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The Chu construction and generalized formal concept analysis

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We continue studying the connections between the Chu construction on the category ChuCors of formal contexts and Chu correspondences, and generalizations of Formal Concept Analysis (FCA). All the required constructions like categorical product, tensor product, together with its bifunctor properties are introduced and proved. The final section focuses on how the second-order generalization of FCA can be built up in terms of the Chu construction.

Keywords: formal concept analysis, category theory, Chu construction

1. Introduction

This paper focuses on categorical developments of Formal Concept Analysis (FCA): on the one hand, the importance of category theory as a foundational tool was discovered soon after its very introduction by Eilenberg and MacLane about seventy years ago; on the other hand, FCA has largely shown both its practical applications and its capability to be generalized to more abstract frameworks, and this is why it has become a very active research topic in the recent years. Just to name a few examples, in (Stell 2014) one can see a framework for FCA in which the sets of objects and attributes are no longer unstructured but have a hypergraph structure interpreted using certain ideas from mathematical morphology; in (Huang, Li, and Guo 2014) we can see an application of the FCA formalism to other areas, specifically, a representation of algebraic domains in terms of FCA.

The main theoretical tool in this paper is the Chu construction (Chu 1979). This notion has interesting applications: on the one hand, it generates *-autonomous categories (which turn out to give models of linear logic); on the other hand, the Chu construction together with the closely related notion of Chu space have already been applied to represent quantum physical systems and their symmetries (Abramsky 2012, 2013).

Roughly speaking, in this work, we continue our study of the categorical foundations of formal concept analysis. It is worth to note that there are other authors also interested in this research line, for instance, in (Hitzler and Zhang 2004; Zhang and Shen 2006) it has been proved that certain concept structures can be approximated using a cartesian closed category; (Krajči 2007) provided a categorical construction

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of certain extensions of FCA; in (Krötzsch, Hitzler, and Zhang 2005) morphisms have received a categorical treatment as a means for modelling communication; in (Denniston, Melton, and Rodabaugh 2013), links among topological systems, Chu systems, and formal contexts are built in terms of the so-called category of formal context interchanges (studied in their crisp and L-fuzzy versions). The properties of the tensor product between contexts will enable us to consider the Chu construction on the category ChuCors of formal contexts and Chu correspondences between them.

Specifically, we continue the research line of the authors on the categorical foundation of FCA (Krídlo, Krajči, and Ojeda-Aciego 2012; Krídlo and Ojeda-Aciego 2011, 2014). The goal of this paper is to highlight the importance of the Chu construction in the categorical description of the theory of FCA and its generalisations, in particular, the second-order formal concept analysis, introduced by Krídlo, Krajči, and Antoni (2016), can be represented in terms of the arrows of CHU(ChuCors, $\perp$). The Chu construction plays here the role of some recipe for constructing a suitable category that covers the second-order generalisation of FCA.

The structure of this paper is the following: in Section 2 we recall the preliminary notions required both from category theory and FCA. Then, the various categorical properties of the input category which are required (like the existence of categorical and tensor products) are developed in detail in Sections 3 and 4. Finally, we elaborate on one application of the Chu construction, presented in Section 5 where it is shown how to construct second-order formal contexts starting from the category of classical formal contexts and Chu correspondences (ChuCors).

2. Preliminaries

In order to make the manuscript self-contained, the fundamental notions and their main properties are recalled in this section.

Definition 1. *A* formal context *is any triple* $\mathcal{C} = \langle B, A, R\rangle$ *where* B *and* A *are finite sets and* $R \subseteq B \times A$ *is a binary relation. It is customary to say that* B *is a set of* objects, A *is a set of* attributes *and* R *represents a relation between objects and attributes.*

Given a formal context $\langle B, A, R\rangle$*, the derivation (or concept-forming) operators are a pair of mappings* $\uparrow : 2^B \to 2^A$ *and* $\downarrow : 2^A \to 2^B$ *such that if* $X \subseteq B$*, then* ${\uparrow}X$ *is the set of all attributes which are related to every object in* X *and, similarly, if* $Y \subseteq A$*, then* ${\downarrow}Y$ *is the set of all objects which are related to every attribute in* Y*.*

In order to simplify the description of subsequent computations, and specially in the L-fuzzy case, it is convenient to describe the concept forming operators in terms of characteristic functions, namely, considering the subsets as functions on the set of Boolean values. Specifically, given $X \subseteq B$ and $Y \subseteq A$, we can consider the sets ${\uparrow}X$ and ${\downarrow}Y$ as mappings ${\uparrow}X : A \to \{0, 1\}$ and ${\downarrow}Y : B \to \{0, 1\}$ defined by

(1) ${\uparrow}X(a) = \bigwedge_{b\in B} \big((b \in X) \Rightarrow ((b, a) \in R)\big)$ for any $a \in A$

(2) ${\downarrow}Y(b) = \bigwedge_{a\in A} \big((a \in Y) \Rightarrow ((b, a) \in R)\big)$ for any $b \in B$

where the infimum $\bigwedge$ is considered in the set of Boolean values and $\Rightarrow$ is the truth-function of the implication of classical logic.

Definition 2. *A* formal concept *of a formal context* $\mathcal{C} = \langle B, A, R\rangle$ *is a pair of* *sets* $\langle X, Y \rangle \in 2^B \times 2^A$ *which is a fixpoint of the pair of concept-forming operators, namely,* $\uparrow X = Y$ *and* $\downarrow Y = X$*. The object part* X *is called the* extent *and the attribute part* Y *is called the* intent*. The set of all formal concepts of a context* $\mathcal{C}$ *will be denoted by* $\mathrm{CL}(\mathcal{C})$*.*

Two main constructions have been traditionally considered in order to relate two formal contexts: the bonds and the Chu correspondences.

Definition 3. *Let* $\mathcal{C}_1 = \langle B_1, A_1, R_1 \rangle$ *and* $\mathcal{C}_2 = \langle B_2, A_2, R_2 \rangle$ *be two formal contexts. A* bond *between* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *is any relation* $\beta \in 2^{B_1 \times A_2}$ *such that, when interpreted as a table, its columns are extents of* $\mathcal{C}_1$ *and its rows are intents of* $\mathcal{C}_2$*. All bonds between such contexts will be denoted by* $\mathrm{Bonds}(\mathcal{C}_1, \mathcal{C}_2)$*.*

The notion of Chu correspondence between contexts can be seen as an alternative inter-contextual structure which, instead, links intents of $\mathcal{C}_1$ and extents of $\mathcal{C}_2$.

Definition 4. *Consider* $\mathcal{C}_1 = \langle B_1, A_1, R_1 \rangle$ *and* $\mathcal{C}_2 = \langle B_2, A_2, R_2 \rangle$ *two formal contexts. A* Chu correspondence *between* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *is any pair* $\varphi = \langle \varphi_L, \varphi_R \rangle$ *of mappings* $\varphi_L \colon B_1 \to \mathrm{Ext}(\mathcal{C}_2)$ *and* $\varphi_R \colon A_2 \to \mathrm{Int}(\mathcal{C}_1)$ *such that for all* $(b_1, a_2) \in B_1 \times A_2$ *it holds that* $\uparrow_2\big(\varphi_L(b_1)\big)(a_2) = \downarrow_1\big(\varphi_R(a_2)\big)(b_1)$*.*
All Chu correspondences between such contexts will be denoted by $\mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$*.*

The notions of bond and Chu correspondence are interchangeable; specifically, we can consider the bond β_φ associated to a Chu correspondence φ from $\mathcal{C}_1$ to $\mathcal{C}_2$ defined for $b_1 \in B_1, a_2 \in A_2$ as follows

$$\beta_\varphi(b_1, a_2) = \uparrow_2 \big(\varphi_L(b_1)\big)(a_2) = \downarrow_1 \big(\varphi_R(a_2)\big)(b_1)$$

Similarly, we can consider the Chu correspondence φ_β associated to a bond φ defined by the following pair of mappings:

$$\varphi_{\beta L}(b_1) = \downarrow_2\big(\beta(b_1)\big) \text{ for all } o_1 \in B_1$$
$$\varphi_{\beta R}(a_2) = \uparrow_1\big(\beta^t(a_2)\big) \text{ for all } a_2 \in A_2$$

The set of all bonds (resp. Chu correspondences) between any two formal contexts endowed with the ordering given by set inclusion has a complete lattice structure. Moreover, both complete lattices are dually isomorphic.

