TAMING NON-COMMUTATIVITY IN THE FRAMEWORK OF MULTI-ADJOINT CONCEPT LATTICES

Jesús Medina
Dept. Matemáticas. Univ. Cádiz
jesus.medina@uca.es

Manuel Ojeda-Aciego
Dept. Matemática Aplicada. Univ. Málaga
aciego@uma.es

Abstract

Sometimes, in real applications, we have to consider the use of non-commutative operators. However, it is interesting to be able to "balance" of "soften" the non-commutative character of the involved operators.

There exist some approaches to the construction of concept lattices based on non-commutative conjunctors $L \times L \to L$, but are based on the fact that the supports of the fuzzy subsets of both objects and attributes has to coincide.

Our contribution in this work is to present sufficient conditions in order to be able to construct concepts in a generalized fuzzy context in which the domain of the underlying conjunctors can be $L_1 \times L_2$ with $L_1 \neq L_2$.

Keywords: concept lattices, multi-adjoint lattices, Galois connection, implication triples.

1 Introduction

Since its introduction by Wille in the eighties, formal concept analysis has become an important and appealing research topic both from a theoretical perspective [17,26,29] and from the applicative one. Regarding applications, we can find papers ranging from ontology merging [11, 24], to applications to the Semantic Web by using the notion of concept similarity [12], and from processing of medical records in the clinical domain [14] to the development of recommender systems [9].

Soon after the introduction of "classical" formal concept analysis, a number of different approaches for its generalization were introduced and, nowadays, there are works which extend the theory with ideas from fuzzy set theory [3, 20, 21] or fuzzy logic reasoning [2, 5, 10] or from rough set theory [19,27,30] or some integrated approaches such as fuzzy and rough [28], or rough and domain theory [18].

In this paper we concentrate on the fuzzy extensions of formal concept analysis, for which a number of different approaches have been presented. To the best of our knowledge, the first one was given in [7], although they did not advance much beyond the basic definitions, probably due to the fact that they did not use residuated implications. Later, in [3, 25] the authors independently used complete residuated lattices as structures for the truth degrees; for this approach, a representation theorem was proved directly in a fuzzy framework in [4], setting the basis of most of the subsequent direct proofs.

In recent years there has been an increased interest in studying formal concept analysis on the perspective of using non-commutative conjunctors. This is not a mere mathematical generalization, but a real need since, for instance, when one learns a conjunction from examples it is not unusual that the resulting conjunction does not satisfy commutativity.

Different authors have argued in favour of considering non-commutative conjunctors. Actually, there exist quite reasonable examples of non-commutative and even non-associative conjunctors defined on a regular partition of the unit interval, see [23]. An often used example considers the case in which we are looking for a hotel which is close to downtown, with reasonable price and being new or recently renovated, then classical fuzzy approaches would assign a user "his/her" particular interpretation of "close", "reasonable" and "new". Therefore, as in practice, we can only recognize finitely many degrees of being close, reasonable, new, for instance, for *closeness* five degrees might be convenient in order to specify: far, non-far, foot-distance, close and very-close; for *reasonable*, three values might be enough to refer to expensive, admissible and cheap, respectively; and a similar treatment could be given to *new*; as a result, the corresponding fuzzy sets have always a stepwise shape. This motivates the lattice-valued approach we will assume in this paper: it is just a matter of representation that the outcome is done by means of intervals of granulation and/or indistinguishability.

Another example of non-conmutative conjunctors is given when the user wants to give preference to some of its arguments, for instance, when considering the conjunctor $\&\colon [0,1] \times [0,1] \to [0,1]$, defined as $x \,\&\, y = x^2 y$ for each $x, y \in [0,1]$.

Non-commutative logic and similarity were used to develop new kinds of concept lattices in [13]. This approach, but in a different direction, was also extended in an asymmetric way in [16] where the so-called generalised concept lattices were introduced.

More recently, we can find even further generalisations, such as the variable threshold concept lattices [31] and multi-adjoint concept lattices [23]. Such a number of versions have been introduced that it is not surprising to discover relationships between them (see for instance [6,15,22]).

Multi-adjoint concept lattices were introduced [23] to formal concept analysis is applied. With the idea of providing a general framework in which the different approaches stated above could be conveniently accommodated, the authors worked in a general non-commutative environment; and this naturally lead to the consideration of adjoint triples, also called implication triples [1] as the main building blocks of a multi-adjoint concept lattice.

Following the general techniques of formal concept analysis and based on the initial work [13], given a non-commutative conjunctor, it is possible to provide generalizations of the mappings for the intension and the extension in two different ways, generating a pair of concept lattices. This approach was subsumed by the so-called t-concept lattice in [22], whose multi-adjoint extension is the framework we will consider in this paper.

