Finitary coalgebraic multisemilattices and multilattices[☆,☆☆]

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Abstract

In this paper we continue the coalgebraization of the structure of multilattice. Specifically, we introduce a coalgebraic characterization of the notion of finitary multi(semi)lattice, a generalization of that of semilattice which arises naturally in several areas of computer science and provides the possibility of handling non-determinism.

Key words: Lattices; Coalgebras; Non-determinism; Multilattice.
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1. Introduction

Coalgebras have received special interest in recent years, specially because they are simple mathematical structures capable to theoretically describe state-based dynamical systems. Coalgebras have been applied in very disparate areas, ranging from genetics (as in [33] where a coalgebraic study of genetics is presented) to several research lines in Computer Science: for instance, coalgebras have been used to describe automata or transition systems [16, 30]. In [14, 27] the specification of object-oriented systems are given via coalgebraic operations and initial states, satisfying certain properties which are defined using bisimilarity. Other application of the theory of coalgebras is the modelling of finite interactive computing agents [36].

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They are also used as semantic structures for logical languages [24], to develop automatic theorem provers [9], and to build semantical structures for various non-classical logics, such as modal logics, temporal logics, logics for multi-agents system [18, 1, 32, 26, 25, 15], etc. Last but not least, coalgebras have been applied in quantum physics to formalize quantum spectra underlying quantum measurements [5].

On the other hand, this paper deals with certain non-deterministic generalization of the notion of (complete) lattice. In the area of fuzzy logic programming, numerous works have focused on the generalization of well-known results by substituting in the semantics the structure of complete lattice by weaker conditions. In this sense, residuated lattices are used in [8, 35]; multi-adjoint lattices in [21, 22]; a more general structure, namely algebraic domains in [28]; another generalization, based on the ordered structure of multilattice, was given in [20].

The main idea underlying the notion of multilattice is to change the notion of supremum by the minimal elements of the set of upper bounds in such a way that every upper bound is greater than or equal to a minimal element of the upper bounds. Dually, every lower bound is less than or equal to a maximal element of the lower bounds. Thus, the existence of a unique supremum is substituted by the existence of several minimal upper bounds (multisuprema), allowing for an alternative approach to non-determinism, as the supremum operation is no longer deterministic: a least upper bound need not exist, but several minimal ones.

The structure of multilattice is an example of hyperstructure [7], which is interesting in several fields of Computer Science [29, 34]. For instance, this structure has been used to design ATPs in temporal logics [6]; on the other hand, the free monoid X^* over a set X forms a multilattice under the *substring order* defined as follows: given $\tau, \omega \in X^*$, the string τ is said to be a substring of ω if $\omega = \varsigma\tau\upsilon$ for some $\varsigma, \upsilon \in X^*$ (see [17, 23] and the references therein, just to mention a few). In the same monoid, we also obtain a multilattice with the *subsequence order* given by: the string τ is said to be a subsequence of ω if we can obtain τ by deleting elements in ω [10, 23]. These examples are considered as important in the theory of multilattices as the power set $\mathcal{P}(X)$ is in lattice theory.

The original notion of ordered multilattice was introduced by Benado in [2] and other equivalent algebraic characterizations can be found in the literature [13]. Unlike in the case for lattices, the existence of multisuprema (and, dually, multiinfima) produces different structures when they are formulated for pairs of elements, as in Benado's approach, and for finite subsets of any cardinality. The first definition of a similar generalized structure which demands the existence of multisuprema for every nonempty finite subset, the *finitary* multilattice,[1] appears in [6, 19] together with an equivalent algebraic characterization.

In [3] a coalgebraic characterization of multilattices was proposed, and it can be viewed as a starting point for the research line carried out in this work, whose aim is to provide a coalgebraic characterization for finitary multisemilattices and multilattices as a step-stone towards *complete*[2] multilattices which will lead to a thorough study of coalgebraic semantics for fuzzy logics.

Concerning the application of the coalgebraic approach to multilattices in the realm of fuzzy logics, our efforts pursue to fuse together two different lines. On the one hand, it is well-known that soon after Zadeh used values of the unit interval $[0, 1]$ as degrees of membership of elements in a fuzzy set, his approach was extended by Goguen [11] to the notion of L-fuzzy set, in which a complete lattice L is substituted for the unit interval; recently, several authors have advocated for the greater level of generality, and hence more flexibility, provided when substituting a complete lattice L by a complete multilattice M, considering a theory of M-fuzzy sets as a basis for fuzzy logics [29, 20]. On the other hand, the extensive use of coalgebraic methods in modal and other non-classical logics suggests the possibility of developing a coalgebraic approach to fuzziness; some initial and promising results have been presented recently in [31], where the generic framework of coalgebraic logic has been adapted to the specific setting of fuzzy description logics in order to prove decidability of this kind of logics with a modality to express the adverb "*probably*".

A key property in the study of this non-deterministic extension of the notion of lattice is that of associativity. The natural extension to the non-deterministic case turns out to be excessively restrictive, since in [19] it was proved that a multilattice satisfying the natural definition of associativity collapses to a lattice.

Although non-associativity arises naturally in different physical phenomena, and there exist applicable non-associative algebraic structures (for

[1]The original term used in [19] was *universal*, but we prefer to change the name in order to prevent possible misunderstandings with universal algebras.

[2]In a *complete* multilattice for *any* subset X of elements and a lower (resp. upper) bound b there exist multiinfima (resp. multisuprema) between b and X.

instance Lie algebras are used in the study of particle physics), we have focused our attention on alternative definitions of associativity. Some approaches were proposed by several authors [2, 13], but they were not really a generalization of associativity. In [19] the notion of weak-associativity allowed for studying multilattices in terms of interlaced multisemilattices, in a similar form to the classical case. However, this type of associativity does not work when changing binary by finitary multisuprema (that is, assuming a flexible-arity in the operator which computes multisuprema, i.e. postulating the existence of multisuprema of finite sets of arbitrary size).

In the coalgebraic approach to binary multilattices presented in [3], a weaker definition of associativity was introduced, the m-associativity, which allows for providing the coalgebraic characterization of multilattices and multisemilattices. In that reference, binary ND-coalgebras were introduced as a new kind of coalgebras capable of providing a suitable definition of binary multilattices. In this paper we show that finitary multilattices also can be properly described as a coalgebra, specifically as a finitary ND-coalgebra using this kind of associativity.

In this manuscript, we focus on the adaptation of the notion of m-associativity to handle finitary non-deterministic operations in a way that the coalgebraic characterization of finitary multi(semi)lattices is preserved. The extension is not straightforward in that, opposite to what occurs with the other properties, the *recursive* extension of the binary m-associativity turns out to be too weak to characterize coalgebraically the finitary multi(semi)lattices.

The paper is organized as follows. Section 2 contains notational conventions and recalls some technical preliminaries. In Section 3, finitary ND-coalgebras are introduced; the natural extension of basic properties used for coalgebraic characterization of multilattices are presented in Section 4. The next section is devoted to analyze the most adequate extension of m-associativity for finitary ND-coalgebras. Section 6 includes the main results of the paper and it leads to the definition of finitary coalgebraic multisemilattice as the coalgebraic version of finitary multisemilattices. The notion of multilattice, which arises naturally from multisemilattices, also has a coalgebraic interpretation, which is developed in Section 7. Finally, Section 8 summarizes the results presented and outlines future work.

2. Notation and mathematical background

Given $(A, \leq)$ a partially ordered set (henceforth, *poset*) and $B \subseteq A$, we write $\textsc{UBounds}(B)$ to denote the set of upper bounds of B and, similarly, $\textsc{LBounds}(B)$ to denote the set of lower bounds of B. Moreover, $\textsc{Maximal}(B)$ and $\textsc{Minimal}(B)$ stand for the set of maximal and minimal elements of B, respectively.