In order to formally define the composition of two Chu correspondences, we need to introduce the extension principle below:

Definition 5. *Given a mapping* $\varphi \colon X \to 2^Y$ *we define its extended mapping* $\varphi_+ \colon 2^X \to 2^Y$ *defined by* $\varphi_+(M) = \bigcup_{x \in M} \varphi(x)$*, for all* $M \in 2^X$*.*

The set of formal contexts together with Chu correspondences as morphisms forms a category denoted by ChuCors. Specifically:

- *objects* formal contexts
- *arrows* Chu correspondences
- *identity arrow* $\iota \colon \mathcal{C} \to \mathcal{C}$ of context $\mathcal{C} = \langle B, A, R \rangle$
  - $\iota_L(o) = \downarrow\uparrow(\{b\})$, for all $b \in B$
  - $\iota_R(a) = \uparrow\downarrow(\{a\})$, for all $a \in A$
- *composition* $\varphi_2 \circ \varphi_1 \colon \mathcal{C}_1 \to \mathcal{C}_3$ *of arrows* $\varphi_1 \colon \mathcal{C}_1 \to \mathcal{C}_2$, $\varphi_2 \colon \mathcal{C}_2 \to \mathcal{C}_3$ (where $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$, $i \in \{1, 2, 3\}$)
  - $(\varphi_2 \circ \varphi_1)_L : B_1 \to 2^{B_3}$ and $(\varphi_2 \circ \varphi_1)_R : A_3 \to 2^{A_1}$
  - $(\varphi_2 \circ \varphi_1)_L(b_1) = \downarrow_3\uparrow_3(\varphi_{2L+}(\varphi_{1L}(b_1)))$

- $(\varphi_2 \circ \varphi_1)_R(a_3) = \uparrow_1\downarrow_1(\varphi_{1R^+}(\varphi_{2R}(a_3)))$

Recall that the category ChuCors is *-autonomous and equivalent to the category of complete lattices and isotone Galois connections, more results on this category and its L-fuzzy extensions can be found in (Krídlo and Ojeda-Aciego 2011; Krídlo, Krajči, and Ojeda-Aciego 2012; Krídlo and Ojeda-Aciego 2014; Mori 2008).

3. Categorical product on ChuCors

In this section, the category ChuCors is proved to contain all finite categorical products, that is, it is a Cartesian category. To begin with, it is convenient to recall the notion of categorical product.

Definition 6. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two objects in a category. A* product *of $\mathcal{C}_1$ and $\mathcal{C}_2$ is an object $\mathcal{P}$ with arrows $\pi_i\colon \mathcal{P} \to \mathcal{C}_i$ for $i \in \{1,2\}$ satisfying the following condition: For any object $\mathcal{D}$ and arrows $\delta_i\colon \mathcal{D} \to \mathcal{C}_i$ for $i \in \{1,2\}$, there exists a unique arrow $\gamma\colon \mathcal{D} \to \mathcal{P}$ such that $\gamma \circ \pi_i = \delta_i$ for all $i \in \{1,2\}$.*

The construction will use the notion of disjoint union of two sets $S_1 \uplus S_2$ which can be formally described as $(\{1\} \times S_1) \cup (\{2\} \times S_2)$ and, therefore, their elements will be denoted as ordered pairs (i,s) where $i \in \{1,2\}$ and $s \in S_i$. Now, we can proceed with the construction:

Definition 7. *Consider $\mathcal{C}_1 = \langle B_1, A_1, R_1\rangle$ and $\mathcal{C}_2 = \langle B_2, A_2, R_2\rangle$ two formal contexts. The* product *of such contexts is a new formal context $\mathcal{C}_1 \times \mathcal{C}_2 = \langle B_1 \uplus B_2, A_1 \uplus A_2, R_{1\times 2}\rangle$ where the relation $R_{1\times 2}$ is given by*

$$((i,b),(j,a)) \in R_{1\times 2} \text{ if and only if } \big((i=j) \Rightarrow (b,a) \in R_i\big)$$

for any $(b,a) \in B_i \times A_j$ and $(i,j) \in \{1,2\} \times \{1,2\}$.

Lemma 1. *Given $\mathcal{C}_1 = \langle B_1, A_1, R_1\rangle$ and $\mathcal{C}_2 = \langle B_2, A_2, R_2\rangle$ two formal contexts, the product context $\mathcal{C}_1 \times \mathcal{C}_2$ fulfills the property of the categorical product on* ChuCors*.*

Proof. We just have to define the projection arrows $\langle \pi_{iL}, \pi_{iR}\rangle \in \mathrm{Chu}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{C}_i)$ for $i \in \{1,2\}$ as follows

- $\pi_{iL}\colon B_1 \uplus B_2 \to \mathrm{Ext}(\mathcal{C}_i) \subseteq 2^{B_i}$
- $\pi_{iR}\colon A_i \to \mathrm{Int}(\mathcal{C}_1 \times \mathcal{C}_2) \subseteq 2^{A_1 \cup A_2}$
- satisfying that for any $(k,x) \in B_1 \uplus B_2$ and $a_i \in A_i$ it holds that

$$\uparrow_i(\pi_{iL}(k,x))(a_i) = \downarrow_{1\times 2}(\pi_{iR}(a_i))(k,x)$$

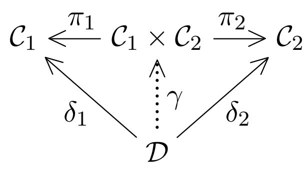
The definition of the projections is given below

$$\pi_{iL}(k,x)(b_i) = \begin{cases} \downarrow_i\uparrow_i(\chi_x)(b_i) \text{ for } k = i \\ \downarrow_i\uparrow_i\left(\overline{0}\right)(b_i) \text{ for } k \neq i \end{cases} \text{for any } (k,x) \in B_1 \uplus B_2 \text{ and } b_i \in B_i$$

$$\pi_{iR}(a_i)(k,y) = \begin{cases} \uparrow_i\downarrow_i(\chi_{a_i})(y) \text{ for } k = i \\ \uparrow_k\downarrow_k\left(\overline{0}\right)(y) \text{ for } k \neq i \end{cases} \text{for any } (k,y) \in A_1 \uplus A_2 \text{ and } a_i \in A_i.$$

The proof that the definitions above actually provide a Chu correspondence is just a long, although straightforward, computation and it is omitted.