2 Preliminary definitions

The main notion in this contribution refers to the notion of L-connection between two complete lattices. As we will see later, this condition will allow to somehow conciliate the different values generated by the consideration of a non-commutative conjunctor in the construction of a concept lattice.

Definition 1 *Given complete lattices* $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ *and* $(L, \preceq)$, *we say that* L_1 *and* L_2 *are* L-connected *if there exist increasing mappings* $i_1 \colon L_1 \to L$, $\phi_1 \colon L \to L_1$, $i_2 \colon L_2 \to L$ *and* $\phi_2 \colon L \to L_2$ *verifying that*

1. $\phi_1(i_1(x)) = x$, *and* $\phi_2(i_2(y)) = y$, *for all* $x \in L_1$, $y \in L_2$;
2. $t \preceq i_1(\phi_1(t))$, *and* $t \preceq i_2(\phi_2(t))$, *for all* $t \in L$.

Example 1 *Any pair of lattices* $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ *with top elements* $\top_1$ *and* $\top_2$ *, respectively, are* $(L_1 \times L_2, \preceq)$*-connected, with the pairwise ordering, and by considering the mappings* ϕ_1, ϕ_2 *as the projections and* i_1, i_2 *as the inclusions defined as* $i_1(x) = (x, \top_2)$, $i_2(y) = (\top_1, y)$, *for all* $x \in L_1$, $y \in L_2$.

A more complex example is presented below:

Example 2 *Assume that, in order to perform an evaluation of a product, for which we have to assign one value out of four possible ones. We ask two experts to collaborate in this task and, only when collecting the feedback from each expert, we notice that one expert has considered the ordering of values as in Figure 1, whereas the other has considered that in Figure 2. In both cases, the expert has used a suitable adjoint triple on these lattices in order to obtain the final result of the evaluation.*

In order to unify both evaluations, we want to embed the lattices above into another one, for example, we might

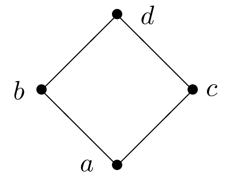

Figure 1. Lattice $(L_1, \preceq_1)$

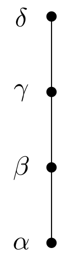

Figure 2. Lattice $(L_2, \preceq_2)$

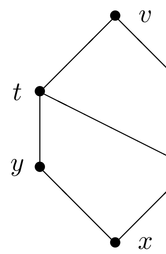

Figure 3. Lattice $(L, \preceq)$

consider the given in Figure 3. In this case we can define two mappings $i_1 \colon L_1 \to L$, $i_2 \colon L_2 \to L$ *as in Figure 4.*

Therefore, we can define the maps $\phi_1 \colon L \to L_1$, $\phi_2 \colon L \to L_2$ *in order to satisfy the properties in Definition 1. There exist several possibilities, one of them is shown below:*

	x	y	z	t	u	v
ϕ_1	a	b	c	d	c	d

	x	y	z	t	u	v
ϕ_2	α	β	γ	γ	δ	δ

Therefore, L_1 *and* L_2 *are* L*-connected.*

Example 3 *A different example arises when we consider the lattices* $([0,1]_2, \leq)$, $([0,1]_4, \leq)$, *where* $[0,1]_n$ *means the granulated on* $n+1$ *different values of* $[0,1]$, *which provide a regular partition of* $[0,1]$ *into* n *pieces, for instance* $[0,1]_2 = \{0, 0.5, 1\}$, $[0,1]_4 = \{0, 0.25, 0.5, 0.75, 1\}$.

We have that $[0,1]_2$, $[0,1]_4$ *are* $[0,1]$*-connected, under the lexicographic ordering, considering the mappings* i_1, i_2 *as the inclusions* $i_1(x) = x$, $i_2(y) = y$, *for all* $x \in L_1$, $y \in L_2$; *and* ϕ_1, ϕ_2 *defined as* $\phi_1(t) = \lceil 2 \cdot t \rceil / 2$, $\phi_2(t) = \lceil 4 \cdot t \rceil / 4$, *where* $\lceil \_ \rceil$ *is the ceiling function. For example, if* $t = 0.55$, $\phi_1(0.55) = 1$, $\phi_2(0.55) = 0.75$,

2.1 Recalling t-concepts

Assuming non-commutativity on the conjunctor, directly provides two different ways of generalising the well-known adjoint property between a t-norm and its residuated implication, depending on which argument is fixed.

	a	b	c	d
i_1	x	y	u	v

	α	β	γ	δ
i_2	x	y	t	v

Figure 4. Definition of i_1 and i_2

Definition 2 *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$ is an* adjoint triple *with respect to P_1, P_2, P_3 if:*

1. *$\&$ is order-preserving in both arguments.*
2. *$\swarrow$ and $\nwarrow$ are order-preserving in the consequent and order-reversing in the antecedent.*
3. *$x \leq_1 z \swarrow y$ iff $x \,\&\, y \leq_3 z$ iff $y \leq_2 z \nwarrow x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$.*

The multi-adjoint framework allows the existence of several adjoint triples for a given triplet of lattices.