Likewise, $\uparrow$ and $\downarrow$ denote the upper and lower closure operators respectively. That is, for all $B \subseteq A$

$$B\uparrow = \bigcup_{b \in B} \{x \in A \mid x \geq b\} \qquad \text{and} \qquad B\downarrow = \bigcup_{b \in B} \{x \in A \mid x \leq b\}$$

In order to introduce the notion of multilattice as an ordered structure, firstly it is necessary to define the concept of multi-supremum as an extension of supremum (resp. multi-infimum). In a poset A, a *multi-supremum* of a subset B is a minimal element of the set of upper bounds of B and through the paper, $\textsc{Msup}(B)$ denotes the set of multi-suprema of B; the notion of *multi-infima* is similarly defined.

The definition of multilattice will be based on that of multisemilattice, which is given below:

Definition 2.1. *A* join-multisemilattice *is a poset $(M, \leq)$ in which, for all $a, b, x \in M$ with $a \leq x$ and $b \leq x$, there exists $z \in \textsc{Msup}\{a, b\}$ such that $z \leq x$. The dual concept of a join-multisemilattice is called* meet-multisemilattice.

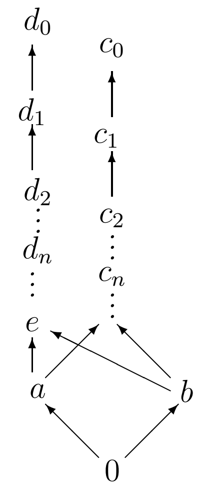
A poset $(M, \leq)$ is said to be a multilattice *if it is a join and meet-multisemilattice.*

Notice that $(A, \leq)$ is a join-multisemilattice whenever $\textsc{UBounds}\{a, b\} \subseteq (\textsc{Msup}\{a, b\})\uparrow$ for all $a, b \in A$, and is a meet-multisemilattice whenever $\textsc{LBounds}\{a, b\} \subseteq (\textsc{Minf}\{a, b\})\downarrow$ for all $a, b \in A$.

The previous definition is consistent with the existence of two incomparable elements *without* any multi-supremum or multi-infimum. In other words, $\textsc{Msup}\{a, b\}$ and $\textsc{Minf}\{a, b\}$ can be empty. Moreover, if $\textsc{Msup}\{a, b\}$ and $\textsc{Minf}\{a, b\}$ are singletons for all $\{a, b\}$, then $(M, \leq)$ is a lattice, which implies that multilattices are more general structures than lattices.

In the concept of ordered multisemilattice, minimal upper bounds (multi-suprema) play the role of least upper bounds in a lattice (analogously for the dual). The main difference that is noticed is that the operators which compute multi-suprema are not single-valued, because there may be several multi-suprema or may be none. The following examples show several specific features of multilattices.

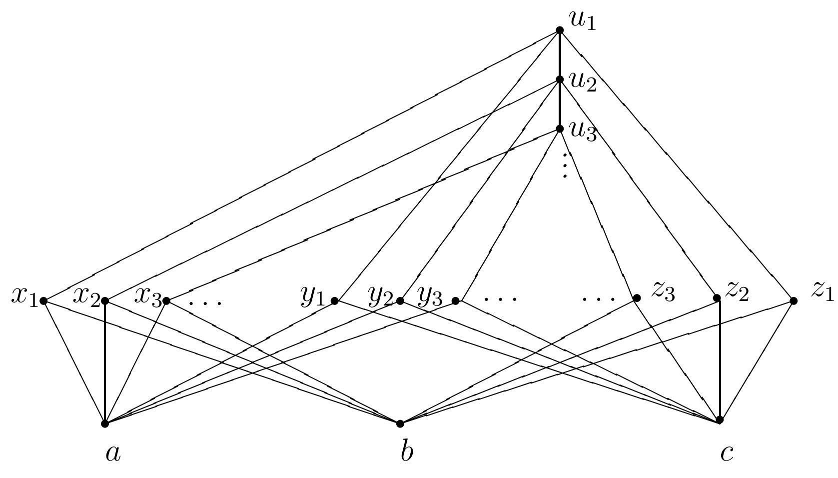
Example 2.1. *Let* $(A, \leq)$ *be the poset whose diagram is:*



$(A, \leq)$ *is a meet-multisemilattice, but not a join-multisemilattice because*

$$\text{UBounds}\{a, b\} = \{e\} \cup \{c_i, d_i\}_{i\in\mathbb{N}} \not\subseteq (\text{Msup}\{a, b\}) \uparrow = \{e\} \cup \{d_i\}_{i\in\mathbb{N}}$$

Example 2.2. *Let us consider the poset* A *whose diagram is*



Notice that $(A, \leq)$ *is a join-multisemilattice. However, the set* $\{u_1, u_2, \dots\}$ *of all upper bounds of the set* $H = \{a, b, c\}$ *has no minimal elements.*

It is well-known that an equivalent definition of a semilattice involves finite nonempty subsets instead of pairs of elements [12]. However, in the realm of multisemilattices, the corresponding extension to finite subsets generates a *new* structure which will be called *finitary multisemilattice* [19].

Definition 2.2. *A* finitary join-multisemilattice *is a poset* $(M, \leq)$ *such that, for all finite* $H \subseteq M$ *there exists* $z \in \textsc{Msup}(H)$ *satisfying* $z \leq x$ *for all* $x \in \text{UB}\textsc{ounds}(H)$. *The dual concept of finitary join-multisemilattice is called* finitary meet-multisemilattice.

A poset $(M, \leq)$ *is said to be a* finitary multilattice *if it is both a finitary join and meet-multisemilattice.*

Notice that the definition above is equivalent to saying that the poset $(M, \leq)$ is a finitary join-multisemilattice if $\text{UB}\textsc{ounds}(H) \subseteq (\textsc{Msup}(H))\uparrow$ for all finite subset $H \subseteq M$, and is a finitary meet-multisemilattice if $\text{LB}\textsc{ounds}(H) \subseteq (\textsc{Minf}(H))\downarrow$ for all finite subset $H \subseteq A$

In order to deal with finitary multilattices from an algebraic point of view, let us introduce the notion of non-deterministic operator of *flexible arity* as a mapping from n-tuples, for $n \in \mathbb{N}$, to subsets. This notion generalizes that of the non-deterministic operator originally introduced in [19].

Definition 2.3. *A non-deterministic operator with* flexible arity *in* X *is a function*

$$F : X^* \longrightarrow \mathcal{P}(X)$$

where $X^* = \bigcup_{n\in\mathbb{N}} X^n$ *is the set of strings in* X.

Given $U \subseteq X$, we will consider the extension of F as follows

$$F(x_1, \ldots, x_{i-1}, U, x_{i+1}, \ldots, x_n) = \bigcup_{u\in U} F(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n)$$

As a consequence, $F(x_1, \ldots, x_{i-1}, \varnothing, x_{i+1}, \ldots, x_n) = \varnothing$.

Notice that $\textsc{Maximal}(\_)$, $\text{UB}\textsc{ounds}(\_)$ and $\textsc{Msup}(\_)$ and their dual operators can be considered as non-deterministic operators of flexible arity in a poset. For instance, given $(X, \leq)$ a poset, $\textsc{Msup}(\_) : X^* \longrightarrow \mathcal{P}(X)$ is defined as

$$\textsc{Msup}(\omega) = \textsc{Msup}\{x_1, x_2, \ldots, x_n\} \text{ for all } \omega = x_1 x_2 \cdots x_n \in X^*.$$

3. Finitary ND-coalgebras

In [3] binary ND-coalgebras were introduced as a new kind of coalgebras capable of providing a suitable definition of multilattices. As stated in the introduction, in this work we focus on the coalgebraic characterization of *finitary* multilattices, and to begin with, we need a convenient generalization of the notion of binary ND-coalgebra which allows for defining the flexible arity ND-operators. This extended structure, introduced in this section, will be called *finitary* ND-coalgebra.

In order to formally introduce the notion of finitary ND-coalgebra, some knowledge of elementary notions of category theory will be assumed: essentially, the notions of *object* and *morphism* in a category, and *functor* as a morphism between categories.

