A multimodal logic approach to order of magnitude qualitative reasoning

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Abstract. In this work we develop a logic for formalizing qualitative reasoning. This type of reasoning is generally used, for instance, when one has a lot of data from a real world example but the complexity of the numerical model suggests a qualitative (instead of quantitative) approach.

1 Introduction

When working with a real world problem one often encounters a lack of quantitative (numerical) information among the observed facts. A possible solution to this absence of information is simply to develop methods for reasoning under an incompletely specified environment, and logic methods have been applied to give rise to reasoning schemes for fuzzy, imprecise and missing information [**?**].

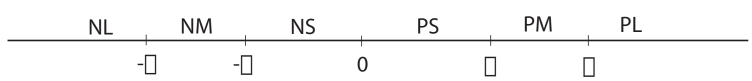
A different approach is to apply ideas from qualitative reasoning and, specifically, order of magnitude reasoning (OMR) introduced in [7] and later extended in [2–4, 9, 11]. The underlying idea is that by reasoning in terms of qualitative ranges of variables, as opposed to precise numerical values, it is possible to compute information about the behavior of a system with very little information about the system and without doing expensive numerical simulation.

Qualitative reasoning works with continuous magnitudes by means of a discretization so that it is possible to distinguish all the relevant aspects required by the context/specification (and only these aspects). On the other hand, as formalized in [7], OMR systems perform inferences based on a calculus of coarse values. These values are abstract representations of precise values taken from a totally ordered set, usually the set of real numbers. A typical OMR calculus is then designed in such a way that it generalises computations over precise values to computations over coarse values. This is of course the same approach taken by any qualitative reasoning system. The distinctive feature of OMR is that the coarse values are generally of different order of magnitude.

Depending on the way the coarse values are defined, different OMR calculi can be generated: It is usual to distinguish between Absolute Order of Magnitude (AOM) and Relative Order of Magnitude (ROM) models. The former is represented by a partition of the real line, in which each element of $\mathbb{R}$ belongs to a qualitative class. The latter type introduces a family of binary order of magnitude relations which establish different comparison relations between numbers. This can be illustrated by means of several important examples.

In [7] and extensions such as [2–4], coarse values are defined by means of ordering relations that express the distance between coarse values on a totally ordered domain in relation to the range they cover on that domain. Specifically, the seminal paper [7], distinguishes three types of qualitative relations, such as x *is close to* y, or x *is negligible w.r.t.* y or x *is comparable to* y; later on, some extensions were proposed in order to improve the original one with the inclusion of quantitative information, and allow for the control of the inference process [2–4].

There exist attempts to integrate both approaches as well, so that an absolute partition is combined with a set of comparison relations between real numbers [9, 11]. For instance, it is customary to divide the real line in seven equivalence classes and use the following labels to denote these equivalence classes of $\mathbb{R}$:



The labels correspond to "negative large", "negative medium", "negative small", "zero", "positive small", "positive medium" and "positive large", respectively. The real numbers α and β are the landmarks used to delimit the equivalence classes (the particular criteria to choose these numbers would depend on the application in mind). In [9] three binary relations (*close to, comparable, negligible*) were defined in the spirit of [7], but using the labels corresponding to quantitative values, and preserving coherence between the relative model they define and the absolute model in which they are defined.

Our aim in this paper is to develop a non-classical logic for handling qualitative reasoning with orders of magnitude. To the best of our knowledge, no formal logic has been developed to deal with order-of-magnitude reasoning. However, non-classical logics have been used as a support of qualitative reasoning in several ways: For instance, in [12, 10] is remarkable the role of multimodal logics to deal with qualitative spatio-temporal representations, and in [8] branching temporal logics have been used to describe the possible solutions of ordinary differential equations when we have limited information about a system.

In this paper, as a starting point of our proposal, we will use an arbitrary set of real numbers, not necessarily all the real line, partitioned in equivalence classes: three classes formed by so-called *observable numbers* (positive or negative) and *non-observable numbers* or *infinitesimals* (including 0). In the class of infinitesimals we will not distinguish between positive or negative. The landmarks are defined by a pair of numbers α^+ and α^-, and the equivalence classes are denoted as follows:

- OBS^+ (positive observable include α^+)
- OBS^- (negative observable include α^-)
- INF (infinitesimals)

Once we have the equivalence classes in the real line, we can make comparison between numbers by using binary relations such as

- *x is less than y*, in symbols $x < y$
- *x is comparable to y*, in symbols $x \sqsubset y$.

where $\sqsubset$ is a restriction of the usual order of the real numbers ($<$) to numbers belonging to the same equivalence class.