Definition 3 *A* multi-adjoint frame *$\mathcal{L}$ is a tuple*

$$(L_1, L_2, L, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

where $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(L, \leq)$ are complete lattices and, for all $i = 1, \dots, n$, $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P.
Multi-adjoint frames are denoted $(L_1, L_2, L, \&_1, \dots, \&_n)$.

Given a frame, a *multi-adjoint context* is a tuple consisting of sets of objects and attributes and a fuzzy relation among them; in addition, the multi-adjoint approach also includes a function which assigns an adjoint triple to each object (or attribute).

Definition 4 *Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, a* context *is a tuple (A, B, R, σ) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon B \to \{1, \dots, n\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.*[1]

In order to make this contribution self-contained and since we will provide a specific construction of a Galois connection, we recall its formal definition below:

Definition 5 *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^{\downarrow}\colon P_1 \to P_2$, ${}^{\uparrow}\colon P_2 \to P_1$ mappings, the pair $({}^{\uparrow}, {}^{\downarrow})$ forms a* Galois connection *between P_1 and P_2 if and only if:*

1. *${}^{\uparrow}$ and ${}^{\downarrow}$ are order-reversing.*
2. *$x \leq_1 x^{\downarrow\uparrow}$ for all $x \in P_1$.*
3. *$y \leq_2 y^{\uparrow\downarrow}$ for all $y \in P_2$.*

[1]A similar theory could be developed by considering a mapping $\tau\colon A \to \{1, \dots, n\}$ which associates any element in A with some particular adjoint triple in the frame.

Given L_1 and L_2, L-connected, a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$, and a context (A, B, R, σ), we can define the following mappings ${}^{\uparrow_{t\sigma}}\colon L^B \to L^A$ and ${}^{\downarrow^{t\sigma}}\colon L^A \to L^B$:

$$g^{\uparrow_{t\sigma}}(a) = i_1(\inf\{R(a,b) \swarrow^{\sigma(b)} \phi_2(g(b)) \mid b \in B\}) \quad (1)$$
$$f^{\downarrow^{t\sigma}}(b) = i_2(\inf\{R(a,b) \nwarrow_{\sigma(b)} \phi_1(f(a)) \mid a \in A\}) \quad (2)$$

Note that we can define, for each adjoint triple $(\&, \swarrow, \nwarrow)$, the mappings $\swarrow^*\colon P \times L_2 \to L_1$, $\nwarrow_*\colon P \times L_1 \to L_2$ as following:

$$\begin{aligned} z \swarrow^* y &= i_1(z \swarrow \phi_2(y)) \\ z \nwarrow_* x &= i_2(z \nwarrow \phi_1(x)) \end{aligned}$$

for all $x, y \in L$ and $z \in P$, and, if i_1 and i_2 are inf-preserving, then the mappings ${}^{\uparrow_{t\sigma}}\colon L^B \to L^A$ and ${}^{\downarrow^{t\sigma}}\colon L^A \to L^B$ are equal to:

$$g^{\uparrow_{t\sigma}}(a) = \inf\{R(a,b) \swarrow^{*b} g(b) \mid b \in B\} \quad (3)$$
$$f^{\downarrow^{t\sigma}}(b) = \inf\{R(a,b) \nwarrow_{*b} f(a) \mid a \in A\} \quad (4)$$

Therefore, if $(\&^*, \swarrow^*, \nwarrow_*)$ is an adjoint triple, then the mappings above are the Galois connection associated to the frame $(L, L, P, \&_1^*, \dots, \&_n^*)$ that were introduced in [23].

These last equalities can only be written if i_1 and i_2 are inf-preserving,[2] which is more restrictive that we need, as we show in the following trivial example. Therefore we will assume Equalities (1), (2).

Example 4 *Given the lattices in Figures 5 and 6, we can define two mappings $i_1\colon L_1 \to L$ and $\phi_1\colon L \to L_1$, as $i_1(a) = e$, $i_1(b) = f$, $i_1(c) = g$, $i_1(d) = g$, and ϕ_1 is the inverse of i_1, which verifying the conditions on Definition 1 but i_1 is not inf-preserving.*

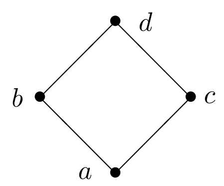

Figure 5. Lattice $(L_1, \preceq_1)$

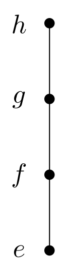

Figure 6. Lattice $(L, \preceq)$

We can also use one of the previous examples in order to obtain another case where i_1 is not inf-preserving.