Now, we have to show that for any formal context $\mathcal{D} = \langle E, F, G\rangle$, where $G \subseteq E \times F$ and any pair of arrows (δ_1, δ_2) with $\delta_i\colon \mathcal{D} \to \mathcal{C}_i$ for all $i \in \{1,2\}$, there exists a unique morphism $\gamma\colon \mathcal{D} \to \mathcal{C}_1 \times \mathcal{C}_2$ such that the following diagram commutes:



We give just the definition of γ as a pair of mappings $\gamma_L\colon E \to 2^{B_1 \uplus B_2}$ and $\gamma_R\colon A_1 \uplus A_2 \to 2^F$

- $\gamma_L(e)(k,x) = \delta_{kL}(e)(x)$ for any $e \in E$ and $(k,x) \in B_1 \uplus B_2$.
- $\gamma_R(k,y)(f) = \delta_{kR}(y)(f)$ for any $f \in F$ and $(k,y) \in A_1 \uplus A_2$.

Checking the condition of categorical product is again straightforward. □

We have just proved that binary products exist, but a Cartesian category requires the existence of *all finite products*. If we recall the well-known categorical theorem which states that if a category has a terminal object and binary product, then it has all finite products, we need to prove just the existence of a terminal object (namely, the nullary product) in order to prove the category ChuCors to be Cartesian.

Any formal context of the form $\langle B, A, B \times A\rangle$ where the incidence relation is the full Cartesian product of the sets of objects and attributes is (isomorphic to) the terminal object of ChuCors. Such a formal context has just one formal concept $\langle B, A\rangle$; hence, from any other formal context there is just one Chu correspondence to $\langle B, A, B \times A\rangle$.

The explicit construction of a general product (not necessarily either binary or nullary) is given below:

Definition 8. *Let $\{\mathcal{C}_i\}_{i\in I}$ be an indexed family of formal contexts $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$, the product of the family is the formal context $\prod_{i\in I} \mathcal{C}_i$ defined by*

$$\prod_{i\in I} \mathcal{C}_i = \left\langle \biguplus_{i\in I} B_i, \biguplus_{i\in I} A_i, R_{\times I} \right\rangle$$

where $\big((k,b),(m,a)\big) \in R_{\times I} \Leftrightarrow \big((k = m) \Rightarrow (b,a) \in R_k\big)$.

It is worth to note that the arbitrary product of contexts commutes with both the concept lattice construction and the bonds between contexts. These two results are explicitly proved below.

Lemma 2. *Let $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$ be a formal context for $i \in I$. It holds that $\mathrm{CL}(\prod_{i\in I} \mathcal{C}_i)$ is isomorphic to $\prod_{i\in I} \mathrm{CL}(\mathcal{C}_i)$.*

Proof (Sketch). We only prove that the intents of both concept lattices coincide; the rest of the proof is straightforward.

Consider an arbitrary subset X of $\biguplus_{i\in I} B_i$, and let us write X_j to refer to the subset of elements of X belonging to B_j, namely, $X_j = X \cap (\{j\} \times B_j)$ for all $j \in I$.

It holds that

$$
\begin{aligned}
(j,a) \in \uparrow(X) &\Leftrightarrow \bigwedge_{(k,b)\in\biguplus_{i\in I} B_i} \big((k,b) \in X \Rightarrow \big((k,b),(j,a)\big) \in R\big) \\
&\Leftrightarrow \bigwedge_{(k,b)\in\biguplus_{i\in I} B_i} \big((k,b) \in X \Rightarrow \big((k=j) \Rightarrow (b,a) \in R_j\big)\big) \\
&\Leftrightarrow 1 \wedge \bigwedge_{b\in B_j} \big((b \in X_j) \Rightarrow (b,a) \in R_j\big) \\
&\Leftrightarrow a \in \uparrow_j(X_j) \text{ for any } j \in I \text{ and } a \in A_j
\end{aligned}
$$

Hence $\uparrow(X) = \prod_{i\in I} \uparrow_i(X_i)$. □

Lemma 3. *Let I and J be two index sets, and consider the two sets of formal contexts $\{\mathcal{C}_i\}_{i\in I}$ and $\{\mathcal{D}_j\}_{j\in J}$. The following isomorphism holds*

$$\text{Bonds}\Big(\prod_{i\in I}\mathcal{C}_i, \prod_{j\in J}\mathcal{D}_j\Big) \cong \prod_{(i,j)\in I\times J} \text{Bonds}(\mathcal{C}_i,\mathcal{D}_j)\,.$$

Proof. Consider $\beta \in \text{Bonds}(\prod_{i\in I}\mathcal{C}_i, \prod_{j\in J}\mathcal{D}_j)$, by the definition of bonds and the previous lemma, every row is from $\prod_{j\in J}\text{Int}(\mathcal{D}_j)$. Similarly every column of β is from $\prod_{i\in I}\text{Ext}(\mathcal{C}_i)$. Hence there exists a set of bonds $\{\beta_{ij} \in \text{Bonds}(\mathcal{C}_i,\mathcal{D}_j) \mid (i,j) \in I\times J\}$ such that $\beta = \prod_{(i,j)\in I\times J}\beta_{ij}$. □

Corollary 1. *Let I and J be two index sets, and consider the two sets of formal contexts $\{\mathcal{C}_i\}_{i\in I}$ and $\{\mathcal{D}_j\}_{j\in J}$. The following isomorphism holds*

$$\text{ChuCors}\Big(\prod_{i\in I}\mathcal{C}_i, \prod_{j\in J}\mathcal{D}_j\Big) \cong \prod_{(i,j)\in I\times J} \text{ChuCors}(\mathcal{C}_i,\mathcal{D}_j)\,.$$

Proof. From the previous lemma and the dual isomorphism between Chu correspondences and bonds. □

4. Tensor product and its bifunctor property

Apart from the categorical product, another product-like construction can be given in the category ChuCors, for which the notion of transposed context $\mathcal{C}^*$ is needed.

Given a formal context $\mathcal{C} = \langle B, A, R\rangle$, its transposed context is $\mathcal{C}^* = \langle A, B, R^t\rangle$, where $R^t(a,b)$ holds iff $R(b,a)$ holds. Note that if $\varphi \in \text{Chu}(\mathcal{C}_1,\mathcal{C}_2)$, then we can consider $\varphi^* \in \text{Chu}(C_2^*, C_1^*)$ defined by $\varphi^*_L = \varphi_R$ and $\varphi^*_R = \varphi_L$.

Definition 9. *The tensor product of formal contexts $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$ for $i \in \{1,2\}$ is defined as the formal context $\mathcal{C}_1 \boxtimes \mathcal{C}_2 = \langle B_1 \times B_2, \text{Chu}(\mathcal{C}_1,\mathcal{C}_2^*), R_\boxtimes\rangle$ where*

$$R_\boxtimes\big((b_1,b_2),\varphi\big) = \downarrow_2\big(\varphi_L(b_1)\big)(b_2).$$

In (Mori 2008) one can find the properties of the tensor product above, together with the result that ChuCors with $\boxtimes$ is a symmetric and monoidal category. Those results were later extended to the L-fuzzy case in (Krídlo, Krajči, and Ojeda-Aciego 2012). In both papers, the structure of the formal concepts of a tensor product context was established as an ordered pair formed by a bond and a set of Chu correspondences. In this respect, it is worth to cite the work by Deiters and Erné (2009), which also studied the categorical and the tensor product of contexts.