We will introduce a minimal system to handle orders of magnitude based on the proposed approach, whose linear ordering will then be extended to $\mathbb{Q}$ and, finally, to $\mathbb{R}$.

In our syntax we will consider the operators $\overrightarrow{\Box}$ and $\overleftarrow{\Box}$ to deal with the usual ordering $<$, and the operators $\overrightarrow{\blacksquare}$ and $\overleftarrow{\blacksquare}$ to deal with $\sqsubset$. The intuitive meanings of each modal operator is as follows:

$\overrightarrow{\Box} A$ means A *is true for all number greater than the current one*

$\overrightarrow{\blacksquare} A$ means A *is true for all number greater than the current one and in the same equivalence class.*

$\overleftarrow{\Box} A$ means A *is true for all number less than the current one*

$\overleftarrow{\blacksquare} A$ means A *is true for all number less than the current one and in the same equivalence class*

Although the treatment presented in this work is considerably simpler than those stated at the beginning of this section, still it is useful as a stepping stone for considering more complex systems, for which the logic has to be enriched by adding new modal operators capable to treat a bigger number of milestones, equivalence classes and/or qualitative relations.

This paper is organized as follows: In Section 2 the syntax and the semantics of the proposed logic is introduced; in Section 3 then a minimal axiom system is presented, whic axiomatizes validity in frames with an arbitrary set of real numbers is defined, then some extensions dealing with $\mathbb{Q}$ or $\mathbb{R}$ are given. In Section 4 the completeness proof is given, following a Henkin-style. In Section 5 some examples are given, which show that, although limited in vocabulary, the presented language can formalize some interesting situations. Finally, in Section 6 some conclusions are drawn and prospects for future work are presented.

2 Syntax and Semantics of the Language $\mathcal{L}(MQ)$

The syntax of our initial language for qualitative reasoning is introduced below:

The alphabet of the language $\mathcal{L}(MQ)$ is defined by using:

- A stock of atoms or propositional variables, $\mathcal{V}$.
- The classical connectives $\neg, \wedge, \vee$ and $\rightarrow$ and the constants $\top$ and $\bot$.
- The unary modal connectives and $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\overrightarrow{\blacksquare}$ and $\overleftarrow{\blacksquare}$.
- The constants α^+ and α^-
- The auxiliary symbols: (,).

Formulas are generated from $\mathcal{V} \cup \{\alpha^+, \alpha^-, \top, \bot\}$ by the construction rules of classical propositional logic adding the following rule: If A is a formula, then so are $\overrightarrow{\Box}A$, $\overleftarrow{\Box}A$, $\overrightarrow{\blacksquare}A$ and $\overleftarrow{\blacksquare}A$. The *mirror image* of A is the result of replacing in A each occurrence of $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\overrightarrow{\blacksquare}$, $\overleftarrow{\blacksquare}$, α^+, α^- by $\overleftarrow{\Box}$, $\overrightarrow{\Box}$, $\overleftarrow{\blacksquare}$, $\overrightarrow{\blacksquare}$, α^-, α^+, respectively. We shall use the symbols $\overrightarrow{\Diamond}$, $\overleftarrow{\Diamond}$, $\overrightarrow{\blacklozenge}$ and $\overleftarrow{\blacklozenge}$ as abbreviations respectively of $\neg\overrightarrow{\Box}\neg$, $\neg\overleftarrow{\Box}\neg$, $\neg\overrightarrow{\blacksquare}\neg$ and $\neg\overleftarrow{\blacksquare}\neg$.

The intended meaning of our language is based on a multi-modal approach, therefore the semantics is given by using the concept of frame.

Definition 1. *A* multimodal qualitative frame *for* $\mathcal{L}(MQ)$ *(or, simply, a* frame*) is a tuple* $\Sigma = (\mathbb{S}, \alpha^+, \alpha^-, <)$*, where*

1. $\mathbb{S}$ *is a nonempty set of real numbers.*
2. $<$ *is a strict linear order on* $\mathbb{S}$.
3. α^+ *and* α^- *are designated points in* $\mathbb{S}$ *(called* frame constants*), and induce a partition in* $\mathbb{S}$*. The equivalence classes* OBS^+*,* INF*, and* OBS^- *are defined as follows:*

$$OBS^- = \{x \in \mathbb{S} \mid x \leq \alpha^-\} \qquad INF = \{x \in \mathbb{S} \mid \alpha^- < x < \alpha^+\}$$
$$OBS^+ = \{x \in \mathbb{S} \mid \alpha^+ \leq x\}$$