For instance, if we consider the definitions of i_1 and ϕ_1 that appear in Example 2, by writing $i_1(b) = t$ instead of $i_1(b) = y$ and $\phi(t) = b$ instead of $\phi(t) = d$, then i_1

[2]We say that a map between two lattice $f\colon L_1 \to L$ is *inf-preserving* if satisfies that $f(\inf x, y) = \inf\{f(x), f(y)\}$, for all $x, y \in L_1$

and ϕ_1 verify the conditions in Definition 1 but i_1 is not inf-preserving because

$$\begin{aligned} i_1(\inf\{b,c\}) &= i_1(a) = x \\ \inf\{i_1(b), i_1(c)\} &= \inf\{t,u\} = z \end{aligned}$$

and $x \neq z$.

Expressions (1), (2) do not coincide with those given in [23], because they are not defined directly from a residuated implication but appear also the mapping i_1, i_1, ϕ_1 and ϕ_2. Hence, we need to prove that these mapping form a Galois connection.

Proposition 1 *Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, where L_1 and L_2 are L-connected, and a context (A, B, R, σ), the pair $({}^{\uparrow_{t\sigma}}, {}^{\downarrow^{t\sigma}})$ is a Galois connection between L^A and L^B.*

Proof: To improve readability, we will write $({}^{\uparrow_t}, {}^{\downarrow^t})$ instead of $({}^{\uparrow_{t\sigma}}, {}^{\downarrow^{t\sigma}})$ and $\swarrow^b$, $\nwarrow_b$ instead of $\swarrow^{\sigma(b)}$, $\nwarrow_{\sigma(b)}$.

By definition, we have to prove that:

1. ${}^{\uparrow_t}$ and ${}^{\downarrow^t}$ are order-reversing. If $f_1, f_2 \in L^A$, $f_1 \preceq_1 f_2$, then $\phi_1(f_1(a)) \preceq_1 \phi_1(f_2(a))$ for all $a \in A$, because ϕ_1 is increasing, now, as the implications are order-reversing in the second argument we obtain that:
$$R(a.b) \nwarrow \phi_1(f_2(a)) \preceq_2 R(a.b) \nwarrow \phi_1(f_1(a))$$
for all $a \in a$ and $b \in B$. Thus, as i_2 is increasing and by the infimum property we obtain that:
$$\begin{aligned} f_2^{\downarrow^t}(b) &= i_2(\inf\{R(a.b) \nwarrow \phi_1(f_2(a)) \mid a \in A\} \\ &\preceq i_2(\inf\{R(a.b) \nwarrow \phi_1(f_1(a)) \mid a \in A\} \\ &= f_1^{\downarrow^t}(b) \end{aligned}$$
for all $b \in B$. The proof for ${}^{\uparrow_t}$ is similar.

2. We need to prove that, given $g \in L^B$, then $g \preceq g^{\uparrow_t \downarrow^t}$. We begin from the definition of ${}^{\uparrow_t}$:
$$g^{\uparrow_t}(a) = i_1(\inf\{R(a,b) \swarrow^{\sigma(b)} \phi_2(g(b)) \mid b \in B\})$$
for all $a \in A$. Now, applying the mapping ϕ_1, we obtain:
$$\phi_1(g^{\uparrow_t}(a)) = \inf\{R(a,b) \swarrow^{\sigma(b)} \phi_2(g(b)) \mid b \in B\}$$
for all $a \in A$. Therefore, given $a \in A$ and $b \in B$ the next chain of inequalities holds by the adjoint property:
$$\begin{aligned} &\phi_1(g^{\uparrow_t}(a)) \preceq_1 R(a,b) \swarrow^b \phi_2(g(b)) \iff \\ &\iff \phi_1(g^{\uparrow_t}(a)) \,\&_b\, \phi_2(g(b)) \leq R(a,b) \\ &\iff \phi_2(g(b)) \preceq_2 R(a,b) \nwarrow_b \phi_1(g^{\uparrow_t}(a)) \end{aligned}$$
As these inequalities hold for all $a \in A$, by applying properties of the infimum we obtain, for all $b \in B$, that
$$\phi_2(g(b)) \quad \preceq_2 \quad \inf\{R(a,b) \nwarrow_b \phi_1(g^{\uparrow_t}(a)) \mid a \in A\}$$
Hence: $g(b) \preceq g^{\uparrow_t \downarrow^t}(b)$ for all $b \in B$, because the map i_2 is increasing.