A *type* (or *signature*) is a non-trivial endofunctor in the category *Set* of sets, $\mathcal{T}\colon Set \to Set$. A *coalgebra of type* $\mathcal{T}$ is a pair (A, α) consisting of a set A and a map $\alpha\colon A \to \mathcal{T}(A)$.

The types of the coalgebras we are concerned with, are related to the (covariant) *powerset functor*, also called *direct image functor*. Specifically, we consider *Set*-functors mappings object X to $\mathcal{P}(X^n)$, for any $n \in \mathbb{N}$. This collection of *Set*-functors allow for many non-deterministic structures to benefit from a coalgebraic treatment.

Definition 3.1 ([3]).

1. *Given $n \in \mathbb{N}$, the functor $\mathcal{T}_n\colon Set \to Set$ is defined as follows:*
   - *if X is a set then $\mathcal{T}_n(X) = \mathcal{P}(X^n)$*
   - *if $f\colon X \to Y$ is a morphism then $\mathcal{T}_n(f)\colon \mathcal{P}(X^n) \to \mathcal{P}(Y^n)$ is the morphism given for all $\mathcal{X} \subseteq X^n$, by*

$$\mathcal{T}_n(f)(\mathcal{X}) = \{(f(x_1), \dots, f(x_n)) \mid (x_1, \dots, x_n) \in \mathcal{X}\}$$

2. *The class of* ND-functors *in the category Set is defined as the least set NDF containing $\mathcal{T}_n$ for all $n \in \mathbb{N}$, and closed for the product of functors, that is if $\mathfrak{T}$ is a subset of NDF, then the product $\prod_{\mathcal{T}\in\mathfrak{T}} \mathcal{T}$ is in NDF.*
3. *An* ND-coalgebra *is a coalgebra of type an ND-functor $\mathcal{T}$, namely, a pair $\mathcal{A} = (A, \alpha)$ where α is a mapping $\alpha\colon A \to \mathcal{T}(A)$.*

Example 3.1. *Functors given by $X \to \mathcal{P}(X^{n_1}) \times \cdots \times \mathcal{P}(X^{n_r})$, where $r \in \mathbb{N}$, $r \geq 1$ and $n_1, \dots, n_r \in \mathbb{N}$, are ND-functors.*

Throughout the paper, given an ND-coalgebra $\mathcal{A} = (A, \alpha)$ of type $\mathcal{T}$, for every $a \in A$, the element $\alpha(a) \in \mathcal{T}(A)$ will be denoted by α_a. We will write $\mathbb{N}^+$ to denote $\mathbb{N} \setminus \{0\}$.

Definition 3.2.

1. *An* n-ary ND-coalgebra *(briefly* (n)*-ND-coalgebra) is an ND-coalgebra of type* $\mathcal{T}_n$*.*
2. *A* finitary ND-coalgebra *is one of type* $\prod_{n \in \mathbb{N}^+} \mathcal{T}_n$*. In other words, a finitary ND-coalgebra is a pair* (A, α) *where* $\alpha\colon A \to \prod_{n \in \mathbb{N}^+} \mathcal{T}_n(A)$*.*

Definition 3.3. *Let* $\mathcal{A} = (A, \alpha)$ *be a finitary ND-coalgebra; given* $n \in \mathbb{N}^+$*, the* n-factor of $\mathcal{A}$ *is defined as the* (n)*-ND-coalgebra* $\mathcal{A}^n = (A, \alpha^n)$*, where* $\alpha^n = \pi_n \circ \alpha$ *and* $\pi_n\colon \prod_{m \in \mathbb{N}^+} \mathcal{P}(A^m) \to \mathcal{P}(A^n)$ *denotes the natural projection.*

Remark 3.4. *Given a finitary ND-coalgebra* (A, α) *and* $a \in A$*, there are several possible interpretations of an element* α_a*:*

1. *Since for* $n \geq 1$, $\mathcal{P}(A^n) \simeq \mathcal{P}(A)^{A^{n-1}}$, α_a *can be determined if we know the subset that* α_a^n *assigns to an arbitrary element of* A^{n-1}*, for all* n*.*
2. *The element* α_a *can be written as* $(\alpha_a^n)_{n \in \mathbb{N}^+}$*, where* $\alpha_a^n \in \mathcal{P}(A^n)$*.*
3. *Finally,* α_a *can be regarded as a non-deterministic operator* $\alpha_a : A^* \to \mathcal{P}(A)$ *by defining*[3] $\alpha_a(\omega) = \begin{cases} \alpha_a^1 & \text{if } \omega = \varepsilon \\ \alpha_a^{n+1}(\omega) & \text{if } \omega \in A^n, n \neq 0 \end{cases}$

Notice that, in particular, a finitary ND-coalgebra can be also considered as a coalgebra whose type is the functor $A \to \mathcal{P}(A)^{A^*}$. In order to distinguish this interpretation from the one given in the first construction (Definition 3.2) it will be henceforth denoted $\mathcal{T}_*$.

Concerning morphisms between finitary ND-coalgebras, as a particular case of ND-coalgebras, all the considerations presented in [3] apply. Specifically, it was shown that the standard notion of morphism in the category of $\mathcal{T}$-coalgebras, for an arbitrary type $\mathcal{T}$,[4] is not suitable for the construction of a category of ND-coalgebras.

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[3]Note that the empty string is denoted by ε.

[4]The class of all $\mathcal{T}$-coalgebras forms a category in which the morphisms are defined as follows: given $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ two $\mathcal{T}$-coalgebras, a map $f\colon A \to B$ is a standard *homomorphism* of coalgebras if $\mathcal{T}f \circ \alpha = \beta \circ f$.

The most adequate definition of morphism, following Benado's initial ideas, is given by relaxing the condition required by substituting an inclusion for the equality sign.

Definition 3.5. *Let $\mathcal{T}$ be an ND-functor. A function $f\colon A \to B$ between two $\mathcal{T}$-coalgebras (A,α) and (B,β) is said to be a* Benado-homomorphism *of coalgebras if $(\mathcal{T}f \circ \alpha)(a) \subseteq (\beta \circ f)(a)$ for all $a \in A$.*

It is straightforward to show that the identity map is always a Benado-homomorphism and the composition of two Benado-homomorphisms is again a Benado homomorphism. Moreover, finitary ND-coalgebras constitute a subcategory of this category of ND-coalgebras.

In adition, it is not difficult to see that any morphism between two finitary ND-coalgebras (A,α) and (B,β) is also a morphism between the n-factors (A,α^n) and (B,β^n), for all $n \in \mathbb{N}^+$.

4. Towards finitary coalgebraic multisemilattices: first results

In [19] it was shown that the generalization of multilattices to finitary multilattices is non-trivial due to the difficulty of obtaining a suitable extension of the associative property. We will have to face a similar problem, in terms of coalgebras, in passing from binary to finitary ND-coalgebras. In fact, some properties of binary ND-coalgebras can be extended as expected to finitary ND-coalgebras, but the notion of m-associativity for binary ND-coalgebras from [3] cannot be extended in a straightforward way to the finitary case. In [3] we obtained a coalgebraic characterization of multilattices as a doubly binary ND-coalgebra consisting of two properly assembled coalgebraic multisemilattices, which are binary ND-coalgebras satisfying certain properties (see Definition 32 in [3]).

In this section our aim is to obtain a coalgebraic characterization of finitary multisemilattices as finitary ND-coalgebras. It is reasonable to believe that we can characterize a finitary multisemilattice as a finitary ND-coalgebra satisfying certain properties similar to those of binary coalgebraic multisemilattices. In fact, a finitary coalgebraic multisemilattice will be a finitary ND-coalgebra such that it is commutative, m-associative, separating, strongly secondary reflexive and all the elements are self-conscious.

On the recursive structure of finitary ND-coalgebras

From Definition 3.2, if (A, α) is an (n)-ND-coalgebra with $n > 0$, then (A, α_a) is an $(n$–$1)$-ND-coalgebra, for all $a \in A$. This property enables the recursive construction of the definitions presented in this section.