Lemma 4. *Given an arbitrary formal concept (β, X) of the tensor product $\mathcal{C}_1 \boxtimes \mathcal{C}_2$, it holds that $\beta = \bigwedge_{\psi \in X} \beta_\psi$ and $X = \{\psi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*) \mid \beta \leq \beta_\psi\}$.*

Proof. Recall that $\langle \beta, X\rangle \in \mathrm{Bonds}(\mathcal{C}_1, \mathcal{C}_2^*) \times 2^{\mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*)}$, and let X be an arbitrary subset of $\mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*)$. Then, for all $(b_1, b_2) \in B_1 \times B_2$, we have

$$\begin{aligned}
\downarrow_{\mathcal{C}_1 \boxtimes \mathcal{C}_2}(X)(b_1, b_2) &= \bigwedge_{\psi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*)} \big((\psi \in X) \Rightarrow \downarrow_2(\psi_L(b_1))(b_2)\big) \\
&= \bigwedge_{\psi \in X} \downarrow_2(\psi_L(b_1))(b_2) = \bigwedge_{\psi \in X} \beta_\psi(b_1, b_2)
\end{aligned}$$

Let β be an arbitrary subset of $B_1 \times B_2$. Then, for all $\psi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*)$

$$\begin{aligned}
\uparrow_{\mathcal{C}_1 \boxtimes \mathcal{C}_2}(\beta)(\psi) &= \bigwedge_{(b_1, b_2) \in B_1 \times B_2} \big(\beta(b_1, b_2) \Rightarrow \downarrow_2(\psi_L(b_1))(b_2)\big) \\
&= \bigwedge_{(b_1, b_2) \in B_1 \times B_2} \big(\beta(b_1, b_2) \Rightarrow \beta_\psi(b_1, b_2)\big)
\end{aligned}$$

Hence $\uparrow_{\mathcal{C}_1 \boxtimes \mathcal{C}_2}(\beta) = \{\psi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2^*) \mid \beta \leq \beta_\psi\}$. □

The notion of product of a context with a Chu correspondence is introduced below.

Definition 10. *Let $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$ for $i \in \{0, 1, 2\}$ be formal contexts, and consider $\varphi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$. The product $(\mathcal{C}_0 \boxtimes \varphi)$ is defined as the pair of mappings*

$$(\mathcal{C}_0 \boxtimes \varphi)_L \colon B_0 \times B_1 \to 2^{B_0 \times B_2} \qquad (\mathcal{C}_0 \boxtimes \varphi)_R \colon \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_2) \to 2^{\mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_1)}$$

defined as follows:

- $(\mathcal{C}_0 \boxtimes \varphi)_L(b, b_1)(o, b_2) = \downarrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2} \uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}(\gamma_\varphi^{b, b_1})(o, b_2)$ *where* $\gamma_\varphi^{b, b_1}(o, b_2) = \big((b = o) \wedge \varphi_L(b_1)(b_2)\big)$ *for any* $b, o \in B_0$, $b_i \in B_i$ *with* $i \in \{1, 2\}$
- $(\mathcal{C}_0 \boxtimes \varphi)_R(\psi_2)(\psi_1) = \big(\psi_1 \geq (\psi_2 \circ \varphi^*)\big)$ *for any* $\psi_i \in \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_i)$

As one could expect, the result is a Chu correspondence. Specifically:

Lemma 5. *$\mathcal{C}_0 \boxtimes \varphi$ is a Chu correspondence between the products of the contexts $\mathcal{C}_0 \boxtimes \mathcal{C}_1$ and $\mathcal{C}_0 \boxtimes \mathcal{C}_2$.*

Proof. By definition, it holds that $(\mathcal{C}_0 \boxtimes \varphi)_L(b, b_1) \in \mathrm{Ext}(\mathcal{C}_0 \boxtimes \mathcal{C}_2)$ for all $(b, b_1) \in B_0 \times B_1$; moreover, by Lemma 4 we also have $(\mathcal{C}_0 \boxtimes \varphi)_R(\psi) \in \mathrm{Int}(\mathcal{C}_0 \boxtimes \mathcal{C}_1)$ for all $\psi \in \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_1)$.

Now, consider arbitrary $b \in B_0$, $b_1 \in B_1$ and $\psi_2 \in \mathrm{Chu}(\mathcal{C}_0, \mathcal{C}_2^*)$:

$$
\begin{aligned}
&\uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}\big((\mathcal{C}_0 \boxtimes \varphi)_L(b, b_1)\big)(\psi_2) \\
&\quad = \uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}\downarrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}\uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}(\gamma_\varphi^{b,b_1})(\psi_2) \\
&\quad = \uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}(\gamma_\varphi^{b,b_1})(\psi_2) \\
&\quad = \bigwedge_{(o,b_2)\in B_0\times B_2} \big(\gamma_\varphi^{b,b_1}(o, b_2) \Rightarrow \downarrow(\psi_{2R}(b_2))(o)\big) \\
&\quad = \bigwedge_{(o,b_2)\in B_0\times B_2} \Big(\big((o = b) \wedge \varphi_L(b_1)(b_2)\big) \Rightarrow \downarrow(\psi_{2R}(b_2))(o)\Big) \\
&\quad = \bigwedge_{o\in B_0} \bigwedge_{b_2\in B_2} \Big((o = b) \Rightarrow \big(\varphi_L(b_1)(b_2) \Rightarrow \downarrow(\psi_{2R}(b_2))(o)\big)\Big) \\
&\quad = \bigwedge_{o\in B_0} \Big((o = b) \Rightarrow \bigwedge_{b_2\in B_2} (\varphi_L(b_1)(b_2) \Rightarrow \downarrow(\psi_{2R}(b_2))(o))\Big) \\
&\quad = \bigwedge_{b_2\in B_2} \big(\varphi_L(b_1)(b_2) \Rightarrow \downarrow(\psi_{2R}(b_2))(b)\big) \\
&\quad = \bigwedge_{b_2\in B_2} \Big(\varphi_L(b_1)(b_2) \Rightarrow \bigwedge_{a\in A} (\psi_{2R}(b_2)(a) \Rightarrow R(b, a))\Big) \\
&\quad = \bigwedge_{a\in A} \Big(\bigvee_{b_2\in B_2} (\varphi_L(b_1)(b_2) \wedge \psi_{2R}(b_2)(a)) \Rightarrow R(b, a)\Big) \\
&\quad = \bigwedge_{a\in A} \Big(\big(\psi_{2R+}(\varphi_L(b_1))(a)\big) \Rightarrow R(b, a)\Big) \\
&\quad = \downarrow(\psi_{2R+}(\varphi_L(b_1))(b) = \downarrow\uparrow\downarrow(\psi_{2R+}(\varphi_L(b_1))(b) = \downarrow((\varphi \circ \psi_2)_R(b_1))(b)
\end{aligned}
$$

Note the use above of the extended mapping as given in Definition 5 in relation to the composition of Chu correspondences.

On the other hand, we have

$$
\begin{aligned}
&\downarrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_1}((\mathcal{C}_0 \boxtimes \varphi)_R(\psi_2))(b, b_1) \\
&\quad = \bigwedge_{\psi_1\in \mathrm{Chu}(\mathcal{C}_0,\mathcal{C}_1)} ((\mathcal{C}_0 \boxtimes \varphi)_R(\psi_2)(\psi_1) \Rightarrow \downarrow(\psi_{1R}(b_1))(b)) \\
&\quad = \bigwedge_{\psi_1\in \mathrm{Chu}(\mathcal{C}_0,\mathcal{C}_1)} ((\psi_1 \geq \varphi \circ \psi_2) \Rightarrow \downarrow(\psi_{1R}(b_1))(b)) \\
&\quad = \bigwedge_{\substack{\psi_1\in \mathrm{Chu}(\mathcal{C}_0,\mathcal{C}_1)\\ \psi_1 \geq \varphi\circ\psi_2}} \downarrow(\psi_{1R}(b_1))(b) = \downarrow((\varphi \circ \psi_2)_R(b_1))(b)
\end{aligned}
$$

