho(a, b)$  for all  $a, b, c \in G$ ; similarly,  $\rho$  is said to be left compatible if  $\rho(ca, cb) \geq \rho(a, b)$  for all  $a, b, c, \in G$ . A congruence on  $G$  is a fuzzy equivalence relation left and right compatible.*

### 3 Fuzzy congruence relations on nd-groupoids

Regarding the extension of the definition of congruence to the non-deterministic case, the following definition was introduced by Bakhshi and Borzooei in [1].

**Definition 3.1** *Let  $(A, \cdot)$  be an nd-groupoid. Then a fuzzy relation  $\rho$  on  $A$  is said to be left (right) compatible if for all  $u \in ax$  ( $u \in xa$ ) there exists  $v \in ay$  ( $v \in ya$ ) and for all  $v \in ay$  ( $v \in ya$ ) there exists  $u \in ax$  ( $u \in xa$ ) such that  $\rho(u, v) \geq \rho(x, y)$ , for all  $x, y, a \in A$  and compatible if it is both fuzzy left and right compatible.*

This definition explicitly uses the fact that the images of the hyperoperator are nonempty. Thus, we propose an alternative definition which generalizes the previous one and adequately handles the empty images.

As a previous step to the consideration of fuzzy congruence relations on a nd-groupoid, let us note that it is possible to extend any fuzzy relation on a set  $A$  to its powerset  $2^A$ ; this construction leads to the definition of an operator  $\hat{\cdot}$  from the set

$FR(A)$  of fuzzy relations on  $A$  to the set  $FR(2^A)$  of fuzzy relations on  $2^A$ . Namely, given a fuzzy relation  $\rho : A \times A \rightarrow [0, 1]$ , its *power extension* is a fuzzy relation  $\widehat{\rho} : 2^A \times 2^A \rightarrow [0, 1]$  defined by

$$\widehat{\rho}(X, Y) = \left( \bigwedge_{x \in X} \bigvee_{y \in Y} \rho(x, y) \right) \wedge \left( \bigwedge_{y \in Y} \bigvee_{x \in X} \rho(x, y) \right)$$

Notice that  $\widehat{\rho}(\emptyset, X) = \widehat{\rho}(X, \emptyset) = 0$ , for all nonempty  $X \subseteq A$ ,  $\widehat{\rho}(\emptyset, \emptyset) = 1$  and  $\widehat{\rho}(\{a\}, \{b\}) = \rho(a, b)$ , for all  $a, b \in A$ .

With this power extension of a fuzzy relation, the definition of fuzzy congruence relation on an nd-groupoid  $(A, \cdot)$  follows exactly the one for the deterministic case:  $\widehat{\rho}(ac, bc) \geq \rho(a, b)$ , for all  $a, b, c \in A$ . It is easy to check that a fuzzy relation that is compatible with  $\cdot$  satisfies this condition but, in general, they are not equivalent as the following example shows:

**Example 3.1** Let  $A = [0, 1]$  be the hypergroupoid endowed with the hyperoperation  $a * b := (0, 1)$  and consider the fuzzy equivalence relation  $\rho(a, b) = 1 - ab$ . Observe that

$$\begin{aligned} \widehat{\rho}(a * c, b * c) &= \left( \bigwedge_{x \in (0,1)} \bigvee_{y \in (0,1)} (1 - xy) \right) \wedge \left( \bigwedge_{y \in (0,1)} \bigvee_{x \in (0,1)} (1 - xy) \right) = \\ &= \left( \bigwedge_{x \in (0,1)} 1 \right) \wedge \left( \bigwedge_{y \in (0,1)} 1 \right) = 1 \geq \rho(a, b) \end{aligned}$$

for all  $a, b, c \in A$ . However, for all  $x \in 0 * c$  and  $y \in b * c$ , we have  $\rho(x, y) < \rho(0, b) = 1$  because otherwise, we would have either  $x = 0$  or  $y = 0$  contradicting that  $x, y \in (0, 1)$ . Thus,  $\rho$  is not compatible with the hyperoperation  $*$ .

Once we have introduced the power extension of a fuzzy relation, in order to use the above condition to define the concept of fuzzy congruence relation, we study the behaviour of the operator  $\widehat{\phantom{x}}$  wrt the properties of reflexivity, simmetry and transitivity.