Schema of recursive extension. The recursive definition of the properties that we will use follows the scheme below:

1. Depending on the property, the base case will be given for unary or binary ND-coalgebras (that is (1)- or (2)-ND-coalgebras).
2. The generalization for an (n)-ND-coalgebra, will be specified in each case, via the so-called *recursive generator*.
3. We will say that the property is verified for a finitary ND-coalgebra $\mathcal{A} = (A, \alpha)$, if it is satisfied for the n-factor of $\mathcal{A}$ for all $n \in \mathbb{N}^+$.

As a first example of extension of binary properties to the finitary case via the schema of recursive extension, let us consider the property of commutativity:

The base case is that given in [3] for a binary ND-coalgebra, that is, $\alpha_a(b) = \alpha_b(a)$ for all $a, b \in A$. Its finitary extension is done in terms of the recursive extension as follows: (A, α) is commutative if (A, α_a) is commutative, for all $a \in A$.

Now, by using the third interpretation of finitary ND-coalgebra showed in Remark 3.4, the following definition naturally arises:

Definition 4.1. *A finitary ND-coalgebra $\mathcal{A} = (A, \alpha)$ is* commutative *if*

$$\alpha_a(x_1 x_2 \cdots x_n) = \alpha_a(x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)})$$

for every permutation σ and $n \in \mathbb{N}^+$ and

$$\alpha_a(x\omega) = \alpha_x(a\omega)$$

for every $a, x \in A$ and $\omega \in A^*$.

In [3], an element a of a (2)-ND-coalgebra (A, α) was defined to be *self-conscious* if $a \in \alpha_a(a)$ and *isolated* if $\{a\} = \alpha_a(a)$. Both concepts can be generalized for finitary ND-coalgebras.

Definition 4.2. *An element a of a finitary ND-coalgebra $\mathcal{A} = (A, \alpha)$ is:*[5]

[5]Note that when we have $((((\alpha^n_{a_1})_{a_2})_{a_3})\ldots)_{a_m}$, using multiplicative notation in the subscripts we will write $\alpha^n_{a_1\cdots a_m}$. In particular, $\alpha^n_{a\ldots a} = \alpha^n_{a^m}$.

1. self-conscious *if for all* $a \in A$ *and all* $n \in \mathbb{N}^+$ $a \in \alpha^n_{a^{n-1}}$.
2. isolated *if for all* $a \in A$ *and all* $n \in \mathbb{N}^+$, $\alpha^n_{a^{n-1}} = \{a\}$.

Example 4.1. *From a poset* $(A, \leq)$*, a finitary ND-coalgebra* (A, α) *can be defined by* $\alpha_a(\omega) = \bigcup_{x \in \omega} \text{UBOUNDS}(a, x)$ *for all* $a \in A$*, and* $\omega \in A^*$*. Every element of* A *is self-conscious but not necessarily isolated.*

Example 4.2. *Considering the set of real numbers* $\mathbb{R}$*, for every* $x \in \mathbb{R}$ *and* $\omega \in \mathbb{R}^*$*, let* $\alpha_a(\omega)$ *denote the smallest closed interval containing a and the elements of* ω*. Clearly, all the elements of the finitary ND-coalgebra* $(\mathbb{R}, \alpha)$ *are isolated.*

Recall from [4] that a binary relation R in a set X is said to be *strongly secondary reflexive* if the following condition holds

$$x \in R(y) \quad \text{implies} \quad R(x) = \{x\}, \quad \text{for all } x, y \in X$$

That is, R is the identity relation when restricted to $R(X) = \bigcup_{x \in X} R(x)$.

A binary ND-coalgebra (A, α) is defined to be *strongly secondary reflexive* (see [3]) if α_x is a strongly secondary reflexive binary relation, for all $x \in A$. So, a finitary ND-coalgebra $\mathcal{A} = (A, \alpha)$ is *strongly secondary reflexive* if it satisfies the schema of recursive construction, with the following recursive generator: an (n)-ND-coalgebra (A, α) with $n > 1$ is *strongly secondary reflexive* if $x \in \alpha^n_{a^n}$ implies $\alpha^n_{x^n} = \{x\}$.

An equivalent presentation in terms of the elements of the coalgebra, is given in the following definition:

Definition 4.3. *A finitary ND-coalgebra* (A, α) *is strongly secondary reflexive if and only if*

a) $x \in \alpha_a(\varepsilon)$ *implies* $\{x\} = \alpha_x(\varepsilon)$ *for all* $x, a \in A$ *and*
b) $x \in \alpha_a(\omega y)$ *implies* $\{x\} = \alpha_a(\omega x)$ *for all* $x, y \in A, \omega \in A^*$.

If the finitary ND-coalgebra is both commutative and strongly secondary reflexive, the previous conditions can be rewritten in a much simpler way as $\{x\} = \alpha_x(\omega)$ for every $x \in \alpha_a(\omega)$.

In a strongly secondary reflexive finitary ND-coalgebra, it is easy to prove that the properties of self-consciousness and isolation coincide. As this fact will be used later, we will formally state it as follows:

Proposition 4.4. *In a strongly secondary reflexive finitary ND-coalgebra an element is self-conscious if and only if it is isolated.*

The simple fact that two different upper (lower) bounds or two different multisuprema (multiinfima) of the same subset cannot be related in a poset, provides an important property to be taken into account when the poset is viewed from a coalgebraic point of view. In binary coalgebras this property was called *separation* (Definition 20 of [3]). We will use the same term for finitary ND-coalgebras.

A binary ND-coalgebra (A, α) is defined to be *separating* if $x, y \in \alpha_a(b)$ and $x \in \alpha_x(y)$ then $x = y$ for all $a, b, x, y \in A$. So, a finitary ND-coalgebra $\mathcal{A} = (A, \alpha)$ is *separating* if it satisfies the schema of recursive extension, with the following recursive generator: an (n)-ND-coalgebra (A, α) with $n > 2$ is *separating* if (A, α_a) is a *separating* $(n-1)$-ND-coalgebra, for all $a \in A$.

As above, this property can also be formulated by means of the elements of the coalgebra. For this reason, we can give the following definition:

Definition 4.5. *A finitary ND-coalgebra* (A, α) *is separating if and only if* (A, α^2) *is separating and*

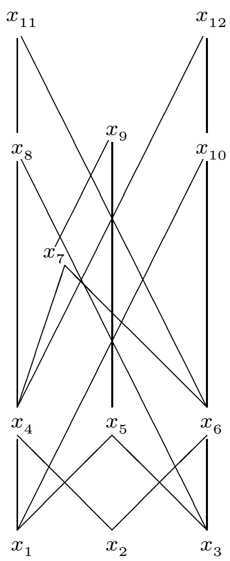
$$\textit{if } \; x, y \in \alpha_a(\omega bc) \; \textit{ and } \; x \in \alpha_a(\omega xy) \; \textit{ then } \; x = y$$

for all $a, b, c, x, y \in A, \omega \in A^*$.

5. On a suitable notion of associativity

As stated in the introduction, a suitable extension of associativity for the coalgebraic characterization of finitary multilattices is not straightforward by using the schema of recursive extension as in the other properties in the previous section. The reason is that the multisuprema of sets of three elements, in general, cannot be described in terms of pairs, see Example 5.1 below, and the general expression of associativity needs three elements to be stated.

Example 5.1. *Consider the multilattice depicted below:*



If we consider $\omega = x_1 x_2 x_3$*, we have that* $\text{MSUP}(x_1 x_2 x_3) = \{x_7, x_8, x_{10}\}$ *and:*

$$\begin{aligned}
&\text{MSUP}(x_1\text{MSUP}(x_2x_3)) = \text{MSUP}(x_1\{x_6, x_8\}) = \{x_7, x_8, x_{10}, x_{11}\}\\
&\text{MSUP}(x_2\text{MSUP}(x_1x_3)) = \text{MSUP}(x_2\{x_5, x_7, x_8, x_{10}\}) = \{x_7, x_8, x_9, x_{10}\}\\
&\text{MSUP}(x_3\text{MSUP}(x_1x_2)) = \text{MSUP}(x_3\{x_4, x_{10}\}) = \{x_7, x_8, x_{10}, x_{12}\}
\end{aligned}$$

The solution we propose considers finitary ND-coalgebras as coalgebras of type $\mathcal{T}_*$ (see at the end of Section 3).