Hence $\uparrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_2}((\mathcal{C}_0 \boxtimes \varphi)_L(b, b_1))(\psi_2) = \downarrow_{\mathcal{C}_0 \boxtimes \mathcal{C}_1}((\mathcal{C}_0 \boxtimes \varphi)_R(\psi_2))(b, b_1)$. Therefore, if $\varphi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$, then $\mathcal{C}_0 \boxtimes \varphi \in \mathrm{Chu}(\mathcal{C}_0 \boxtimes \mathcal{C}_1, \mathcal{C}_0 \boxtimes \mathcal{C}_2)$. □

Given a formal context $\mathcal{C}$, the tensor product $\mathcal{C} \boxtimes -$ forms a mapping between objects of ChuCors assigning to any formal context $\mathcal{D}$ the formal context $\mathcal{C} \boxtimes \mathcal{D}$. Moreover, given any arrow $\varphi \in \mathrm{Chu}(\mathcal{C}_1, \mathcal{C}_2)$ we have $\mathcal{C} \boxtimes \varphi \in \mathrm{Chu}(\mathcal{C} \boxtimes \mathcal{C}_1, \mathcal{C} \boxtimes \mathcal{C}_2)$. We will show that this mapping preserves the unit arrows and the composition of

Chu correspondences. Hence the mapping forms an endofunctor on ChuCors, that is, a covariant functor from the category ChuCors into itself.

To begin with, let us recall the definition of functor between two categories:

Definition 11 (See (Barr and Wells 1995))**.** *A covariant functor* $F\colon \mathrm{C} \to \mathrm{D}$ *between categories* C *and* D *is a mapping of objects to objects and arrows to arrows, in such a way that:*

- *For any morphism* $f\colon A \to B$*, one has* $F(f)\colon F(A) \to F(B)$
- $F(g \circ f) = F(g) \circ F(f)$
- $F(1_A) = 1_{F(A)}$.

Lemma 6. *Given a formal context* $\mathcal{C}$*, it holds that* $\mathcal{C} \boxtimes -$ *is an endofunctor on* ChuCors*.*

Proof. Consider the unit morphism $\iota_{\mathcal{C}_1}$ of a formal context $\mathcal{C}_1 = \langle B_1, A_1, R_1 \rangle$, and let us show that $(\mathcal{C} \boxtimes \iota_{\mathcal{C}_1}) = \iota_{\mathcal{C}\boxtimes\mathcal{C}_1}$. In other words, $\mathcal{C} \boxtimes -$ respects the identity arrows in ChuCors.

$$
\begin{aligned}
&\uparrow_{\mathcal{C}\boxtimes\mathcal{C}_1}\big((\mathcal{C}\boxtimes\iota_{\mathcal{C}_1})(b,b_1)\big)(\psi)\\
&\quad= \bigwedge_{(o,o_1)\in B\times B_1}\Big(\big((o=b)\wedge \iota_{\mathcal{C}_1 L}(b_1)(o_1)\big) \Rightarrow \downarrow_1(\psi_L(o))(o_1)\Big)\\
&\quad= \bigwedge_{o_1\in B_1}\big(\downarrow_1\uparrow_1(\chi_{b_1})(o_1) \Rightarrow \downarrow_1(\psi_L(b))(o_1)\big)\\
&\quad= \bigwedge_{o_1\in B_1}\Big(\downarrow_1\uparrow_1(\chi_{b_1})(o_1) \Rightarrow \bigwedge_{a_1\in A_1}\big(\psi_L(b)(a_1) \Rightarrow R(o_1,a_1)\big)\Big)\\
&\quad= \bigwedge_{o_1\in B_1}\bigwedge_{a_1\in A_1}\Big(\downarrow_1\uparrow_1(\chi_{b_1})(o_1) \Rightarrow \big(\psi_L(b)(a_1) \Rightarrow R(o_1,a_1)\big)\Big)\\
&\quad= \bigwedge_{o_1\in B_1}\bigwedge_{a_1\in A_1}\Big(\psi_L(b)(a_1) \Rightarrow \big(\downarrow_1\uparrow_1(\chi_{b_1})(o_1) \Rightarrow R(o_1,a_1)\big)\Big)\\
&\quad= \bigwedge_{a_1\in A_1}\Big(\psi_L(b)(a_1) \Rightarrow \bigwedge_{o_1\in B_1}\big(\downarrow_1\uparrow_1(\chi_{b_1})(o_1) \Rightarrow R(o_1,a_1)\big)\Big)\\
&\quad= \bigwedge_{a_1\in A_1}\big(\psi_L(b)(a_1) \Rightarrow \uparrow_1\downarrow_1\uparrow_1(\chi_{b_1})(a_1)\big)\\
&\quad= \bigwedge_{a_1\in A_1}\big(\psi_L(b)(a_1) \Rightarrow R_1(b_1,a_1)\big) = \downarrow_1(\psi_L(b))(b_1)
\end{aligned}
$$

and, on the other hand, we have

$$
\begin{aligned}
&\uparrow_{\mathcal{C}\boxtimes\mathcal{C}_1}(\iota_{\mathcal{C}\boxtimes\mathcal{C}_1}(b,b_1))(\psi)\\
&\quad= \uparrow_{\mathcal{C}\boxtimes\mathcal{C}_1}(\chi_{(b,b_1)})(\psi)\\
&\quad= \bigwedge_{(o,o_1)\in B\times B_1}\big(\chi_{(b,b_1)}(o,o_1) \Rightarrow \downarrow_1(\psi_L(o))(o_1)\big)\\
&\quad= \downarrow_1(\psi_L(b))(b_1)
\end{aligned}
$$

As a result, we obtain $\uparrow_{\mathcal{C}\boxtimes\mathcal{C}_1}((\mathcal{C}\boxtimes\iota_{\mathcal{C}_1})(b,b_1))(\psi) = \uparrow_{\mathcal{C}\boxtimes\mathcal{C}_1}(\iota_{\mathcal{C}\boxtimes\mathcal{C}_1}(b,b_1))(\psi)$ for all

$(b, b_1) \in B \times B_1$ and and $\psi \in \mathrm{Chu}(\mathcal{C}, \mathcal{C}_1)$; hence, $\iota_{\mathcal{C}\boxtimes\mathcal{C}_1} = (\mathcal{C} \boxtimes \iota_{\mathcal{C}_1})$.

We will show now that $\mathcal{C} \boxtimes -$ preserves the composition of arrows. Specifically, this means that for any two arrows $\varphi_i \in \mathrm{Chu}(\mathcal{C}_i, \mathcal{C}_{i+1})$ for $i \in \{1, 2\}$ it holds that $\mathcal{C} \boxtimes (\varphi_1 \circ \varphi_2) = (\mathcal{C} \boxtimes \varphi_1) \circ (\mathcal{C} \boxtimes \varphi_2)$.

$$
\begin{aligned}
&\uparrow_{\mathcal{C}\boxtimes\mathcal{C}_3}\big(\big(\mathcal{C} \boxtimes (\varphi_1 \circ \varphi_2)\big)_L(b, b_1)\big)(\psi_3) \\
&\quad = \bigwedge_{(o,b_3)\in B\times B_3} \Big(\big((o = b) \wedge (\varphi_1 \circ \varphi_2)_L(b_1)(b_3)\big) \Rightarrow \downarrow(\psi_{3R}(b_3))(o)\Big) \\
&\quad = \bigwedge_{b_3\in B_3} \big((\varphi_1 \circ \varphi_2)_L(b_1)(b_3) \Rightarrow \downarrow(\psi_{3R}(b_3))(b)\big) \\
&\qquad \text{(by similar operations to those in the first part of the proof)} \\
&\quad = \downarrow\big(\big((\varphi_1 \circ \varphi_2) \circ \psi_3\big)_L(b_1)\big)(b)
\end{aligned}
$$