3. $f \preceq f^{\downarrow^t \uparrow_t}$ for all $f \in L_1^A$. The proof is similar. □

Hence, we define a new Galois connection from a frame where the lattices $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are L-connected, using the mappings i_1, i_2, ϕ_1 and ϕ_2. Therefore, the concepts are different and we obtain a new concept lattice. In this framework, a *concept* is a pair $\langle g^*, f^* \rangle$ satisfying that $g^* \in L^B$, $f^* \in L^A$ and that $(g^*)^{\uparrow_t} = f^*$ and $(f^*)^{\downarrow^t} = g^*$; with $({}^{\uparrow_t}, {}^{\downarrow^t})$ being the Galois connection defined above.

Definition 6 *The* multi-adjoint abelianized concept lattice *associated to a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and a context (A, B, R, σ), where L_1 and L_2 are L-connected, is the set*
$$\mathcal{M}_L = \{\langle g^*, f^* \rangle \mid \langle g^*, f^* \rangle \text{ is a concept}\}$$
in which the ordering is defined by $\langle g_1^*, f_1^* \rangle \preceq \langle g_2^*, f_2^* \rangle$ if and only if $g_1^* \preceq_2 g_2^*$ (equivalently $f_2^* \preceq_1 f_1^*$).

As $({}^{\uparrow_t}, {}^{\downarrow^t})$ is a Galois connection, the pair $(\mathcal{M}_L, \preceq)$ is, indeed, a complete lattice [8,23].

In a similar way that we have defined the Galois connection $({}^{\uparrow_t}, {}^{\downarrow^t})$, permutating the adjoint implications we define the following operators ${}^{\Uparrow_t}\colon L^B \to L^A$ and ${}^{\Downarrow^t}\colon L^A \to L^B$, given a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$, and a context (A, B, R, σ), where L_1 and L_2 are L-connected:

$$\begin{aligned} g^{\Uparrow_t}(a) &= i_2(\inf\{R(a,b) \nwarrow_{\sigma(b)} \phi_1(g(b)) \mid b \in B\}) \\ f^{\Downarrow^t}(b) &= i_1(\inf\{R(a,b) \swarrow^{\sigma(b)} \phi_2(f(a)) \mid a \in A\}) \end{aligned}$$

It is not difficult to check that these two arrows generate a Galois connection, because of Proposition 1, since it coincides with the connection defined by Equations (1), (2), on the multi-adjoint frame $(L_2, L_1, P, \&_1^{op}, \dots, \&_n^{op})$ and context (A, B, R, σ), being $\&_i^{op}\colon L_2 \times L_1 \to P$ and $x \,\&_i^{op}\, y = y \,\&_i\, x$ for all $i \in \{1, \dots, n\}$, since the implications are permuted, if the initial adjoint triples are $(\&_i, \nwarrow_i, \swarrow^i)$ the adjoint triples considered are: $(\&_i^{op}, \swarrow^i, \nwarrow_i)$.

Now, we have two Galois connections $({}^{\uparrow}, {}^{\downarrow})$, $({}^{\uparrow_{op}}, {}^{\downarrow_{op}})$, on which two different multi-adjoint concept lattices $(\mathcal{M}_L, \preceq)$, $(\mathcal{M}_L^{op}, \preceq)$ can be defined.[3] These concept lattices can be proved to embed the concept lattices defined by Georgescu and Popescu in [13].

As both lattices are different if at least one conjunctor $\&_i$ is non-commutative, but are certainly related since both are defined on the same adjoint triples. This suggests to consider the following subsets of $\mathcal{M}_L \times \mathcal{M}_L^{op}$:

$$\begin{aligned} \mathcal{N}_1^L &= \{(\langle g, f_1 \rangle, \langle g, f_2 \rangle) \mid \langle g, f_1 \rangle \in \mathcal{M}_L, \langle g, f_2 \rangle \in \mathcal{M}_L^{op}\} \\ \mathcal{N}_2^L &= \{(\langle g_1, f \rangle, \langle g_2, f \rangle) \mid \langle g_1, f \rangle \in \mathcal{M}_L, \langle g_2, f \rangle \in \mathcal{M}_L^{op}\} \end{aligned}$$

which, together with the orderings

$$\begin{aligned} (\langle g, f_1 \rangle, \langle g, f_2 \rangle) \preceq (\langle g', f_1' \rangle, \langle g', f_2' \rangle) &\quad \text{iff} \quad g \preceq g' \\ (\langle g_1, f \rangle, \langle g_2, f \rangle) \preceq (\langle g_1', f' \rangle, \langle g_2', f' \rangle) &\quad \text{iff} \quad f' \preceq f \end{aligned}$$

[3]Note that the ordering relation is the same for both lattices, although its domain might differ from one to another.

are sublattices of $\mathcal{M}_L \times \mathcal{M}_L^{op}$ and, thus, are complete lattices.