Definition 5.1. *A finitary ND-coalgebra* (A, α) *is said to be* m-associative *if for elements* $a, b \in A$ *and* $\omega \in A^*$ *satisfying* $\alpha_a(b\omega) = \{a\}$*, the following inclusion holds*

$$\alpha_a(\gamma) \subseteq \bigcap_{\omega\gamma=\omega'\gamma'} \alpha_a(\alpha_b(\omega')\gamma') \quad \textit{for all } \gamma \in A^*. \tag{1}$$

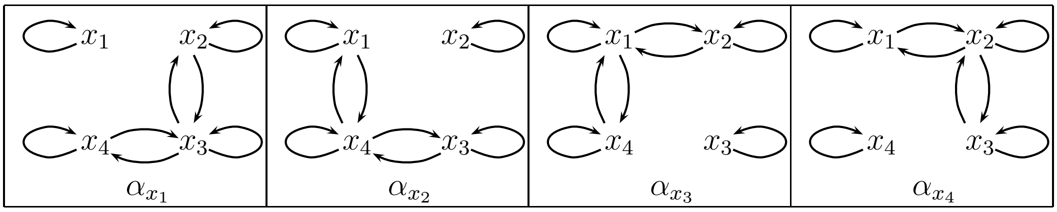
If the inclusion in (1) *is an identity, the finitary ND-coalgebra will be called* strongly m-associative.

Example 5.2. *The finitary ND-coalgebra* $(\mathbb{R}, \alpha)$ *defined in Example 4.2 is m-associative; in fact, in this case, all the sets* $\alpha_a(\alpha_b(\omega')\gamma')$ *in* (1) *coincide with* $\alpha_a(\gamma)$*, for all* $a, b \in \mathbb{R}, \gamma \in \mathbb{R}^*$*, so it is strongly m-associative.*

Example 5.3. *Consider the set* $X = \{x_1, x_2, x_3, x_4\}$ *and* $\alpha : X \to \mathcal{P}(X)^{X^*}$ *as follows:*

$$\alpha_{x_i}(\varepsilon) = \{x_i\}; \quad \alpha_{x_i}(x_i \cdots x_i) = \{x_i\} \quad \text{for all } i \in \{1,2,3,4\}$$

$\alpha_{x_i}(x_j)$ *is described in the figure below for all* $i, j \in \{1,2,3,4\}$



and $\alpha_{x_i}(\omega) = X$ *otherwise. The finitary ND-coalgebra* (X, α) *is m-associative but not strongly m-associative because* $\alpha_{x_1}(x_1) = \{x_1\}$ *but taking* $\gamma = x_2$*, in the left-hand side of the inclusion* (1)*, we have* $\alpha_{x_1}(x_2) = \{x_2, x_3\}$*, whereas in the right-hand side*

$$\bigcap_{x_2=\omega'\gamma'} \alpha_{x_1}(\alpha_{x_1}(\omega')\gamma') = \alpha_{x_1}(\alpha_{x_1}(x_2)) \cap \alpha_{x_1}(x_1x_2) =$$

$$\alpha_{x_1}(\{x_2, x_3\}) \cap X = \{x_2, x_3, x_4\}.$$

Example 5.4. *Let* $\mathbb{Z}$ *be the ring of integer numbers ordered with the divisibility relation. For* $a \in \mathbb{Z}$ *and* $\omega \in \mathbb{Z}^*$*, consider* $\alpha_a(\omega)$ *as the set of prime numbers dividing both* a *and all the elements in* ω*. The finitary ND-coalgebra* $(\mathbb{Z}, \alpha)$ *is strongly secondary reflexive, commutative, separating and m-associative. Only the prime elements are self-conscious.*

Now, we have all the required properties defined in the finitary case. The rest of this section includes some useful technical results which arise as consequences of several combinations of properties.

Proposition 5.2. *Let* (A, α) *be an m-associative finitary ND-coalgebra whose elements are isolated. Then, for all* $a \in A$ *and* $\omega \in A^*$*,*

(i) $\alpha_a(\omega) \subseteq \bigcap_{\omega=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma')$*. In particular,* $\alpha_a(\omega) \subseteq \alpha_a(\alpha_a(\omega))$*.*

(ii) $\alpha_a(\omega) \subseteq \alpha_a(a\omega) = \alpha_a(a \cdots a\omega)$*.*

(iii) $\alpha_a(\omega a) \subseteq \alpha_a(\omega)$ *if* $\alpha_a(\omega) = \{a\}$*.*

PROOF:
(i) Consider $a \in A$ and $\omega \in A^*$. As $\alpha_a(a\varepsilon) = \alpha_a(a) = \{a\}$, by m-associativity

$$\alpha_a(\omega) \subseteq \bigcap_{\varepsilon\omega=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma') = \bigcap_{\omega=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma')$$

(ii) As $\alpha_a(aa) = \{a\}$, then $\alpha_a(\omega) \subseteq \bigcap_{a\omega=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma')$. Taking $\omega' = a$ and $\gamma' = \omega$, we deduce that

$$\alpha_a(\omega) \subseteq \alpha_a(\alpha_a(a)\omega) = \alpha_a(a\omega).$$

Consequently, we also have that $\alpha_a(a\omega) \subseteq \alpha_a(a\cdots a\omega)$. Now, applying $i)$, we have

$$\alpha_a(a\cdots a\omega) \subseteq \bigcap_{a\cdots a\omega=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma') \subseteq \alpha_a(\alpha_a(a\cdots a)\omega) = \alpha_a(a\omega).$$

(iii) Assume that $\alpha_a(\omega) = \{a\}$, then

$$\alpha_a(\omega\, a) \subseteq \bigcap_{\omega\, a=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma') \subseteq \alpha_a(\alpha_a(\omega)a) = \alpha_a(aa) = \{a\} = \alpha_a(\omega).$$

■

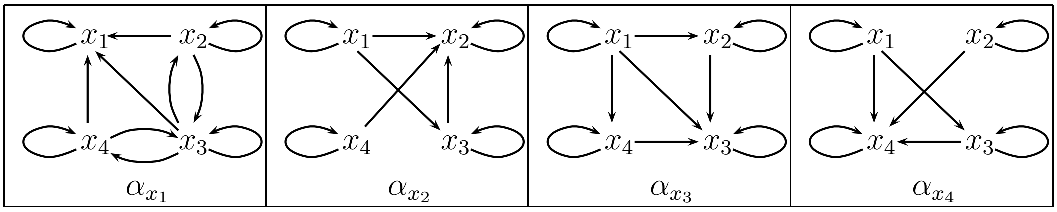
It is easy to see that on the additional assumption of commutativity to the hypotheses of the previous proposition, if $\alpha_a(\omega) = \{a\}$ then $\alpha_a(\omega) = \alpha_a(\omega\, a)$.

The following example shows that the inclusion in Proposition 5.2.(ii) need not be an equality in general. Moreover, even though the finitary ND-coalgebra is commutative, the aforementioned inclusion needn't be an equality.