On the other hand, and writing F for $\mathcal{C} \boxtimes -$ in order to simplify the resulting expressions, we have

$$
\begin{aligned}
&\uparrow_{F\mathcal{C}_3}((F\varphi_1 \circ F\varphi_2)_L(b, b_1))(\psi_3) \\
&= \uparrow_{F\mathcal{C}_3}\downarrow_{F\mathcal{C}_3}\uparrow_{F\mathcal{C}_3}\big((F\varphi_2)_{L+}\big((F\varphi_1)_L(b, b_1)\big)\big)(\psi_3) \\
&= \bigwedge_{(o,b_3)\in B\times B_3} \Bigg(\\
&\qquad \bigvee_{(j,b_2)\in B\times B_2} \Big((F\varphi_1)_L(b, b_1)(j, b_2) \wedge (F\varphi_2)_L(j, b_2)(o, b_3)\Big) \Rightarrow \downarrow(\psi_{3R}(b_3))(o)\Bigg) \\
&= \bigwedge_{b_3\in B_3} \bigwedge_{b_2\in B_2} \Big(\big(\varphi_{1L}(b_1)(b_2) \wedge \varphi_{2L}(b_2)(b_3)\big) \Rightarrow \downarrow(\psi_{3R}(b_3))(b)\Big) \\
&= \bigwedge_{b_3\in B_3} \Big(\bigvee_{b_2\in B_2} \big(\varphi_{1L}(b_1)(b_2) \wedge \varphi_{2L}(b_2)(b_3)\big) \Rightarrow \downarrow(\psi_{3R}(b_3))(b)\Big) \\
&= \bigwedge_{b_3\in B_3} \Big(\varphi_{2L+}(\varphi_{1L}(b_1))(b_3) \Rightarrow \downarrow(\psi_{3R}(b_3))(b)\Big) \\
&= \bigwedge_{b_3\in B_3} \Big((\varphi_1 \circ \varphi_2)_L(b_1)(b_3) \Rightarrow \downarrow(\psi_{3R}(b_3))(b)\Big)
\end{aligned}
$$

From the previous equalities we see that $\mathcal{C} \boxtimes (\varphi_1 \circ \varphi_2) = (\mathcal{C} \boxtimes \varphi_1) \circ (\mathcal{C} \boxtimes \varphi_2)$. Hence, composition is preserved.

As a result, the mapping $\mathcal{C} \boxtimes -$ forms a functor from ChuCors into itself. □

All the previous computations can be applied to the first argument without any problems, hence we can directly state the following proposition.

Proposition 1. *The tensor product forms a bifunctor* $- \boxtimes -$ *from* ChuCors $\times$ ChuCors *to* ChuCors.

5. The Chu construction on ChuCors and second-order FCA

The Chu construction (Chu 1979) is a theoretical process that, starting from a symmetric monoidal closed (autonomous) category and a dualizing object, generates a *-autonomous category. The basic theory of *-autonomous categories and their properties are given in (Barr 1979; Barr and Wells 1995).

In the following, the construction will be applied on ChuCors and the dualizing object $\perp = \langle\{\diamond\}, \{\diamond\}, \neq\rangle$ as inputs. In this section it is shown how second order FCA (Krídlo, Krajči, and Antoni 2016) is connected to the output of such a construction.

The category generated by the Chu construction and ChuCors and $\perp$ will be denoted by $\mathrm{CHU}(\mathrm{ChuCors}, \perp)$:

- Its objects are triplets of the form $\langle \mathcal{C}, \mathcal{D}, \rho\rangle$ where
  - $\mathcal{C}$ and $\mathcal{D}$ are objects of the input category ChuCors (i.e. formal contexts)
  - ρ is an arrow in $\mathrm{ChuCors}(\mathcal{C} \boxtimes \mathcal{D}, \perp)$.
- Its morphisms are pairs of the form $\langle\varphi, \psi\rangle \colon \langle \mathcal{C}_1, \mathcal{C}_2, \rho_1\rangle \to \langle \mathcal{D}_1, \mathcal{D}_2, \rho_2\rangle$ where $\mathcal{C}_i$ and $\mathcal{D}_i$ are formal contexts for $i \in \{1, 2\}$ and
  - φ and ψ are elements from $\mathrm{ChuCors}(\mathcal{C}_1, \mathcal{D}_1)$ and $\mathrm{ChuCors}(\mathcal{D}_2, \mathcal{C}_2)$, respectively, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}_1 \boxtimes \mathcal{D}_2 & \xrightarrow{\mathcal{C}_1 \boxtimes \psi} & \mathcal{C}_1 \boxtimes \mathcal{C}_2 \\ {\scriptstyle \varphi \boxtimes \mathcal{D}_2}\downarrow & & \downarrow{\scriptstyle \rho_1} \\ \mathcal{D}_1 \boxtimes \mathcal{D}_2 & \xrightarrow[\rho_2]{} & \perp \end{array}$$

or, equivalently, the following equality holds

$$(\mathcal{C}_1 \boxtimes \psi) \circ \rho_1 = (\varphi \boxtimes \mathcal{D}_2) \circ \rho_2$$

Connection to second-order FCA

A second-order formal context (Krídlo, Krajči, and Antoni 2016) focuses on external information represented by the formal contexts, and it serves a bridge between the L-fuzzy (Bělohlávek 2004) and heterogeneous (Antoni, Krajči, Krídlo, Macek, and Pisková 2014) frameworks.

The notion of second-order formal context was originally introduced in the L-fuzzy framework, but here we will work just with the crisp version introduced below.

Definition 12. *Consider two (nonempty) indexed sets* $\{\mathcal{C}_i\}_{i\in I}$ *and* $\{\mathcal{D}_j\}_{j\in J}$ *of formal contexts, so-called external formal contexts, where* $\mathcal{C}_i = \langle B_i, T_i, P_i\rangle$ *and* $\mathcal{D}_j = \langle O_j, A_j, Q_j\rangle$. *A* second-order formal context *is a tuple*

$$\Big\langle \biguplus_{i\in I} B_i,\ \{\mathcal{C}_i\}_{i\in I},\ \biguplus_{j\in J} A_j,\ \{\mathcal{D}_j\}_{j\in J},\ R\Big\rangle$$

where $R \subseteq \biguplus_{i\in I} B_i \times \biguplus_{j\in J} A_j$.

Proposition 2. *Objects of* $\mathrm{CHU}(\mathrm{ChuCors}, \perp)$ *are representable as second-order formal contexts.*

Proof. Every object in $\text{CHU}(\text{ChuCors}, \bot)$ has the form $\langle \mathcal{C}, \mathcal{D}, \rho \rangle$ where $\mathcal{C}, \mathcal{D}$ are formal contexts and $\rho \in \text{Chu}(\mathcal{C} \boxtimes \mathcal{D}, \bot)$. As ChuCors is a closed monoidal category[1] we have that for every three formal contexts $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ the following isomorphism holds $\text{Chu}(\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}) \cong \text{ChuCors}(\mathcal{C}, \mathcal{D} \multimap \mathcal{E})$, and recall that $\mathcal{D} \multimap \bot \cong \mathcal{D}^*$ because ChuCors is *-autonomous. Thus, we obtain $\text{Chu}(\mathcal{C} \boxtimes \mathcal{D}, \bot) \cong \text{Chu}(\mathcal{C}, \mathcal{D} \multimap \bot) \cong \text{Chu}(\mathcal{C}, \mathcal{D}^*)$.

