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Linking L-Chu correspondences and completely lattice L-ordered sets

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Abstract Continuing our categorical study of L-fuzzy extensions of formal concept analysis, we provide a representation theorem for the category of L-Chu correspondences between L-formal contexts and prove that it is equivalent to the category of completely lattice L-ordered sets.

Keywords Concept lattice, Category theory, Adjunction, Galois connection, Fuzzy logic, Equivalence functor

1 Introduction

Category theory has become important in many areas of modern mathematics (either as a research area *per se* or as a tool for doing mathematics) and computer science (as a means to unifying several approaches of abstract machines, or type theories, etc), although its use in other areas of computer science tend to find resistance, due to the reluctancy to admit high levels of abstraction; on the other hand, Formal Concept Analysis (FCA) has become an extremely useful theoretical and practical tool for formally describing structural and hierarchical properties of data with "object-attribute" character, and this applicability justifies the need of a deeper knowledge of its underlying mechanisms: and one important way to obtain this extra knowledge turns out to be via generalization and abstraction.

Goguen argues in [22] that research on *concepts* should be thoroughly interdisciplinary, and in particular, should transcend the boundaries between sciences and humanities. One of the tools that he proposes is precisely category theory as a unifying language capable of merging different apparently disparate approaches. Not trying to reach such an ambitious goal (at least on the short/mid term), this paper continues

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previous work of the authors of a categorical study of FCA, and deals with an extremely general form of L-fuzzy FCA, based on categorical constructs and L-fuzzy sets.

The introduction of generalization and abstraction in FCA, as in many other research areas, may lead to new theoretical and applied results. For instance, concerning the use of fuzzy (or, in same cases, L-fuzzy) FCA, one can see papers ranging from resolution of fuzzy relational equations [2] and ontology merging [12], to applications to the Semantic Web by using the notion of concept similarity or rough sets [17, 18], and from noise control in document classification [32] to the development of recommender systems [15], or the study of fuzzy databases, in areas such as functional dependencies [37] or data mining in terms of closure systems [10].

Theoretically, several approaches have been presented for generalizing the framework and the scope of FCA and, nowadays, one can see works which extend it by using ideas from fuzzy set theory [1, 3], rough set theory [30, 31, 42], the multi-adjoint framework [33, 34, 36], or possibility theory [16], or heterogeneous approaches in which concept lattices are based on Galois connections allowing to analyse object-attribute models with different structures for truth values of attributes [11, 35].

The use of category theory to study of ideas related to FCA has proliferated in the recent years; for instance, the Information Flow Framework [25] provides a framework for ontology development making it possible to communicate between categorical and FCA formalisms, or the study of concept structures done by Hitzler *et al* [23, 24] applying categorical methods to define the notion of approximable structure (borrowed from the field of denotational semantics), or the categorical study of fuzzy Galois connections [20] allowing for presenting its theory in a more succinct way, and providing a useful method to study the links between the commutative and the non-commutative worlds, or a more abstract study of the concept lattice functors [40] including the relationship between contexts, closure spaces, and complete lattices, or the categorical view of generalized concept lattices [26].

The categorical treatment of morphisms as fundamental structural properties has been advocated by [29] as a means for the modelling of data translation, communication, and distributed computing, among other applications. Our approach broadly focuses on the research line which links the theory of Chu spaces with concept lattices [44, 45]; in the latter, it is shown that the notion of state in Scott's information system corresponds precisely to that of formal concepts in FCA with respect to all finite Chu spaces, and the entailment relation corresponds to *association rules* (another link between FCA with database theory) and, specifically, on the identification of the categories associated to certain constructions.