Since, if $L_1 = L_2$, the multi-adjoint concept lattices $\mathcal{M}_L \times \mathcal{M}_L^{op}$ allow to prove that the concept lattices defined by Georgescu and Popescu in [13] can be seen as particular cases of multi-adjoint concept lattices $\mathcal{M}_L \times \mathcal{M}_L^{op}$. Specifically,

Theorem 1 *Given a complete generalized residuated lattice $(L, \preceq, \&, \swarrow, \nwarrow)$ and a residuated context (A, B, R) (defined in [13]), then there exist a multi-adjoint concept lattice, $\mathcal{M}$, such that the sublattice $\mathcal{N}_1$ is isomorphic to the non-commutative fuzzy concept lattice $\mathcal{L}$ (defined in [13]).*

From now on, we will only consider $\mathcal{N}_1^L$ (the results given for $\mathcal{N}_1^L$ can be proved for the other one analogously). Moreover, we will write the elements of $\mathcal{N}_1^L$ as $\langle g, f_1, f_2\rangle$ and we will called *triple concept* (t-concept, in short).

Hence, if we consider a t-concept, then $\langle g, f_1, f_2\rangle$ satisfies $(g^*)^{\uparrow_t} = f_1^*$, $(g^*)^{\Uparrow_t} = f_2^*$, $(f_1^*)^{\downarrow_t} = (f_2^*)^{\Downarrow_t} = g^*$, so we consider a concept which the non-commutative of the operators are not determinant with respect to g.

Finally, we need to prove that given a t-concept we can obtain directly a concept in $\mathcal{M}_L$. That is, if $\langle g^*, f^*\rangle \in \mathcal{M}_L$, then we have an associated concept in $\mathcal{M}$ as we show the following result.

Proposition 2 *Given a context (A, B, R, σ), a frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and the multi-adjoint concept lattice $\mathcal{M}_L$. If $\langle g^*, f^*\rangle \in \mathcal{M}_L$, then the mappings $g\colon B \to L_2$, $f\colon A \to L_1$, defined as: $g = \phi_2 \circ g^*$, $f = \phi_1 \circ f^*$, form a concept of the multi-adjoint concept lattice $\mathcal{M}$ associated to the frame $(L_1, L_2, P, \&_1, \dots, \&_n)$.*

Proof: We need to prove that $g^\uparrow = f, f^\downarrow = g$. We will prove the first equality and the second one follows similarly.

Given $a \in A$, as $\langle g^*, f^*\rangle \in \mathcal{M}_L$, we have that the following chain of equalities:

$$\begin{aligned} f^*(a) &= (g^*)^{\uparrow_t} \\ &= i_1(\inf\{R(a,b) \swarrow \phi_2(g^*(b)) \mid b \in B\}) \\ &= i_1(\inf\{R(a,b) \swarrow g(b) \mid b \in B\}) \\ &= i_1(g^\uparrow) \end{aligned}$$

Hence, applying ϕ_1 on both members, we obtain

$$f(a) = \phi_1(f^*(a)) = \phi_1(i_1(g^\uparrow)) = g^\uparrow(a)$$

$\square$

The result above can also be applied to $\mathcal{M}_L^{op}$ and we obtain that, given a t-concept $\langle g, f_1, f_2\rangle$, the elements $\langle \phi_1 \circ g, \phi_1 \circ f_1\rangle$, $\langle \phi_2 \circ g, \phi_2 \circ f_2\rangle$ are concepts of $\mathcal{M}_L$ and $\mathcal{M}_L^{op}$, respectively. Moreover, we have that

$$g = (i_1 \circ \phi_1 \circ f_1)^\downarrow = (i_2 \circ \phi_2 \circ f_2)^\Downarrow$$

Therefore, we are presented a procedure to obtain concepts where the non-commutative character of the operators on the frame is avoided.

3 An application example

To begin with, we cannot use the lattices in Example 1 to obtain non-trivial t-concepts because, in order to calculate a fixed point g of both Galois connections, we will have to compose the mappings i_1 and ϕ_2 ($\phi_2 \circ i_1$). This will give us the trivial constant mapping $\phi_2 \circ i_1(x) = \top_1$, for all $x \in L_1$. Hence, using Cartesian product of two lattices in order to drop the non-commutativity is not interesting.

The lattices in Example 2 can be used but we should define specific adjoint triples and the example will not be clarifying but confusing. Hence, in this application example, we will use lattices similar to those in Example 3 and the related adjoint triples will be defined from the product t-norm.

Let us consider that we have written a scientific paper and we still have to decide which journal the paper will be submitted to. According to the main topics of the paper, a number of journals are considered as potential target. The target journal will be chosen according to several parameters appearing in the ISI Journal Citation Report.