Example 5.5. *Let X be the set $\{x_1, x_2, x_3, x_4\}$ and $\alpha : X \to \mathcal{P}(X)^{X^*}$ be as follows:*

$$\alpha_{x_i}(\varepsilon) = \{x_i\}; \quad \alpha_{x_i}(x_i\cdots x_i) = \{x_i\} \quad \textit{for all } i \in \{1,2,3,4\}$$

$\alpha_{x_i}(x_j)$ is described in the figure below for all $i, j \in \{1,2,3,4\}$



and $\alpha_{x_i}(\omega) = X$ otherwise. The finitary ND-coalgebra (X, α) is commutative, m-associative and all the elements are isolated. However,

$$\alpha_{x_1}(x_2) = \{x_1, x_2, x_3\} \subsetneq \alpha_{x_1}(x_1x_2) = X$$

Notice that (X, α) is not strongly m-associative because $\alpha_{x_1}(x_1) = \{x_1\}$ but setting $\gamma = x_2$, in the first place $\alpha_{x_1}(x_2) = \{x_1, x_2, x_3\}$ and in the second place

$$\bigcap_{x_2=\omega'\gamma'} \alpha_{x_1}(\alpha_{x_1}(\omega')\gamma') = \alpha_{x_1}(\alpha_{x_1}(x_2)) \cap \alpha_{x_1}(x_1x_2) =$$

$$= \alpha_{x_1}(\{x_1, x_2, x_3\}) \cap X = X$$

If the ND-coalgebra is commutative and also strongly m-associative, the equality always holds in Proposition 5.2. (ii). Moreover, under certain conditions, which will be specified in the theorem below, the previous equality characterizes strong m-associativity.

Theorem 5.3. *Let $\mathcal{A} = (A, \alpha)$ be a commutative and m-associative finitary ND-coalgebra whose elements are isolated. The following conditions are equivalent:*

i) $\alpha_a(\omega) = \alpha_a(a\omega)$, *for all* $a \in A, \omega \in A^*$.
ii) $\alpha_a(\gamma) = \alpha_a(\omega\gamma)$ *if* $\alpha_a(\omega) = \{a\}$, *for all* $a \in A, \omega, \gamma \in A^*$.
iii) $\mathcal{A}$ *is strongly m-associative.*

PROOF:

i) $\Rightarrow$ *ii)* If $\alpha_a(\omega) = \{a\}$ then $\alpha_a(\omega\gamma) \overset{\dagger}{\subseteq} \alpha_a(\alpha_a(\omega)\gamma) = \alpha_a(a\gamma) = \alpha_a(\gamma)$ where inclusion † is consequence of Proposition 5.2.(i).

On the other hand, as $\alpha_a(a\omega) = \{a\}$, then

$$\alpha_a(\gamma) \subseteq \bigcap_{\omega\gamma=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma') \subseteq \alpha_a(\alpha_a(\varepsilon)\omega\gamma) = \alpha_a(a\omega\gamma) = \alpha_a(\omega\gamma)$$

ii) $\Rightarrow$ *iii)* If $\alpha_a(b\omega) = \{a\}$,

$$\alpha_a(\gamma) \subseteq \bigcap_{\omega\gamma=\omega'\gamma'} \alpha_a(\alpha_b(\omega')\gamma') \subseteq \alpha_a(\alpha_b(\varepsilon)\omega\gamma) = \alpha_a(b\omega\gamma) = \alpha_a(\gamma).$$

which proves that $\mathcal{A}$ is strongly m-associative.

iii) $\Rightarrow$ *i)* As $\alpha_a(a) = \{a\}$ then $\alpha_a(a\omega) = \bigcap_{a\omega=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma')$.

Finally, from $\alpha_a(aa) = \{a\}$ we have that $\alpha_a(\omega) = \bigcap_{a\omega=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma')$ which completes the proof.

∎

Corollary 5.4. *Let (A, α) be a commutative, strongly m-associative finitary ND-coalgebra whose elements are isolated. Then*

i) *If $\alpha_a(x) = \{a\}$ for every $x \in \omega$, then $\alpha_a(\omega) = \{a\}$, for all $a \in A, \omega \in A^*$.*
ii) *Let $\omega \in A^*$ and $\hat{\omega}$ be any string obtained from ω deleting some elements repeated in the sequence. Then, $\alpha_a(\omega) = \alpha_a(\hat{\omega})$.*

Proof: The proof of item *i)* is straightforward, reasoning by induction on the length of the string ω and applying Theorem 5.3-*ii)*. Likewise, item *ii)* is also proved by using repeatedly Theorem 5.3-*i)* and commutativity.

∎

6. Finitary coalgebraic multisemilattices

When considering several properties simultaneously, such as strong secondary reflexivity, together with m-associativity, commutativity and self-conscious elements, a more general equality holds (Proposition 6.2 and Corollary 6.3) which resembles the idempotency and is closer to the binary version.

Definition 6.1. *A* finitary coalgebraic quasimultisemilattice *is a a finitary ND-coalgebra (A, α), satisfying the following properties*

a) *All the elements of $\mathcal{A}$ are self-conscious*
b) *$\mathcal{A}$ is commutative*
c) *$\mathcal{A}$ is strongly secondary reflexive*
d) *$\mathcal{A}$ is m-associative*

Notice that, in difference with the result in [3], the property "uncoupling" in the binary case is replaced here by "strongly secondary reflexive". The reason is that the first one is a property of a binary relation and makes no sense for finitary ND-coalgebras.

Proposition 6.2. *Let (A, α) be a finitary coalgebraic quasimultisemilattice. If $x \in \alpha_a(\omega)$ then $\{x\} = \alpha_a(\gamma x)$ for all $\gamma \subseteq \omega$, where the inclusion $\gamma \subseteq \omega$ between chains should be interpreted in set-theoretic terms, that is, $x \in \omega$ for all $x \in \gamma$.*

PROOF: Firstly, if $\gamma \subseteq \omega$ and the commutative law holds, we can iteratively apply Proposition 5.2.(ii) and obtain that $\alpha_a(\omega) \subseteq \alpha_a(\omega\gamma)$.

If $x \in \alpha_a(\omega)$, then $x \in \alpha_a(\omega a)$ and the strong secondary reflexivity ensures that $\alpha_a(\omega x) = \{x\}$. Hence,

$$\{x\} = \alpha_a(\omega x) \subseteq \alpha_a(\omega x \gamma) \overset{\dagger}{\subseteq} \alpha_a(\alpha_a(\omega x)\gamma) = \alpha_a(x\gamma)$$

where, in †, we use Proposition 5.2.(i). Now, by strong secondary reflexivity, again, we have $x \in \alpha_a(x\gamma)$ implies $\{x\} = \alpha_a(\gamma x)$.

■

A consequence of the previous proposition, also verified in the binary case (Lemma 48 [3]), can be stated here in terms of the idempotency of α_a.

Corollary 6.3. *Let (A, α) be a finitary coalgebraic quasimultisemilattice. Then for all $a \in A$ and $\omega \in A^*$,*

i) $\alpha_a(\alpha_a(\omega)) = \alpha_a(\omega)$.
ii) $\alpha_a(\omega) = \bigcap_{\omega=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma')$

PROOF: To prove item i), it suffices to take $\gamma = \varepsilon$ in Proposition 6.2 and apply Proposition 5.2.i). For item ii), note that

$$\alpha_a(\omega) \subseteq \bigcap_{\omega=\omega'\gamma'} \alpha_a(\alpha_a(\omega')\gamma') \subseteq \alpha_a(\alpha_a(\omega)) = \alpha_a(\omega)$$

■

Now we have all the pieces in order to obtain a coalgebraic characterization of finitary multisemilattices as a finitary ND-coalgebra satisfying certain properties similar to those of the Definition 32 in [3].

Definition 6.4. *A* finitary coalgebraic multisemilattice *is a finitary coalgebraic quasimultisemilattice $\mathcal{A} = (A, \alpha)$, such that $\mathcal{A}$ is separating.*

In a similar way as Proposition 6.2 provides an alternative definition of the strong secondary reflexivity, next lemma provides a more suitable use of separation which is reminiscent of the binary case.