Our approach is particularly based on the notion of Chu correspondences between formal contexts, developed by Mori, which we briefly sketch below:

In [38], the author focused on the great number of structures having certain duality which can be formalized in terms of a formal context (B, A, r). Homomorphisms between two of these structures, with contexts (B_i, A_i, r_i) for $i \in \{1, 2\}$, induce Chu mappings, that is pairs of mappings $\varphi\colon B_1 \to B_2$ and $\psi\colon A_2 \to A_1$ such that $r_2(\varphi(b_1), a_2) = r_1(b_1, \psi(a_2))$; note the similarity with the adjoint property of isotone Galois connections. A functor, the Galois functor, can be naturally defined from the category of Chu mappings and the category of join-preserving mappings between complete lattices; unfortunately, this functor is neither full (surjective) nor faithful (injective). The main contribution of [38] was the introduction of the notion of Chu correspondence and proving, on the one hand, the fullness and faithfulness of the Galois functor and, on the other hand, the *-autonomous structure of the category of Chu correspondences.

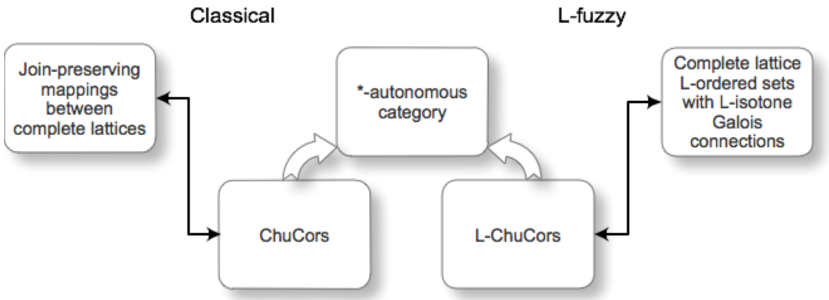

Fig. 1 Classical ChuCors vs L-(fuzzy)ChuCors.

Previous work in this categorical approach has been already developed by the authors. In [28], the notion of L-Chu correspondence between L-contexts was introduced; in addition, the resulting set of L-Chu correspondences was shown to be a complete lattice anti-isomorphic to that of L-bonds between formal contexts. More recently [27], the authors started the categorical study of L-contexts and their morphisms by introducing the category L-ChuCors, having L-contexts as objects and L-Chu correspondences as morphisms, providing a further abstraction with the aim of formally describing structural properties of intercontextual relationships. In addition, in that paper it was proved that the resulting category is *-autonomous and, therefore, its underlying logic is classical linear logic [5, 39].

In the present work, we continue our study of L-ChuCors, seen as a common categorical umbrella for several fuzzy extensions of the classical notion concept lattice, initiated mainly by Bělohlávek [6–9], who extended the underlying interpretation on classical logic to the more general framework of L-fuzzy logic [21].

Pictorially, we can represent the contribution of this work as the right arrow in Figure 1, which somehow closes the initial study of L-ChuCors, in that we already have completed the picture of the behavior of Chu correspondences in an L-fuzzy environment. Specifically, the main result in this work is a constructive proof of the equivalence between the category L-ChuCors and a category of completely lattice L-ordered sets (L-CLOS[1]) with isotone Galois connections between them. This result, on the one hand, reinforces the notion of L-CLOS as the most adequate fuzzy version of complete lattice, since in the crisp case the equivalence is with the category of join-preserving maps between complete lattices, on the other hand, paves the way for future work on finding further connections following the thread of L-CLOS; an interesting possibility might be studying the topic of approximable concepts, because of the existing relationship between them and L-CLOS [13].

In order to obtain a reasonably self-contained document, Section 2 introduces the basic definitions concerning the L-fuzzy extension of formal concept analysis, as well as those concerning L-Chu correspondences; then, the categories associated to L-formal contexts and L-CLOS are defined in Section 3 and, finally, the proof of equivalence is in Section 4.

[1] Although the proper acronym should be CLLOS, we prefer to use the prefix L in the acronym to better reflect that we are working on an L-fuzzy extension.

2 Preliminaries

2.1 Basics of L-fuzzy FCA

In this section we introduce the preliminary definitions of L-fuzzy FCA and the theory of completely L-lattice ordered sets. In this respect, we are assuming the same motivations used in [8, 9].