The set of attributes A is the following:

$$\{\mathrm{IF}, \mathrm{II}, \mathrm{CHL}, \mathrm{BP}\}$$

where we consider the "Impact Factor" (IF), the "Immediacy Index" (II) the "Cited Half-Life" (CHL) and the "best position" (BP) which means the best quartile of the different categories under which the journal is included. The set of objects B is:

$$\{\mathrm{AMC}, \mathrm{CAMWA}, \mathrm{FSS}, \mathrm{IEEE\text{-}FS}, \mathrm{IJGS}, \mathrm{IJUFKS}, \mathrm{JIFS}\}$$

where the journals considered are Applied Mathematics and Computation (AMC), Computer and Mathematics with Applications (CAMWA), Fuzzy Sets and Systems (FSS), IEEE transactions on Fuzzy Systems (IEEE-FS), International Journal of General Systems (IJGS), International Journal of Uncertainty Fuzziness and Knowledge-based Systems (IJUFKS), Journal of Intelligent and Fuzzy Systems (JIFS).

We will consider a multi-adjoint frame with three different lattices: one for handling the information taken from the JCR, which is rounded to the second decimal digit; a second one to handle information about the attributes, in which we estimate steps of 0.05 in order to distinguish to appreciate a qualitative difference; and a third one, used to set the different levels of preference of the journal, which is considered to be of 0.125 (hence the unit interval is divided into eight equal pieces)

Let $\mathcal{L} = ([0,1]_{20}, [0,1]_8, [0,1]_{100}, \le, \le, \le, \&_P^*)$ be a multi-adjoint frame where[4] $\&_P^*\colon [0,1]_{20} \times [0,1]_8 \to [0,1]_{100}$ is the discretisations of the product conjunctor, defined as:

$$\&_P^*(a,b) = \frac{\lceil 100 \cdot a \cdot b \rceil}{100}$$

[4]Recall that $[0,1]_m$ denotes a regular partition of $[0,1]$ into m pieces.

where $\cdot$ denotes the usual product of real numbers and $\lceil \_ \rceil$ is the ceiling function.

The corresponding residuated implications of $\&_P^*$ are $\swarrow^*_P \colon [0,1]_{100} \times [0,1]_8 \to [0,1]_{20}$ and $\nwarrow^*_P \colon [0,1]_{100} \times [0,1]_{20} \to [0,1]_8$, which are defined as:

$$
\begin{aligned}
b \swarrow^*_P a &= \frac{\lfloor 20 \cdot \min\{1, b/a\} \rfloor}{20} \\
b \nwarrow^*_P c &= \frac{\lfloor 8 \cdot \min\{1, b/c\} \rfloor}{8}
\end{aligned}
$$

where $\lfloor \_ \rfloor$ is the floor function.

The fuzzy relation between them, $R \colon A \times B \to P$, is the normalization to the unit interval $[0,1]$ of the information in the JCR, and can be seen in Table 1.

Table 1. Fuzzy relation between objects and attributes.

R	IF	II	CHL	BP
AMC	0.34	0.13	0.31	0.75
CAMWA	0.21	0.09	0.71	0.5
FSS	0.52	0.36	0.92	1
IEEE-FS	0.85	0.17	0.65	1
IJGS	0.43	0.1	0.89	0.5
IJUFKS	0.21	0.04	0.47	0.25
JIFS	0.09	0.06	0.93	0.25

The problem of choosing a *suitable* journal to submit depends on the definition of "suitability" we have in mind. For example, a fuzzy notion of suitability can be defined as a journal with high impact factor, relatively big immediacy index, more than 5.5 years of half-life and with not a bad position in the listing of the category. Such a notion of suitability can be defined, in the context (A, B, R, σ) where $\sigma(b) = \&_P$ for every $b \in B$, by the fuzzy subset $f \colon A \to [0,1]$ below:

$$
\begin{aligned}
f(\text{IF}) &= 0.75, & f(\text{II}) &= 0.25, \\
f(\text{CHL}) &= 0.5, & f(\text{BP}) &= 0.5
\end{aligned}
$$

Now, the problem consists in finding a multi-adjoint concept which represents the suitable journal as defined by the fuzzy set f.