Definition 2. *Let Σ be a multimodal qualitative frame, a* multimodal qualitative model on Σ *(or Σ-*model, *for short) is an ordered pair $\mathcal{M} = (\Sigma, h)$, where h is a* meaning function *(or,* interpretation*) $h\colon \mathcal{V} \longrightarrow 2^{\mathbb{S}}$. Any interpretation can be uniquely extended to the set of all formulas in $\mathcal{L}(MQ)$ (also denoted by h) by using the usual conditions for the classical boolean connectives and the constants $\top$ and $\bot$, and the following conditions for the modal operators and frame constants:*[1]

$$
\begin{aligned}
h(\overrightarrow{\Box} A) &= \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x < y\}\\
h(\overrightarrow{\blacksquare} A) &= \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x \sqsubset y\}\\
h(\overleftarrow{\Box} A) &= \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } y < x\}\\
h(\overleftarrow{\blacksquare} A) &= \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } y \sqsubset x\}\\
h(\alpha^+) &= \alpha^+ \qquad h(\alpha^-) = \alpha^-
\end{aligned}
$$

The concepts of truth and validity are defined in a straightforward manner.

3 Axiomatic systems for $\mathcal{L}(MQ)$

In this section we define several axiomatic systems for multimodal qualitative logic. A list of axiom schemes and inference rules are presented in order to build the different systems. We also consider all the tautologies of classical propositional logic.

Schemes of axioms for $\overrightarrow{\Box}, \overrightarrow{\Diamond}$:

K1 $\overrightarrow{\Box}(A \to B) \to (\overrightarrow{\Box} A \to \overrightarrow{\Box} B)$
K2 $A \to \overrightarrow{\Box}\overleftarrow{\Diamond} A$
K3 $\overrightarrow{\Box} A \to \overrightarrow{\Box}\overrightarrow{\Box} A$
K4 $\big(\overrightarrow{\Diamond} A \wedge \overrightarrow{\Diamond} B\big) \to \big(\overrightarrow{\Diamond}(A \wedge B) \vee \overrightarrow{\Diamond}(\overrightarrow{\Diamond} A \wedge B) \vee \overrightarrow{\Diamond}(A \wedge \overrightarrow{\Diamond} B)\big)$
K5 $\overrightarrow{\Box}\overrightarrow{\Box} A \to \overrightarrow{\Box} A$
K6 $\overrightarrow{\Box} A \to \overrightarrow{\Diamond} A$
K7 $(\overrightarrow{\Diamond} A \wedge \overrightarrow{\Diamond}\overrightarrow{\Box}\neg A\big) \to \overrightarrow{\Diamond}(\overleftarrow{\Box}\overrightarrow{\Diamond} A \wedge \overrightarrow{\Box}\neg A))$

Schemes of axioms for $\overrightarrow{\blacksquare}, \overrightarrow{\blacklozenge}$:

C1 $\overrightarrow{\blacksquare}(A \to B) \to (\overrightarrow{\blacksquare} A \to \overrightarrow{\blacksquare} B)$
C2 $A \to \overrightarrow{\blacksquare}\overleftarrow{\blacklozenge} A$
C3 $(\overrightarrow{\blacksquare} A \wedge \neg\alpha^-) \to \overrightarrow{\blacklozenge} A$

Mixed axiom:

[1] In the following we will use $x \sqsubset y$ as an abbreviation of "$x < y$ and $x, y \in EQ$, where $EQ \in \{INF, OBS^+, OBS^-\}$".

M1 $\overrightarrow{\Box} A \to \overrightarrow{\blacksquare} A$

Schemes of axioms for constants:

c1 $\overleftarrow{\Diamond}\xi \vee \xi \vee \overrightarrow{\Diamond}\xi$
c2 $\xi \to (\overleftarrow{\Box}\neg\xi \wedge \overrightarrow{\Box}\neg\xi)$
c3 $\alpha^- \to \overrightarrow{\Diamond}\alpha^+$
c4 $\alpha^- \to \overrightarrow{\blacksquare} A$
c5 $(\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+) \to \overrightarrow{\blacksquare}(\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+)$
c6 $\overrightarrow{\Diamond}\alpha^- \to \overrightarrow{\blacklozenge}\alpha^-$
c7 $(\alpha^+ \wedge \overrightarrow{\blacksquare} A) \to \overrightarrow{\Box} A$
c8 $\overrightarrow{\blacksquare} A \to \overrightarrow{\Box}((\alpha^- \vee \overrightarrow{\Diamond}\alpha^-) \to A)$
c9 $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\blacksquare} A) \to \overrightarrow{\Box} A$
c10 $(\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\blacksquare} A) \to \overrightarrow{\Box}((\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+) \to A)$

We also consider as schemes of axioms the mirror images corresponding to K1, K2, K4, K6, K7, C1–C3, M1 and c4–10.