Definition 1 An algebra $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ is said to be a **complete residuated lattice** if

- $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1,
- $\langle L, \otimes, 1\rangle$ is a commutative monoid,
- $\otimes$ and $\rightarrow$ are adjoint, i.e. $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$, for all $a, b, c \in L$, where $\leq$ is the ordering in the lattice generated from $\wedge$ and $\vee$.

For a good overview of the theory of complete residuated lattices, the reader is referred to [19].

Definition 2 Let L be a complete residuated lattice, an **L-fuzzy context** is a triple $\langle B, A, r\rangle$ consisting of a set of objects B, a set of attributes A and an L-fuzzy binary relation r, i.e. a mapping $r\colon B \times A \longrightarrow L$, which can be alternatively understood as an L-fuzzy subset of $B \times A$. The set of all L-sets of objects from B will be denoted by L^B, and similarly for any base set.

Definition 3 Consider an L-fuzzy context $\langle B, A, r\rangle$. Mappings $\uparrow\colon L^B \longrightarrow L^A$ and $\downarrow\colon L^A \longrightarrow L^B$ can be defined for every $f \in L^B$ and $g \in L^A$ as follows:

$$\uparrow (f)(a) = \bigwedge_{o\in B} \big(f(o) \rightarrow r(o, a)\big) \qquad \downarrow (g)(o) = \bigwedge_{a\in A} \big(g(a) \rightarrow r(o, a)\big) \qquad (1)$$

Definition 4 An **L-fuzzy concept** is a pair $\langle f, g\rangle$ such that $\uparrow (f) = g$ and $\downarrow (g) = f$. The first component f is said to be the **extent** of the concept, whereas the second component g is the **intent** of the concept.

The set of all L-fuzzy concepts associated to a fuzzy context $\langle B, A, r\rangle$ will be denoted as L-FCL(B, A, r).

An ordering between L-fuzzy concepts is defined as follows: $\langle f_1, g_1\rangle \leq \langle f_2, g_2\rangle$ if and only if $f_1 \subseteq f_2$ (namely, $f_1(o) \leq f_2(o)$ for all $o \in B$) if and only if $g_1 \supseteq g_2$ (that is, $g_1(a) \geq g_2(a)$ for all $a \in A$).

Example 1 Consider two L-contexts C_1 and C_2, where $L = \{1, 0.5, 0\}$. Any value from L in any cell of the following tables represents a relationship of the corresponding object-attribute pair. It is a formalization of the information about the degree that an object has some attribute or, conversely, how any attribute is shared by some object.

C_1	a_{11}	a_{12}	a_{13}	a_{14}
o_{11}	1	1	0.5	0
o_{12}	1	0.5	1	0.5

C_2	a_{21}	a_{22}	a_{23}
o_{21}	1	1	1
o_{22}	0.5	1	1
o_{23}	0	1	0.5

If we consider Łukasiewicz logic connectives $\langle\otimes,\rightarrow\rangle$, defined as

$$k \otimes m = \max\{0, k+m-1\} \qquad k \rightarrow m = \min\{1, 1-k+m\}$$

and with derivation operators $\langle\uparrow,\downarrow\rangle$ defined in (1) above we obtain the set of all L-concepts of C_1 and C_2 that are shown in the following tables. Each row represents one L-concept. For the sake of readability, we will denote L-concepts of C_1 as $p_1, \ldots, p_5$ and L-concepts of C_2 as q_1, q_2, q_3.