As any concept gets completely determined by any of its components, it is sufficient to compute the component $f^{\downarrow}$ which, in addition, will provide information about the suitability (modulo f) of every journal. As explained in previous sections, the required computations are as follows:

$$
\begin{aligned}
f^{\downarrow}(\text{AMC}) &= \inf\{R(a, \text{AMC}) \nwarrow^*_P f(a) \colon a \in A\} \\
&= \inf\{0.34 \nwarrow^*_P 0.75, 0.13 \nwarrow^*_P 0.3, \\
&\qquad 0.31 \nwarrow^*_P 0.55, 0.75 \nwarrow^*_P 0.5\} \\
&= \frac{\lfloor 8 \cdot \min\{1, 0.13/0.3\} \rfloor}{8} \\
&= 0.375
\end{aligned}
$$

For the rest of the journals, the computation is similar, obtaining the following results

$$
\begin{aligned}
&f^{\downarrow}(\text{AMC}) = 0.375 && f^{\downarrow}(\text{CAMWA}) = 0.25 \\
&f^{\downarrow}(\text{FSS}) = 0.625 && f^{\downarrow}(\text{IEEE-FS}) = 0.625 \\
&f^{\downarrow}(\text{IJGS}) = 0.4 && f^{\downarrow}(\text{IJUFKS}) = 0.125 \\
&f^{\downarrow}(\text{JIFS}) = 0
\end{aligned}
$$

based on which, there are two suitable journals because FSS and IEEE-FS have the same value.

Now, the user should think about other possibilities which allow to better discriminate among the target journals. One of this can be to consider a new frame in which the information about the attributes is considered to be in $[0,1]_8$ and, in order to appreciate a qualitative difference, the information about the journal in $[0,1]_{20}$. Therefore, the new frame is

$$
\mathcal{L}^{op} = ([0,1]_{20}, [0,1]_8, [0,1]_{100}, \leq, \leq, \leq, (\&_P^*)^{op})
$$

the new Galois conection is the dual one of above, and we obtain from the particular definition for "suitability", f, that:

$$
\begin{aligned}
&f^{\Downarrow}(\text{AMC}) = 0.45 && f^{\Downarrow}(\text{CAMWA}) = 0.25 \\
&f^{\Downarrow}(\text{FSS}) = 0.65 && f^{\Downarrow}(\text{IEEE-FS}) = 0.65 \\
&f^{\Downarrow}(\text{IJGS}) = 0.4 && f^{\Downarrow}(\text{IJUFKS}) = 0.15 \\
&f^{\Downarrow}(\text{JIFS}) = 0.1
\end{aligned}
$$

and we have a tie again.

Finally, we will consider a new possibility, that is trying to completely remove the granularity and to consider a lattice L which embeds both $[0,1]_8$ and $[0,1]_{20}$. Hence, we can consider $L = [0,1]$, the mappings i_8, i_{20}, ϕ_8 and ϕ_{20} as in Example 3, and the following Galois connections:

$$
\begin{aligned}
g^{\uparrow^{20}_{8}}(a) &= i_{20}(\inf\{R(a,b) \swarrow^*_P \phi_8(g(b)) \mid b \in B\}) \\
f^{\downarrow^{20}_{8}}(b) &= i_8(\inf\{R(a,b) \nwarrow^*_P \phi_{20}(f(a)) \mid a \in A\}) \\
g^{\Uparrow^{8}_{20}}(a) &= i_8(\inf\{R(a,b) \nwarrow^*_P \phi_{20}(g(b)) \mid b \in B\}) \\
f^{\Downarrow^{8}_{20}}(b) &= i_{20}(\inf\{R(a,b) \swarrow^*_P \phi_8(f(a)) \mid a \in A\})
\end{aligned}
$$

We want to obtain a triple $\langle f, g_1, g_2 \rangle \in \mathcal{N}_2^L$ such that $\langle g_1, f \rangle$ is a concept of M_L and $\langle g_2, f \rangle$ is a concept of M_L^{op}.

Table 2. t-concepts from f_0

	f_0		f		$f^{\downarrow^{20}_{8}}$	$f^{\Downarrow^{8}_{20}}$
IF	0.75		1	AMC	0.25	0.3
II	0.25		0.25	CAMWA	0.125	0.2
CHL	0.5		1	FSS	0.5	0.5
BP	0.5		1	IEEE-FS	0.5	0.65
				IJGS	0.375	0.4
				IJUFKS	0.125	0.15
				JIFS	0	0

Iterating the mapping $\_\downarrow^{20}_{8}\uparrow^{20}_{8}\Downarrow^{8}_{20}\Uparrow^{8}_{20}$ from the function above, which we will rewrite it as f_0, we will obtain a fixpoint and, as a consequence, the required triple.

In Table 2, we have the values of the triple $\langle f, g_1, g_2 \rangle$, and we can check that, finally, the best target journal is IEEE-FS.

4 Conclusions

We have presented one approach to the construction of concept lattices based on non-commutative conjunctors which allows the underlying conjunctors to have the type $L_1 \times L_2 \to L$ with $L_1 \neq L_2$.

The idea is to present sufficient conditions in order to be able to construct concepts in a generalized fuzzy context in which the domain of the underlying conjunctors $L_1 \times L_2$ can be adequately embedded in a common one. As a result, our approach enables a conciliation between the different values obtained due to the non-commutativity.

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