Lemma 6.5. *For a finitary coalgebraic multisemilattice $\mathcal{A} = (A, \alpha)$, the following condition holds:*

$$\textit{if } \ x, y \in \alpha_a(\omega) \ \textit{and } \ x \in \alpha_x(y) \ \textit{then } \ x = y \ \textit{for all } \ a, x, y \in A, \ \omega \in A^*$$

Proof: First, $\alpha_a(\omega) \subseteq \alpha_a(\omega\, a) = \alpha_a(\omega\, aa)$, by Proposition 5.2.$(ii)$ and commutativity. Thereby, $x, y \in \alpha_a(\omega\, aa)$. Likewise, as $\mathcal{A}$ is strongly secondary reflexive, $x \in \alpha_a(\omega\, a)$ implies $\{x\} = \alpha_a(\omega\, x) = \alpha_x(a\omega)$. Applying now the m-associative property, we have that

$$x \in \alpha_x(y) \subseteq \bigcap_{\omega y = \omega'\gamma'} \alpha_x(\alpha_a(\omega')\gamma') \subseteq \alpha_x(\alpha_a(\varepsilon)\omega\, y) = \alpha_x(a\omega\, y) = \alpha_a(\omega\, xy)$$

Therefore, the conditions required in Definition 4.5 are satisfied and hence $x = y$.

∎

As mentioned previously, under certain ambient hypothesis different properties can collapse to the same thing. For instance, in a strongly secondary reflexive finitary ND-coalgebra, an element is self-conscious if and only if it is isolated (Proposition 4.4). Concerning finitary coalgebraic multisemilattices, we can state the following proposition.

Proposition 6.6. *Strongly m-associative holds in any finitary coalgebraic multisemilattice.*

Proof: Theorem 5.3 enables one just to check the equality $\alpha_a(\omega) = \alpha_a(a\omega)$ for all $a \in A$, $\omega \in A^*$. Recall that $\alpha_a(\omega) \subseteq \alpha_a(a\omega)$ according to

Proposition 5.2. Given $x \in \alpha_a(a\omega)$, since $\alpha_a(a\omega) = \alpha_a(\omega a) \subseteq \alpha_a(\alpha_a(\omega)a)$, there exists $y \in \alpha_a(\omega)$ (and so $y \in \alpha_a(a\omega)$) such that $x \in \alpha_a(ya) = \alpha_y(aa)$. By Proposition 6.2, we have that $\alpha_y(x) = \{x\}$. Finally, Lemma 6.5 guarantees $x = y$ and, therefore, $x \in \alpha_a(\omega)$.

■

We already stated the conditions required to link finitary coalgebraic multisemilattices and ordered sets.

Proposition 6.7. *Let (A, α) be a commutative, m-associative finitary ND-coalgebra whose elements are isolated. Then $(A, \leq)$ is a poset with the binary relation defined by $x \leq y$ iff $\alpha_y(x) = \{y\}$.*

If (A, α) is also strongly m-associative and strongly secondary reflexive then, $\text{UBOUNDS}(a\omega) \subseteq \alpha_a(\omega)\uparrow$*, for all $a \in A$, $\omega \in A^*$.*

PROOF: Reflexivity and antisymmetry are directly deduced from commutativity and the fact that all the elements are isolated. Assume now that $\alpha_y(x) = \{y\}$ and $\alpha_z(y) = \{z\}$. Then, $\alpha_z(x) \subseteq \alpha_z(\alpha_y(x)) = \{z\}$. As $\alpha_z(x)$ is nonempty because $\{z\} = \alpha_y(z) \subseteq \alpha_y(\alpha_x(z))$, thus $\alpha_z(x) = \{z\}$, which proves transitivity.

Assume that (A, α) is also strongly m-associative and strongly secondary reflexive. Let z be an upper bound of $a\omega$. Then, $\alpha_z(a) = \{z\}$ and also $\alpha_z(x) = \{z\}$ for all $x \in \omega$. By Corollary 5.4.i), we have that $\alpha_z(a\omega) = \{z\}$ which allows us to apply m-associativity and obtain $\{z\} = \alpha_z(\varepsilon) \subseteq \bigcap_{\omega=\omega'\gamma'} \alpha_z(\alpha_a(\omega')\gamma') \subseteq \alpha_z(\alpha_a(\omega))$. So, there exists $y \in \alpha_a(\omega)$ such that $z \in \alpha_z(y)$ which implies that $\{z\} = \alpha_z(y)$, that is $z \geq y$. Hence, $z \in \alpha_a(\omega)\uparrow$.

■

Proposition 6.8. *Given a poset $(A, \leq)$, the finitary ND-coalgebra (A, α) defined by setting $\alpha_a(\omega) = \text{MSUP}(a\omega)$, for all $a \in A$, $\omega \in A^*$, verifies:*

i) *All the elements of $\mathcal{A}$ are self-conscious.*
ii) *$\mathcal{A}$ is commutative.*
iii) *$\mathcal{A}$ is strongly secondary reflexive.*
iv) *$\mathcal{A}$ is separating.*

Furthermore, $(A, \leq)$ is a finitary join-multisemilattice if and only if (A, α) is m-associative.

PROOF: Properties $i) - iv)$ are directly obtained from the definition. Assume now that $(A, \leq)$ is a finitary join-multisemilattice and $\{a\} = \alpha_a(b\omega) = \text{MSUP}(ab\,\omega)$. Given $z \in \alpha_a(\gamma)$ and ω', γ' such that $\omega\gamma = \omega'\gamma'$, it is clear that z is an upper bound of $b\omega'$. Thus, there exists $t \in \text{MSUP}(b\omega')$ such that $t \leq z$. Observe that $z \in \text{MSUP}(a\,t\gamma') = \alpha_a(t\gamma')$ because every upper bound of $a\,t\gamma'$ is also an upper bound of $a\gamma$. To prove the converse, it is sufficient to apply Proposition 6.6 and 6.7.

■

Theorem 6.9.

i) *If $(A, \leq)$ is a finitary join-multisemilattice, then the finitary ND-coalgebra (A, α) defined by setting $\alpha_a(\omega) = \text{MSUP}(a\omega)$, for all $a \in A$, $\omega \in A^*$, is a finitary coalgebraic multisemilattice.*

ii) *If (A, α) is a finitary coalgebraic multisemilattice, then $(A, \leq)$ where $x \leq y \Leftrightarrow \alpha_y(x) = \{y\}$ is a finitary join-multisemilattice such that $\alpha_a(\omega) = \text{MSUP}(a\omega)$, for all $a \in A$, $\omega \in A^*$.*

PROOF:

$i)$ This item is deduced by applying Proposition 6.8.

$ii)$ Due to Proposition 6.8, it suffices to prove that $\alpha_a(\omega) = \text{MSUP}(a\omega)$, for all $a \in A$, $\omega \in A^*$. On the one hand, we have that $\alpha_a(\omega) \subseteq \text{UBOUNDS}(a\omega)$ by Proposition 6.2. Besides, $\text{UBOUNDS}(a\omega) \subseteq \alpha_a(\omega)\uparrow$ by Proposition 6.7, whence $\alpha_a(\omega)\uparrow = \text{UBOUNDS}(a\omega)$. Lemma 6.5 establishes that $\alpha_a(\omega)$ is an antichain and therefore $\alpha_a(\omega) = \text{MINIMAL}(\text{UBOUNDS}(a\omega)) = \text{MSUP}(a\omega)$.

■

Remark 6.10. *The dual result for a finitary meet-multisemilattice (A, β) is true as well, considering $x \leq y$ if and only if $\beta_x(y) = \{x\}$.*

7. Finitary coalgebraic multilattices

In the same way as a lattice can be constructed from two different semilattices which are conveniently assembled, our purpose in this section is to define a finitary multilattice as a coalgebra which can be separated as two finitary coalgebraic multisemilattices and next analyze how they both can be properly connected.

Definition 7.1. *A* doubly finitary ND-coalgebra *is an ND-coalgebra of type* $\prod_{n\in\mathbb{N}^+\oplus\mathbb{N}^+} \mathcal{T}_n$ *where* $\oplus$ *denotes the disjoint union.*[6]

Thus, a double finitary ND-coalgebra is a pair (A, γ) with

$$\gamma : A \longrightarrow \prod_{\theta\in\mathbb{N}^+\oplus\mathbb{N}^+} \mathcal{T}_{\nu(\theta)}(A)$$

being $\nu : \mathbb{N}^+ \oplus \mathbb{N}^+ \to \mathbb{N}$ the mapping that assigns $(n, 0)$ and $(n, 1)$ to n.