**Proposition 3.2** Let  $\rho$  be a fuzzy relation in a non-empty set  $A$  and let  $\widehat{\rho}$  be its power extension as defined above. If  $\rho$  is a fuzzy equivalence relation then so is  $\widehat{\rho}$ .

Summarizing the previous considerations we can state the following definition and theorem.

**Definition 3.3** A fuzzy equivalence relation  $\rho$  on an nd-groupoid  $(A, \cdot)$  is said to be a right (resp. left) congruence relation if  $\widehat{\rho}(ac, bc) \geq \rho(a, b)$  (resp.  $\widehat{\rho}(ca, cb) \geq \rho(a, b)$ ) for all  $a, b, c \in A$ . A fuzzy relation is said to be a congruence relation if it is a left and right congruence relation.

**Theorem 3.4** Let  $\rho$  be a fuzzy equivalence relation on an nd-groupoid  $(A, \cdot)$ . Then,  $\rho$  is a fuzzy congruence relation if and only if  $\widehat{\rho}$  is a fuzzy congruence relation in the induced power groupoid  $(2^A, \cdot)$ .

The sup property, which was introduced in Definition 2.3, guarantees the equivalence between our definition of fuzzy congruence relation and the one given in [1].

**Lemma 3.5** *Let  $\rho$  be a fuzzy equivalence relation on an nd-groupoid  $(A, \cdot)$  which satisfies sup property. Then,  $\rho$  is a fuzzy congruence relation if and only if  $\rho$  is compatible with the nd-operation.*

## 4 On the lattice structure of fuzzy congruence relations

In the previous section, we introduce the map  $\hat{\cdot}$  defined over the lattices of fuzzy equivalence relations on an nd-groupoid  $A$  and powerset  $2^A$ . Let us now consider this map on  $FCon(A)$ , the subset of  $FEq(A)$  given by the fuzzy congruence relations. First, notice that Theorem 3.4 guarantees that  $\hat{\cdot}: FCon(A) \rightarrow FCon(2^A)$  is well defined.

In the crisp case, Murali proved in [15] that the set of fuzzy congruence relations on a groupoid  $X$  is a complete sublattice of the set of all fuzzy equivalence relations. This result might suggest that the lattice structure of  $FCon(2^A)$  can be reproduced on  $FCon(A)$ , via the map  $\hat{\cdot}$ . However, although  $\hat{\cdot}$  is injective, since  $\hat{\rho}(\{a\}, \{b\}) = \rho(a, b)$ , for all  $a, b \in A$ , it is not surjective. If it were surjective, then for all  $\Theta \in FCon(2^A)$  the following equality would hold

$$\Theta(X, Y) = \left( \bigwedge_{x \in X} \bigvee_{y \in Y} \Theta(\{x\}, \{y\}) \right) \wedge \left( \bigwedge_{y \in Y} \bigvee_{x \in X} \Theta(\{x\}, \{y\}) \right)$$

but, in general, this is not the case.

**Example 4.1** *Let  $(A, \cdot)$  be the nd-groupoid with  $A = \{a, b\}$  and  $x \cdot y = \{a\}$ , for all  $x, y \in A$ . Consider  $\Theta$  the reflexive and symmetric fuzzy relation on  $2^A$  given by  $\Theta(\{a\}, \{b\}) = 1$ ;  $\Theta(\{a\}, A) = \Theta(\{b\}, A) = 1/2$  and  $\Theta(\emptyset, \{a\}) = \Theta(\emptyset, \{b\}) = \Theta(\emptyset, A) = 0$ . It is routine calculation that  $\Theta$  is a congruence relation, but*

$$\begin{aligned} & \left( \bigwedge_{a \in \{a\}} \bigvee_{y \in A} \Theta(\{a\}, \{y\}) \right) \wedge \left( \bigwedge_{y \in A} \bigvee_{a \in \{a\}} \Theta(\{a\}, \{y\}) \right) = \\ & \left( \bigvee_{y \in A} \Theta(\{a\}, \{y\}) \right) \wedge \left( \bigwedge_{y \in A} \Theta(\{a\}, \{y\}) \right) = \bigwedge_{y \in A} \Theta(\{a\}, \{y\}) = 1 \neq \frac{1}{2} = \Theta(\{a\}, A). \end{aligned}$$

Under the additional assumption of commutativity with respect to the usual composition of binary relations, Bakhshi and Borzooei [1], stated that the set of all fuzzy congruence relations on a hypergroupoid  $(H, \cdot)$  is a complete lattice. The following example proves that this result is not true even in the crisp case and, thus, neither in a fuzzy framework.