Rules of inference:

(MP) Modus Ponens for $\to$
(N$\overrightarrow{\Box}$) If $\vdash A$ then $\vdash \overrightarrow{\Box} A$ (N$\overrightarrow{\blacksquare}$) If $\vdash A$ then $\vdash \overrightarrow{\blacksquare} A$
(N$\overleftarrow{\Box}$) If $\vdash A$ then $\vdash \overleftarrow{\Box} A$ (N$\overleftarrow{\blacksquare}$) If $\vdash A$ then $\vdash \overleftarrow{\blacksquare} A$

Definition 3. *The minimal system for $\mathcal{L}(MQ)$ is denoted MQ. It consists of the axiom schemes K1–K4 plus M1, C1, C2, c1–c10 and the corresponding mirror images. $MQ_{\mathbb{Q}}$ is the extension of MQ by adding K5, K6 and C3. Finally, $MQ_{\mathbb{R}}$ is the extension of $MQ_{\mathbb{Q}}$ by adding K7.*

The concepts of *proof* and *theorem* are defined in a standard way.

4 Soundness and completeness

The proof of soundness is straightforward, since validity of the axioms and preservation of validity by inference rules is simply a standard calculation. Thus, we need only to focus on completeness, for which a Henkin-style proof can be constructed.

The proof of completeness follows the step-by-step method as in [1]; specifically, some results about *consistent* (*maximal consistent*) sets of formulas are needed. Some familiarity with the basic properties of maximal consistent sets is assumed, we shall use $\mathcal{MC}$ to denote the set of all maximal consistent sets of formulas (*mc-sets*) of any of the systems introduced in the previous section. We denote by $\mathcal{S}$ any such axiomatic system.

Definition 4. *Let $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$. Then:*

1. *$\Gamma_1 \vartriangleright \Gamma_2$ if and only if $\{A \mid \overrightarrow{\square} A \in \Gamma_1\} \subseteq \Gamma_2$*
2. *$\Gamma_1 \blacktriangleright \Gamma_2$ if and only if $\{A \mid \overrightarrow{\blacksquare} A \in \Gamma_1\} \subseteq \Gamma_2$*

The three lemmas below state some modal properties of the operators $\vartriangleright$ and $\blacktriangleright$: the behaviour with respect to the relations just introduced, the transitivity and linearity of those orderings, and the existence of mc-sets with suitable properties. The statements only contain the behaviour of the specific (black) modal connectives, the usual (white) modalities have the same properties:

Lemma 1. *Let $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$, then:*

1. *$\Gamma_1 \blacktriangleright \Gamma_2$ if and only if $\{A \mid \overleftarrow{\blacksquare} A \in \Gamma_2\} \subseteq \Gamma_1$*
2. *$\Gamma_1 \blacktriangleright \Gamma_2$ if and only if $\{\overrightarrow{\blacklozenge} A \mid A \in \Gamma_2\} \subseteq \Gamma_1$*
3. *$\Gamma_1 \blacktriangleright \Gamma_2$ if and only if $\{\overleftarrow{\blacklozenge} A \mid A \in \Gamma_1\} \subseteq \Gamma_2$*
4. *(Lindenbaum's Lemma) Any consistent set of formulas in $\mathcal{S}$ can be extended to an mc-set in $\mathcal{S}$.*

Lemma 2. *Consider $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$, then*

1. *If $\Gamma_1 \blacktriangleright \Gamma_2$ and $\Gamma_2 \blacktriangleright \Gamma_3$, then $\Gamma_1 \blacktriangleright \Gamma_3$.*
2. *If $\Gamma_1 \blacktriangleright \Gamma_2$ and $\Gamma_1 \blacktriangleright \Gamma_3$, then either $\Gamma_2 \blacktriangleright \Gamma_3$, or $\Gamma_3 \blacktriangleright \Gamma_2$, or $\Gamma_2 = \Gamma_3$.*

Lemma 3. *Assume $\Gamma_1 \in \mathcal{MC}$:*

1. *If $\overrightarrow{\blacklozenge} A \in \Gamma_1$, then there exists $\Gamma_2 \in \mathcal{MC}$ such that $\Gamma_1 \blacktriangleright \Gamma_2$ and $A \in \Gamma_2$.*

The following two lemmas are specific of our logic, since the behaviour of specific and general connectives is studied.