By considering the isomorphism

$$\prod_{\theta\in\mathbb{N}^+\oplus\mathbb{N}^+} \mathcal{T}_{\nu(\theta)}(A) \simeq \left(\prod_{n\in\mathbb{N}^+} \mathcal{T}_n(A)\right) \times \left(\prod_{n\in\mathbb{N}^+} \mathcal{T}_n(A)\right)$$

for every $a \in A$, the element γ_a can be considered as a pair $\gamma_a = (\alpha_a, \beta_a)$, where $\alpha_a = (\alpha_a^n)_{n\in\mathbb{N}^+}$ and $\beta_a = (\beta_a^n)_{n\in\mathbb{N}^+}$.

This allows us to separate both components as a pair of finitary *ND*-coalgebras, namely (A, α) and (A, β). Henceforth, given a doubly finitary ND-coalgebra (A, γ), we will write $\gamma = (\alpha, \beta)$.

Theorem 7.2. *Let* $\mathcal{A} = (A, \gamma)$ *be a doubly finitary ND-coalgebra, where* $\gamma = (\alpha, \beta)$*, such that*

a) *both* (A, α) *and* (A, β) *are finitary coalgebraic multisemilattices and*
b) $\alpha_y(x) = \{y\}$ *if and only if* $\beta_x(y) = \{x\}$*, for all* $x, y \in A$. (duality)

Then, A is a finitary multilattice under the ordering relation

$$x \le y \quad \Leftrightarrow \quad \alpha_y(x) = \{y\} \quad \textit{where}$$

$$\alpha_a(\omega) = \textrm{MSUP}(a\omega) \textit{ and } \beta_a(\omega) = \textrm{MINF}(a\omega) \quad \textit{for all } a \in A, \omega \in A^*$$

Proof: It is immediate fromTheorem 6.9 and Remark 6.10.

■

[6] That is, $\mathbb{N}^+ \oplus \mathbb{N}^+ = \{(n, 0) \mid n \in \mathbb{N}^+\} \cup \{(n, 1) \mid n \in \mathbb{N}^+\}$.

Definition 7.3. *Two finitary ND-coalgebras, (A, α) and (A, β), are said to be* assembled *if, for all $a \in A$ and $\omega \in A^*$, the following conditions hold:*

a) *If $\beta_a(\omega) \neq \varnothing$ then $\alpha_a(\beta_a(\omega)) = \{a\}$.*
b) *If $\alpha_a(\omega) \neq \varnothing$ then $\beta_a(\alpha_a(\omega)) = \{a\}$.*

A doubly finitary ND-coalgebra (A, γ), where $\gamma = (\alpha, \beta)$, satisfies the assembly *property if (A, α) and (A, β) are assembled.*

Remark 7.4. *Let (A, α) and (A, β) be two commutative binary ND-coalgebras. If they are assembled, clearly the duality condition holds. That is,*

$$\alpha_y(x) = \{y\} \Leftrightarrow \beta_x(y) = \{x\} \quad \textit{for all } x, y \in A$$

Obviously, the converse is not true, in general, because the assembly property refers to arbitrary strings whereas duality condition involves just the 2-factors of the finitary ND-coalgebras.

Definition 7.5. *A doubly finitary ND-coalgebra (A, γ), where $\gamma = (\alpha, \beta)$, satisfies the* absorption *property if, for all $a \in A$, the following conditions hold:*

a) *If $z \in \beta_a(A^*)$ then $\alpha_a(z) = \{a\}$.*
b) *If $z \in \alpha_a(A^*)$ then $\beta_a(z) = \{a\}$.*

By the assembly property, for every string ω the following equality holds

$$\alpha_a(\beta_a(\omega)) = \bigcup_{x \in \beta_a(\omega)} \alpha_a(x) = \{a\}$$

Thus, $\alpha_a(z) \subseteq \{a\}$, for all $z \in \beta_a(A^*)$. Notice that since $\alpha_a(z)$ can be empty, it does not necessarily coincide with $\{a\}$.

Proposition 7.6. *Let (A, γ) be a doubly finitary ND-coalgebra, where $\gamma = (\alpha, \beta)$ such that (A, α) and (A, β) are commutative, strongly m-associative and all their elements are isolated. Then, the following conditions are equivalent:*

i) *(A, γ) satisfies the absorption property.*
ii) *(A, α) and (A, β) are strongly secondary reflexive and assembled.*

iii) (A,α) *and* (A,β) *are strongly secondary reflexive and the duality condition holds.*

PROOF: *i)* $\Rightarrow$ *ii)* Firstly, if $x \in \alpha_a(\varepsilon)$, as x is isolated, $\alpha_x(\varepsilon) = \{x\}$. If $x \in \alpha_a(\omega\, b)$, we have $\beta_a(x) = \{a\}$ and also $\beta_u(x) = \{u\}$ for all $u \in \omega$, whence $\alpha_x(a) = \{x\}$ and $\alpha_x(u) = \{x\}$. Applying Corollary 5.4 *i)*, one obtains $\alpha_x(a\omega) = \{x\}$ which is equivalent to $\alpha_a(\omega\, x) = \{x\}$ by commutativity. *ii)* $\Rightarrow$ *iii)* It is straightforward. *iii)* $\Rightarrow$ *i)* If $z \in \beta_a(\omega)$, by Proposition 6.2 we have that $\beta_a(z) = \{z\}$. Duality condition guarantees that $\alpha_a(z) = \{a\}$. Analogously, the other condition of absorption property is proved.

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Definition 7.7. *A doubly finitary ND-coalgebra,* $\mathcal{A} = (A,\gamma)$ *where* $\gamma = (\alpha,\beta)$*, is said to be a* finitary coalgebraic multilattice *if*

a) *both* (A,α) *and* (A,β) *are finitary coalgebraic multisemilattices and*
b) (A,γ) *satisfies the assembly property (absorption property or duality condition).*

Theorem 7.8. *There exists a one-to-one correspondence between finitary coalgebraic multilattices and finitary ordered multilattices.*

PROOF:

As Theorem 7.2 states, given (A,γ) a finitary coalgebraic multilattice, where $\gamma = (\alpha,\beta)$, an ordering relation can be defined by $x \le y \Leftrightarrow \alpha_y(x) = \{y\} \Leftrightarrow \beta_x(y) = \{x\}$, for all $x, y \in A$ such that $(A,\le)$ is a finitary ordered multilattice where $\alpha_a(\omega) = \mathrm{MSUP}(a\omega)$ and $\beta_a(\omega) = \mathrm{MINF}(a\omega)$ for all $a \in A$, $\omega \in A^*$.

Conversely, given $(A,\le)$ an ordered finitary multilattice, by Theorem 6.9 and Remark 6.10, one can obtain two finitary coalgebraic multisemilattices (A,α) and (A,β) defining $\alpha_a(\omega) = \mathrm{MSUP}(a\omega)$ and $\beta_a(\omega) = \mathrm{MINF}(a\omega)$ for all $a \in A$, $\omega \in A^*$ in such a way the duality condition trivially holds. Then, (A,γ) where $\gamma_a = (\alpha_a,\beta_a)$, for all $a \in A$, is a finitary coalgebraic multilattice.

∎

8. Conclusions

In previous works, we introduced a coalgebraic characterization for multisemilattices and multilattices which involved subsets of cardinality 2. In doing so, we defined a collection of coalgebras related to non-determinism, namely, the ND-coalgebras. The properties which specify multisemilattices as a particular type of binary ND-coalgebras were determined precisely. Since the generalization to finite subsets with any cardinality does not produce the same structure, we have provided the coalgebraic characterization for the so-called finitary multisemilattices and multilattices. Initially, they are introduced as a class of ND-coalgebras which can be recursively defined from the binary case. However, a more elegant development is obtained in terms of non-deterministic operators with flexible arity. Our next aim is to find both algebraic and coalgebraic definitions for complete multilattices.

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