**Example 4.2** *Let  $H$  be the set  $\{a, b, c, u_0, u_1, v_0, v_1\}$  provided with a commutative hyperoperation  $*$  which is defined as follows:*

$$\begin{aligned} a * a &= a * b = b * b = \{a, b\}; & a * c &= \{u_0, u_1\}; \\ b * c &= \{v_0, v_1\} & \text{and } x * y &= \{c\}, \quad \text{elsewhere} \end{aligned}$$

Consider  $R, S : H \times H \rightarrow \{0, 1\}$  two binary relations, where  $R$  is the least equivalence relation containing  $\{(a, b), (u_0, v_0), (u_1, v_1)\}$  and  $S$  the least equivalence relation containing  $\{(a, b), (u_0, v_1), (u_1, v_0)\}$ . A tedious check shows that  $R$  and  $S$  commute and are compatible with the hyperoperation  $*$  (they are congruence relations). However, the intersection  $R \cap S$  is not a congruence relation.

As a result of the previous example, the rest of the paper studies conditions that must be satisfied by the nd-groupoid in order to guarantee that  $(FCon(A), \subseteq)$  is a lattice.

**Theorem 4.1** *Let  $(A, \cdot)$  be an nd-groupoid satisfying idempotency and property  $C_1$ , and let  $\rho$  be a fuzzy equivalence relation satisfying the supremum property. Then  $\rho$  is a congruence relation if and only if the following holds:*

$$\text{For all } a, b, c \in A \text{ with } a \leq b \text{ we have that } \widehat{\rho}(ac, bc) \geq \rho(a, b).$$

From now on we focus on the search of properties that ensure the condition of the previous theorem.

**Proposition 4.2** *Let  $(A, \cdot)$  be an  $m$ -associative nd-groupoid that satisfies  $C_1$  and, for  $a, b, c \in A$ , consider  $a \leq b$  and  $z \in bc$ :*

1. *There exists  $w \in ac$  such that  $w \leq z$ .*
2. *Furthermore, if  $(A, \cdot)$  is commutative and  $C_2$  holds and  $\rho$  is a fuzzy congruence relation in  $A$ , then every element  $w$  as in the previous item satisfies that  $\rho(w, z) \geq \rho(a, b)$ .*

In order to obtain the converse result, we need to introduce the following definition.

**Definition 4.3** *An nd-operation  $\cdot$  in a set  $A$  is said to be  **$m$ -distributive** when, for all  $a, b, c \in A$ , if  $a \leq b$  and  $w \in ac$ , then  $bw \cap bc \neq \emptyset$ .*

The justification of this name is that a multilattice  $(A, \vee, \wedge)$  in which both operations are  $m$ -distributive satisfies the following property: for all  $a, b \in A$  with  $a \leq b$  and  $c \in A$ :

1.  $(a \wedge b) \vee c \subseteq (a \vee c) \wedge (b \vee c)$
2.  $(a \vee b) \wedge c \subseteq (a \wedge c) \vee (b \wedge c)$

**Proposition 4.4** *Let  $(M, \cdot)$  be an  $m$ -distributive nd-groupoid that satisfies  $C_1$  and  $a, b, c \in M$ . If  $a \leq b$  and  $w \in ac$  then there exists  $z \in bc$  such that  $w \leq z$ .*

Notice that the properties required as hypotheses of Proposition 4.4 and Proposition 4.2 are those of a multiseuilattice without idempotency. The following result, stated in terms of a multiseuilattice, is a straightforward consequence of these two propositions.

**Proposition 4.5** *Let  $(M, \cdot)$  be an  $m$ -distributive multisemilattice,  $\rho$  be a fuzzy congruence relation and  $a, b, c \in M$ . If  $a \leq b$ ,  $w \in ac$  and  $z \in bc$  with  $w \leq z$  then  $\rho(w, z) \geq \rho(a, b)$ .*

Now, we have all the required properties and lemmas needed in order to face the main goal of this paper, namely, to prove that under certain circumstances the set of congruences of an nd-groupoid is a complete lattice.