L-FCL(C_1)	o_{11}	o_{12}	a_{11}	a_{12}	a_{13}	a_{14}
p_1	1	1	1	0.5	0.5	0
p_2	1	0.5	1	1	0.5	0
p_3	0.5	1	1	0.5	1	0.5
p_4	0.5	0.5	1	1	1	0.5
p_5	0	0.5	1	1	1	1

L-FCL(C_2)	o_{21}	o_{22}	o_{23}	a_{21}	a_{22}	a_{23}
q_1	1	1	1	0	1	0.5
q_2	1	1	0.5	0.5	1	1
q_3	1	0.5	0	1	1	1

There are four extremal L-concepts, namely, p_1, p_5, q_1 and q_3 formalizing the situation in which the (crisp) set of all objects has the whole set corresponding L-set of common attributes or, vice versa, that all attributes are shared by the corresponding L-set of objects. There are also examples of L-concepts covering the information about some L-set of objects satisfying an L-set of common attributes, for instance p_2. Finally, concept p_4 has objects o_{11} and o_{12} both with membership degree 0.5 which share the common attributes a_{11}, a_{12} and a_{13}, attribute a_{14} is common attribute with membership degree 0.5. □

Bělohlávek has extended the fundamental theorem of concept lattices by Dedekind-MacNeille completion in fuzzy settings by using the notions of L-equality and L-ordering. All the definitions and related constructions given until the end of this section are from [9].

Definition 5 A binary L-relation $\approx$ on X is called an ***L*-equality** if it satisfies

1. $(x \approx x) = 1$, (reflexivity),
2. $(x \approx y) = (y \approx x)$, (symmetry),
3. $(x \approx y) \otimes (y \approx z) \leq (x \approx z)$, (transitivity),
4. $(x \approx y) = 1$ implies $x = y$

L-equality is a natural generalization of the classical (bivalent) notion.

Definition 6 An ***L*-ordering** (or fuzzy ordering) on a set X endowed with an L-equality relation $\approx$ is a binary L-relation $\preceq$ which is compatible w.r.t. $\approx$ (i.e. $f(x) \otimes (x \approx y) \leq f(y)$, for all $x, y \in X$) and satisfies

1. $x \preceq x = 1$, (reflexivity),
2. $(x \preceq y) \wedge (y \preceq x) \leq (x \approx y)$, (antisymmetry),
3. $(x \preceq y) \otimes (y \preceq z) \leq (x \preceq z)$, (transitivity).

If $\preceq$ is an L-order on a set X with an L-equality $\approx$, we call the pair $\langle\langle X, \approx\rangle \preceq\rangle$ an L-ordered set.

Clearly, if $L = 2$, the notion of L-order coincides with the usual notion of (partial) order.

Definition 7 An L-set $f \in L^X$ is said to be an L**-singleton** in $\langle X, \approx\rangle$ if it is compatible w.r.t. $\approx$ and the following holds:

1. There exists $x_0 \in X$ with $f(x_0) = 1$
2. $f(x) \otimes f(y) \le (x \approx y)$, for all $x, y \in X$.

Definition 8 For an L-ordered set $\langle\langle X, \approx\rangle \preceq\rangle$ and $f \in L^X$ the L-sets $\inf(f)$ and $\sup(f)$ in X are defined by

1. $\inf(f)(x) = (\mathcal{L}(f))(x) \wedge (\mathcal{UL}(f))(x)$
2. $\sup(f)(x) = (\mathcal{U}(f))(x) \wedge (\mathcal{LU}(f))(x)$

where

$$\mathcal{L}(f)(x) = \bigwedge_{y\in X} \big(f(y) \to (x \preceq y)\big) \qquad \text{and} \qquad \mathcal{U}(f)(x) = \bigwedge_{y\in X} \big(f(y) \to (y \preceq x)\big)$$

The L-sets $\inf(f)$ and $\sup(f)$ are called **infimum** and **supremum**, respectively.

Definition 9 An L-ordered set $\langle\langle X, \approx\rangle \preceq\rangle$ is said to be **completely lattice** L**-ordered set** if for any $f \in L^X$ both $\sup(f)$ and $\inf(f)$ are $\approx$-singletons.