Therefore, given $\rho \in \text{Chu}(\mathcal{C} \boxtimes \mathcal{D}, \bot)$, if we write $\mathcal{C} = \langle B, T, P \rangle$ and $\mathcal{D} = \langle O, A, Q \rangle$, then we have a pair of mappings

$$\rho_L \colon B \times O \to 2^{\{\diamond\}} \qquad \rho_R \colon \{\diamond\} \to \text{Int}(\mathcal{C} \boxtimes \mathcal{D}) \subseteq 2^{\text{Chu}(\mathcal{C}, \mathcal{D}^*)}$$

satisfying that $\uparrow_\bot (\rho_L(b, o))(\diamond) = \neg \rho_L(b, o)(\diamond) = \downarrow_{\mathcal{C} \boxtimes \mathcal{D}} (\rho_R(\diamond))(b, o)$. From Lemma 4, it follows that $\downarrow_{\mathcal{C} \boxtimes \mathcal{D}} (\rho_R(\diamond))$ is equal to some bond $\beta \in \text{Bonds}(\mathcal{C}, \mathcal{D}^*)$ and then ρ_L is its negation defined as $\rho_L(b, o)(\diamond) = \neg \beta(b)(o)$

As a result, any object $\langle \mathcal{C}, \mathcal{D}, \rho \rangle$ of $\text{CHU}(\text{ChuCors}, \bot)$ can be represented as a second-order formal context $\langle B, \{\mathcal{C}\}, O, \{\mathcal{D}^*\}, \{\neg \rho_L\} \rangle$. □

Proposition 3. *Any second-order formal context is representable as an object of* $\text{CHU}(\text{ChuCors}, \bot)$.

Proof. Let $\left\langle \biguplus_{i \in I} B_i,\ \{\mathcal{C}_i\}_{i \in I},\ \biguplus_{j \in J} A_j,\ \{\mathcal{D}_j\}_{j \in J},\ R \right\rangle$ be an arbitrary second-order formal context. For all $(i, j) \in I \times J$, consider its closest covering bond given by

$$\rho_{ij} = \bigwedge \{\beta \in \text{Bonds}(\mathcal{C}_i, \mathcal{D}_j) \mid \beta(o_i)(a_j) \geq R(o_i, a_j) \text{ for all } (o_i, a_j) \in B_i \times A_j)\} .$$

Corollary 1 states a property of categorical product on ChuCors which allows us to use just one external formal context $\prod_{i \in I} \mathcal{C}_i$ instead of $\{\mathcal{C}_i\}_{i \in I}$ and similarly for $\{\mathcal{D}_j\}_{j \in J}$. In this case, we need to consider just one covering bond $\rho = \prod_{(i,j) \in I \times J} \rho_{ij}$ as guaranteed by Lemma 3. Hence, the original second-order formal context can be expressed in a rather simplified form.

Now we can define $\varphi_\rho \in \text{Chu}\Big(\prod\limits_{i \in I} \mathcal{C}_i \boxtimes \prod\limits_{j \in J} \mathcal{D}_j^*, \bot\Big)$ as the following pair of mappings

- $\varphi_{\rho L} : \biguplus\limits_{i \in I} B_i \times \biguplus\limits_{j \in J} A_j \to 2^{\{\diamond\}}$ defined as $\varphi_{\rho L}((i, b_i), (j, a_j))(\diamond) = \neg \rho_{ij}(b_i)(a_j)$.
- $\varphi_{\rho R} : \{\diamond\} \to 2^{\text{Chu}(\prod_{i \in I} \mathcal{C}_i, \prod_{j \in J} \mathcal{D}_j)}$ defined as $\varphi_{\rho R}(\diamond) = \{\varphi \in \text{Chu}(\prod_{i \in I} \mathcal{C}_i, \prod_{j \in J} \mathcal{D}_j) \mid \beta_\varphi \leq \rho\}$

Hence any second-order formal context of $\langle \biguplus_{i \in I} B_i,\ \{\mathcal{C}_i\}_{i \in I},\ \biguplus_{j \in J} A_j,\ \{\mathcal{D}_j\}_{j \in J},\ R \rangle$ is representable in $\text{CHU}(\text{ChuCors}, \bot)$ as its object $\langle \prod_{i \in I} \mathcal{C}_i, \prod_{j \in J} \mathcal{D}_j, \varphi_\rho \rangle$. □

The propositions above show a one-to-one connection between objects of $\text{CHU}(\text{ChuCors}, \bot)$ and second-order formal contexts. Hence it seems that the Chu construction applied on ChuCors is a good way to categorically describe a highly abstract extension of FCA like the second-order extension.

The following lemma confirms the correctness of the categorical description proposal. Its meaning is based on the well known fact about *-autonomous categories which states that for any object X of such a category it holds that $X \cong \top \multimap X$. In the case of ChuCors and the Chu-based extension, the isomorphism $\cong$ relies on the isomorphism between their concept lattices.

[1]See (Barr and Wells 1995) and more concretely (Mori 2008; Krídlo, Krajči, and Ojeda-Aciego 2012).

Proposition 4. *Given formal contexts $\mathcal{C}$ and $\mathcal{D}$, a bond $\rho \in \mathrm{Bonds}(\mathcal{C},\mathcal{D})$ and $\varphi_\rho \in \mathrm{Chu}(\mathcal{C} \boxtimes \mathcal{D}^*, \bot)$, the following isomorphism holds*

$$\langle \bot, \bot, \varphi_\neg \rangle \multimap \langle \mathcal{C}, \mathcal{D}^*, \varphi_\rho \rangle \cong \mathrm{CL}(\langle \mathcal{C}, \mathcal{D}, \rho \rangle)$$

i.e. the arrows in $\mathrm{CHU}(\mathrm{ChuCors}, \bot)$ *have a lattice structure which is isomorphic to the concept lattice of a second-order formal context $\langle \mathcal{C}, \mathcal{D}, \rho \rangle$.*

Proof. Let $\langle \Phi, \Psi \rangle$ be an arrow between $\langle \bot, \bot, \neq \rangle$ and $\langle \mathcal{C}, \mathcal{D}, \varphi_\rho \rangle$, such that the following diagram commutes

$$\begin{array}{ccc} \bot \boxtimes \mathcal{D} & \xrightarrow{\ \bot \boxtimes \psi\ } & \bot \boxtimes \bot \\ {\scriptstyle \Phi \boxtimes \mathcal{D}} \downarrow & & \downarrow {\scriptstyle \varphi_\neg} \\ \mathcal{C} \boxtimes \mathcal{D} & \xrightarrow[\ \varphi_\rho\ ]{} & \bot \end{array}$$

or, equivalently, the following holds

$$\downarrow_{\bot \boxtimes \mathcal{D}^*}(((\Phi \boxtimes \mathcal{D}^*) \circ \varphi_\rho)_R(\diamond))(\diamond, a) = \downarrow_{\bot \boxtimes \mathcal{D}^*}(((\bot \boxtimes \Psi) \circ \varphi_\neg)_R(\diamond))(\diamond, a)$$

We will now show that the previous characterization of any arrow is equal to some intent of a second-order formal context $\langle \mathcal{C}, \mathcal{D}, \rho \rangle$, and for that, we will introduce some useful notations:

- Let us write $\mathcal{C} = \langle B, T, P \rangle$ and $\mathcal{D} = \langle O, A, Q \rangle$
- $\xi_{\rho L} : B \to \mathrm{Ext}(\mathcal{D})$, defined by $\xi_{\rho L}(b)(o) = \downarrow_{\mathcal{D}} (\rho(b))(o)$
- $\xi_{\rho R} : A \to \mathrm{Int}(\mathcal{C})$, defined by $\xi_{\rho R}(a)(t) = \uparrow_{\mathcal{C}} (\rho^{\mathrm{t}}(a))(t)$
- $\varphi_\rho \in \mathrm{Chu}(\mathcal{C} \boxtimes \mathcal{D}^*, \bot)$
- $\varphi_{\rho L} : B \times A \to 2^{\{\diamond\}}$, defined by $\varphi_{\rho L}(b, a)(\diamond) = \neg\rho(b)(a)$
- $\varphi_{\rho R} : \{\diamond\} \to 2^{\mathrm{Chu}(\mathcal{C},\mathcal{D})}$, defined by $\varphi_{\rho R}(\diamond) = \{\varphi \in \mathrm{Chu}(\mathcal{C},\mathcal{D}) \mid \varphi \geq \xi_\rho\}$

$$\begin{aligned}
&\downarrow_{\bot \boxtimes \mathcal{D}^*} (((\Phi \boxtimes \mathcal{D}^*) \circ \varphi_\rho)_R(\diamond))(\diamond, a) = \\
&= \bigwedge_{\delta \in \mathrm{Chu}(\bot,\mathcal{D})} (((\Phi \boxtimes \mathcal{D}^*) \circ \varphi_\rho)_R(\diamond)(\delta) \Rightarrow \beta_\delta(\diamond)(a)) \\
&= \bigwedge_{\delta \in \mathrm{Chu}(\bot,\mathcal{D})} (\bigvee_{\xi \in \mathrm{Chu}(\mathcal{C},\mathcal{D})} ((\Phi \boxtimes \mathcal{D}^*)_R(\xi)(\delta) \wedge \varphi_{\rho R}(\diamond)(\xi) \Rightarrow \beta_\delta(\diamond)(a)) \\
&= \bigwedge_{\delta \in \mathrm{Chu}(\bot,\mathcal{D})} \bigwedge_{\xi \in \mathrm{Chu}(\mathcal{C},\mathcal{D})} ((\Phi \boxtimes \mathcal{D}^*)_R(\xi)(\delta) \wedge \varphi_{\rho R}(\diamond)(\xi) \Rightarrow \beta_\delta(\diamond)(a)) \\
&= \bigwedge_{\delta \in \mathrm{Chu}(\bot,\mathcal{D})} \bigwedge_{\xi \in \mathrm{Chu}(\mathcal{C},\mathcal{D})} (\varphi_{\rho R}(\diamond)(\xi) \Rightarrow ((\Phi \boxtimes \mathcal{D}^*)_R(\xi)(\delta) \Rightarrow \beta_\delta(\diamond)(a))) \\
&= \bigwedge_{\xi \in \mathrm{Chu}(\mathcal{C},\mathcal{D})} (\varphi_{\rho R}(\diamond)(\xi) \Rightarrow \bigwedge_{\delta \in \mathrm{Chu}(\bot,\mathcal{D})} ((\Phi \boxtimes \mathcal{D}^*)_R(\xi)(\delta) \Rightarrow \beta_\delta(\diamond)(a))) \\
&= \bigwedge_{\xi \in \mathrm{Chu}(\mathcal{C},\mathcal{D})} (\varphi_{\rho R}(\diamond)(\xi) \Rightarrow \bigwedge_{\delta \in \mathrm{Chu}(\bot,\mathcal{D})} ((\delta \leq \xi \circ \Phi^*) \Rightarrow \beta_\delta(\diamond)(a)))
\end{aligned}$$

$$= \bigwedge_{\xi \in \mathrm{Chu}(\mathcal{C},\mathcal{D})} (\varphi_{\rho R}(\diamond)(\xi) \Rightarrow \bigwedge_{\delta \in \mathrm{Chu}(\perp,\mathcal{D});\delta \leq \xi \circ \Phi^*} \beta_\delta(\diamond)(a))$$

$$= \bigwedge_{\xi \in \mathrm{Chu}(\mathcal{C},\mathcal{D})} (\varphi_{\rho R}(\diamond)(\xi) \Rightarrow \beta_{\xi \circ \Phi^*}(\diamond)(a))$$

$$= \bigwedge_{\xi \in \mathrm{Chu}(\mathcal{C},\mathcal{D})} (\varphi_{\rho R}(\diamond)(\xi) \Rightarrow \uparrow_{\mathcal{D}} ((\xi \circ \Phi^*)_L(\diamond))(a))$$

$$= \bigwedge_{\xi \in \mathrm{Chu}(\mathcal{C},\mathcal{D})} (\varphi_{\rho R}(\diamond)(\xi) \Rightarrow \bigwedge_{o \in O} ((\xi \circ \Phi^*)_L(\diamond)(o) \Rightarrow Q(o,a))$$

$$= \bigwedge_{\xi \in \mathrm{Chu}(\mathcal{C},\mathcal{D})} ((\xi \geq \xi_\rho) \Rightarrow \bigwedge_{o \in O} ((\xi \circ \Phi^*)_L(\diamond)(o) \Rightarrow Q(o,a))$$

$$= \bigwedge_{o \in O} ((\xi_\rho \circ \Phi^*)_L(\diamond)(a) \Rightarrow Q(o,a))$$

$$= \bigwedge_{o \in O} (\bigvee_{b \in B} (\xi_{\rho L}(b)(o) \wedge \Phi_L(\diamond)(b)) \Rightarrow Q(o,a))$$

$$= \bigwedge_{b \in B} (\Phi_L(\diamond)(b) \Rightarrow \bigwedge_{o \in O} (\xi_{\rho L}(b)(o) \Rightarrow Q(o,a)))$$

$$= \bigwedge_{b \in B} (\Phi_L(\diamond)(b) \Rightarrow \rho(b)(a))$$

$$= \uparrow_\rho(\Phi_L(\diamond))(a)$$

Recalling that Φ_L is defined as a mapping $\{\diamond\} \to \mathrm{Ext}(\mathcal{C})$, we obtain that $\Phi_L(\diamond)$ is always an extent of $\mathcal{C}$, and the previously analyzed arrow is in fact an intent of a second-order formal context.

In the previous computation, we have used a simplified version of the second-order formal context. In the general case $\Phi_L \colon \{\diamond\} \to \mathrm{Ext}(\prod_{i \in I} \mathcal{C}_i)$ and, hence, $\rho \in \mathrm{Bonds}(\prod_{i \in I} \mathcal{C}_i, \prod_{j \in J} \mathcal{D}_j)$, the proof follows the same structure. □

6. Conclusion

Two different product constructions, namely the categorical product and the tensor product, have been studied in the category ChuCors of formal contexts and Chu correspondences.

On the one hand, the existence of products enables us to represent tables and, hence, binary relations as those used in formal concept analysis; on the other hand, the tensor product is proved to fulfill the required properties of a bifunctor, which enables us to consider the Chu construction on the category ChuCors. These two results pave the way to conjecturing that different existing generalizations of FCA might be captured as suitable instantiations of the Chu construction, for instance using subcategories of $\mathrm{CHU}(\mathrm{ChuCors}, \perp)$.

This conjecture has been substantiated in Section 5, where it has been shown that the second-order formal concept analysis, introduced by Krídlo, Krajči, and Antoni (2016), can be represented in terms of the arrows of $\mathrm{CHU}(\mathrm{ChuCors}, \perp)$. The application given above seems to be just a first-step in the search for representation results of different generalizations of FCA in terms of the Chu construction applied on different input categories. So far, we already have sketched versions of the representation results for the L-fuzzy extension of FCA. For future work, we are planning to provide representations based on the Chu construction for one-sided FCA, heterogeneous FCA, multi-adjoint FCA, et cetera.

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