**Theorem 4.6** *The set of the fuzzy congruence relations in an  $m$ -distributive multisemilattice  $M$ ,  $FCon(M)$ , is a sublattice of  $FEq(M)$  and, moreover is a complete lattice wrt the fuzzy inclusion ordering.*

## 5 Conclusions and future work

Starting with the usual notion of fuzzy congruence relation in a groupoid, we have introduced the definition of fuzzy congruence relation in an nd-groupoid by means of the power extension of the relation to the power set of the carrier. Our definition is proved to be an adequate generalization of that introduced by Bakhshi and Borzooei in [1]. Moreover, contrariwise to their claim, we have proved that, if  $(A, \cdot)$  is a hypergroupoid (and thus an nd-groupoid), in general,  $(FCon(A), \subseteq)$  is not a lattice.

Finally, we introduce conditions on the nd-groupoid so that we can guarantee the structure of lattice, moreover, of complete lattice of its set of fuzzy congruences. Such conditions are those of an  $m$ -distributive multisemilattice.

As future work on this research line, our plan is to keep investigating new or analogue results concerning congruences on generalized algebraic structures, specially in a non-deterministic sense; in this topic, it seems to be important to study the so-called power structures from a universal standpoint [3, 9]. We will also focus on the corresponding fuzzifications of concepts such as ideal, closure systems and homomorphisms over nd-structures, in the line of [16].

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## References

- [1] M. Bakhshi and R. A. Borzooei. Lattice structure on fuzzy congruence relations of a hypergroupoid. *Inform. Sci.*, 177(16):3305–3313, 2007.
- [2] M. Benado. Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier. I. *Čehoslovack. Mat. Ž.*, 4(79):105–129, 1954.
- [3] I. Bošnjak and R. Madarász. On power structures. *Algebra Discrete Math.*, (2):14–35, 2003.



- [4] P. Cordero, G. Gutiérrez, J. Martínez, M. Ojeda-Aciego, and I. de las Peñas. Congruence relations on hypergroupoids and nd-groupoids. In *14th Spanish Conference on Fuzzy Logic and Technology*, 2008. (To appear).
- [5] P. Cordero, G. Gutiérrez, J. Martínez, M. Ojeda-Aciego, and I. de las Peñas. Congruence relations on multilattices. In *8th International FLINS Conference on Computational Intelligence in Decision and Control*, 2008. (To appear).
- [6] P. Das. Lattice of fuzzy congruences in inverse semigroups. *Fuzzy Sets and Systems*, 91(3):399–408, 1997.
- [7] T. Dutta and B. Biswas. On fuzzy congruence of a near-ring module. *Fuzzy Sets and Systems*, 112(2):399–408, 2000.
- [8] J. Goguen. L-fuzzy sets. *J. Math. Anal. Appl.*, 18:145–174, 1967.
- [9] R. S. Madarász. Remarks on power structures. *Algebra Universalis*, 34(2):179–184, 1995.
- [10] J. Martínez, G. Gutiérrez, I. P. de Guzmán, and P. Cordero. Generalizations of lattices via non-deterministic operators. *Discrete Math.*, 295(1-3):107–141, 2005.
- [11] J. Martínez, G. Gutiérrez, I. P. de Guzmán, and P. Cordero. Multilattices via multiseimlattices. In *Topics in applied and theoretical mathematics and computer science*, pages 238–248. WSEAS, 2001.
- [12] F. Marty. Sur une généralisation de la notion de groupe. In *8th Congress Math. Scandinaves*, pages 45–49, Stockholm, 1934.
- [13] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Fuzzy logic programming via multilattices. *Fuzzy Sets and Systems*, 158(6):674–688, 2007.
- [14] V. Murali. Fuzzy equivalence relations. *Fuzzy Sets and Systems*, 30(2):155–163, 1989.
- [15] V. Murali. Fuzzy congruence relations. *Fuzzy Sets and Systems*, 41(3):359–369, 1991.
- [16] U. M. Swamy and D. V. Raju. Fuzzy ideals and congruences of lattices. *Fuzzy Sets and Systems*, 95(2):249–253, 1998.
- [17] Y. Tan. Fuzzy congruences on a regular semigroup. *Fuzzy Sets and Systems*, 117(3):399–408, 2001.
- [18] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.