In the proof of the following chain of lemmas some well-known properties of residuated lattices are used (details can be found in [8]). Some of the needed properties are listed below.

$$(k \to (l \to m)) = ((k \otimes l) \to m) = ((l \otimes k) \to m) = (l \to (k \to m)) \tag{2}$$

$$k \to \bigwedge_{i\in I} m_i = \bigwedge_{i\in I} (k \to m_i) \tag{3}$$

$$(\bigvee_{i\in I} m_i) \to k = \bigwedge_{i\in I} (m_i \to k) \tag{4}$$

Lemma 1 *For any pair of L-concepts $\langle f_i, g_i\rangle \in L\text{-FCL}(B, A, r)$ $(i \in \{1, 2\})$ of any L-context $\langle B, A, r\rangle$ the following equality holds.*

$$\bigwedge_{o\in B} \big(f_1(o) \to f_2(o)\big) = \bigwedge_{a\in A} \big(g_2(a) \to g_1(a)\big)$$

Proof

$$\begin{aligned}
\bigwedge_{o\in B} (f_1(o) \to f_2(o)) &= \bigwedge_{o\in B} (f_1(o) \to \downarrow(g_2)(o)) \\
&\overset{(1)}{=} \bigwedge_{o\in B} \left(f_1(o) \to \bigwedge_{a\in A} (g_2(a) \to r(o, a)) \right) \\
&\overset{(3)}{=} \bigwedge_{o\in B} \bigwedge_{a\in A} (f_1(o) \to (g_2(a) \to r(o, a)))
\end{aligned}$$

$$\stackrel{(2)}{=} \bigwedge_{o\in B} \bigwedge_{a\in A} (g_2(a) \to (f_1(o) \to r(o,a)))$$

$$\stackrel{(3)}{=} \bigwedge_{a\in A} \left(g_2(a) \to \bigwedge_{o\in B} (f_1(o) \to r(o,a)) \right)$$

$$\stackrel{(1)}{=} \bigwedge_{a\in A} (g_2(a) \to \uparrow(f_1)(a))$$

$$= \bigwedge_{a\in A} (g_2(a) \to g_1(a)) \qquad \square$$

Definition 10 L-equality $\approx$ and L-ordering $\preceq$ on the set of formal concepts L-FCL(C) of L-context C are defined as follows:

$$\langle f_1, g_1\rangle \preceq \langle f_2, g_2\rangle = \bigwedge_{o\in B} \big(f_1(o) \to f_2(o)\big) = \bigwedge_{a\in A} \big(g_2(a) \to g_1(a)\big) \tag{5}$$

$$\langle f_1, g_1\rangle \approx \langle f_2, g_2\rangle = \bigwedge_{o\in B} \big(f_1(o) \leftrightarrow f_2(o)\big) = \bigwedge_{a\in A} \big(g_2(a) \leftrightarrow g_1(a)\big) \tag{6}$$

where $k \leftrightarrow m$ is defined as $(k \to m) \wedge (m \to k)$ for any $k, m \in L$.

Example 2 In the following tables we can see the L-ordering on the sets of L-concepts of C_1 and C_2 from Example 1.

$\preceq_1$	p_1	p_2	p_3	p_4	p_5
p_1	1	0.5	0.5	0.5	0
p_2	1	1	0.5	0.5	0
p_3	1	0.5	1	0.5	0
p_4	1	1	1	1	0.5
p_5	1	1	1	1	1

$\preceq_2$	q_1	q_2	q_3
q_1	1	0.5	0
q_2	1	1	0.5
q_3	1	1	1

Definition 11 Let $C = \langle B, A, r\rangle$ be an L-fuzzy formal context and γ be an L-set from $L^{L\text{-FCL}(C)}$. L-sets of objects and attributes $\bigcup_B \gamma$ and $\bigcup_A \gamma$ are defined as follows:

1. $(\bigcup_B \gamma)(o) = \bigvee_{\langle f,g\rangle \in L\text{-FCL}(C)} (\gamma(\langle f,g\rangle) \otimes f(o))$, for $o \in B$
2. $(\bigcup_A \gamma)(a) = \bigvee_{\langle f,g\rangle \in L\text{-FCL}(C)} (\gamma(\langle f,g\rangle) \otimes g(a))$, for $a \in A$