Lemma 4. *Consider $\Gamma_1, \Gamma_2 \in \mathcal{MC}$ such that $\Gamma_1 \vartriangleright \Gamma_2$, then $\Gamma_1 \blacktriangleright \Gamma_2$ holds if and only if one of the following conditions below is fulfilled:*

1. *There is $\varphi \in \{\overleftarrow{\Diamond} \alpha^- \wedge \overrightarrow{\Diamond} \alpha^+, \overleftarrow{\Diamond} \alpha^+, \overrightarrow{\Diamond} \alpha^-\}$ satisfying $\varphi \in \Gamma_1$ and $\varphi \in \Gamma_2$*
2. *$\alpha^+ \in \Gamma_1$*
3. *$\alpha^- \in \Gamma_2$*

Lemma 5. *Given $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$ we have:*

1. If $\Gamma_1 \blacktriangleright \Gamma_2$, then $\Gamma_1 \triangleright \Gamma_2$
2. If $\Gamma_1 \blacktriangleright \Gamma_2$, $\Gamma_1 \triangleright \Gamma_3$ and it is not the case that $\Gamma_1 \blacktriangleright \Gamma_3$, then $\Gamma_2 \triangleright \Gamma_3$
3. If $\Gamma_1 \triangleright \Gamma_2 \triangleright \Gamma_3$ and $\Gamma_1 \blacktriangleright \Gamma_3$, then $\Gamma_1 \blacktriangleright \Gamma_2 \blacktriangleright \Gamma_3$

The information provided by the previous lemmas allows to give a step-by-step proof of completeness, the details of the proof are not included due to lack of space.

Theorem 1. *If A is a valid formula of MQ, then A is a theorem of MQ.*

5 Examples

In this section we introduce an example based on the physical problem of the separation of a solution of two fluids; just note that the formulas presented below do not intend to formally represent the underlying physical behaviour, which appears in an oversimplified version; instead they are introduced with the purpose of showing how the minimal language MQ can be used as a tool for the specification.

Example 1. Let us assume that we have a solution of two fluids (say, A and B). The idea is to heat the solution until the most volatile fluid has been evaporated; and, obviously, we are also interested in minimizing the cost of the heating process.

We are introducing the actions of two different agents on the process, with slightly different behaviors. The equivalence classes OBS^-, INF, and OBS^+ will be denoted, respectively, as LIQ, LIQ_GAS, and GAS, and the physical interpretation of each of these labels is *the physical state of the most volatile fluid*, which can be completely liquid, or part in liquid state and part in gas, or completely evaporated. The landmarks represent the initial and final boiling points of the solution (which depend on the concentration of the solution), which are assumed to be almost indistinguishable.

Agent 1. The behavior of this agent is specified by the following set of proper axioms:

1. $LIQ \to heat$
2. $GAS \to cool$
3. $LIQ\_GAS \to \overrightarrow{\blacksquare} heat$
4. $cool \to \neg heat$

The underlying intuition of the behavior of this agent is that the heating of the solution is being checked as the most volatile fluid is being evaporated. Somehow, we could say that this is the *parsimonious* approach.

Agent 2. This agent has its behavior specified by the following set of proper axioms:

1. $GAS \to \overleftarrow{\blacksquare} cool$
2. $(LIQ \vee LIQ\_GAS) \to heat$
3. $cool \to \neg heat$

The behavior of this agent is different from the previous one in that the first priority is to obtain the evaporation of the most volatile fluid and, later, it is cooled so that it attains the state of minimum energy while preserving the gas state.

6 Conclusions and future work

The introduction of a minimal multimodal language for the handling of qualitative reasoning has been justified. A sound and complete system for multimodal qualitative reasoning has been presented, and its completeness theorem has been sketched. Finally, a small example taken from Physics has been used to show the expressive power of the language.

Obviously, this minimal language is still very poor in order to represent more interesting problems of qualitative reasoning, however as future work it is expected to integrate further modalities expressed in terms of the Absolute and/or Relative Orders of Magnitude, such as closeness or negligibility.

Nevertheless, the importance of using the logical apparatus in the treatment of qualitative reasoning is the possibility of mechanization of its reasoning system. To this end, a first approach to the automatization of the deduction in MQ is envisaged in terms of a tableau calculus.

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