Theorem 1 *Let $C = \langle B, A, r\rangle$ be an L-context. $\langle\langle L\text{-FCL}(C), \approx\rangle, \preceq\rangle$ is a completely lattice L-ordered set in which infima and suprema can be described as follows: for an L-set $\gamma \in L^{L\text{-FCL}(C)}$ we have:*

$$^1\inf(\gamma) = \Big\{\langle \downarrow(\bigcup_A \gamma), \uparrow\downarrow(\bigcup_A \gamma)\rangle\Big\} \qquad ^1\sup(\gamma) = \Big\{\langle \downarrow\uparrow(\bigcup_B \gamma), \uparrow(\bigcup_B \gamma)\rangle\Big\}$$

Moreover a completely lattice L-ordered set $\mathcal{V} = \langle\langle V, \approx\rangle, \preceq\rangle$ is said to be isomorphic to $\langle\langle L\text{-FCL}(\langle B, A, r\rangle), \approx_1\rangle, \preceq_1\rangle$ iff there are mappings $\gamma : B \times L \longrightarrow V$ and $\mu : A \times L \longrightarrow V$, such that $\gamma(B \times L)$ is $\{0,1\}$-supremum dense and $\mu(A \times L)$ is $\{0,1\}$-infimum dense in $\mathcal{V}$, and $((k \otimes l) \to r(o,a)) = (\gamma(o,k) \preceq \mu(a,l))$ for all $o \in B$, $a \in A$ and $k, l \in L$. In particular, $\mathcal{V}$ is isomorphic to $\langle\langle L\text{-FCL}(V, V, \preceq), \approx_1\rangle, \preceq_1\rangle$.

If $L = 2$, the previous theorem coincides with the standard version of the fundamental theorem of concept lattices.

2.2 L-Isotone Galois connection

Now an L-fuzzy extension of the notion of isotone Galois connection will be introduced. Firstly we will define an L-isotone mapping between two L-ordered sets.

Definition 12 Let $\langle\langle V_i, \approx_i\rangle, \preceq_i\rangle$ for $i \in \{1,2\}$ be two L-ordered sets. A mapping $s\colon V_1 \longrightarrow V_2$ is said to be *L-isotone* if for any $u, v \in V_1$ the following holds

$$\big(u \preceq_1 v\big) \le \big(s(u) \preceq_2 s(v)\big).$$

It is not difficult to check that this definition extends that in the classical case.

Lemma 2 *2-isotone mappings correspond to classical isotone mappings.*

The definition of L-isotone Galois connection is given below:

Definition 13 Let $\langle\langle V_i, \approx_i\rangle, \preceq_i\rangle$ for $i \in \{1,2\}$ be two L-ordered sets. An *L-isotone Galois connection* is a pair of L-isotone mappings $\langle s, z\rangle$ such that $s\colon V_1 \longrightarrow V_2$ and $z : V_2 \longrightarrow V_1$ and for any pair $(v_1, v_2) \in V_1 \times V_2$ the following equality holds

$$\big(s(v_1) \preceq_2 v_2\big) = \big(v_1 \preceq_1 z(v_2)\big).$$

Lemma 3 *The 2-isotone Galois connections correspond to classical isotone Galois connections.*

Proof From the previous lemma, we know that if $L = 2$ then s and z are *classical* isotone mappings between ordered sets $\langle V_1, \le_1\rangle$ and $\langle V_2, \le_2\rangle$. Equality of values $(s(v_1) \le_2 v_2)$ and $(v_1 \le_1 z(v_2))$ that are from $\{0,1\}$ makes the equivalence $(s(v_1) \le_2 v_2) \Leftrightarrow (v_1 \le_1 z(v_2))$ hold true. Hence $\langle s, z\rangle$ forms an isotone Galois connection. □

2.3 L-Chu correspondences

For the sake of self-containment, in this section we recall the main definitions concerning L-Chu correspondences, which were already used in our previous works [27, 28].

Definition 14 An *L-multifunction* from set X to set Y is a mapping from X to L^Y.

Definition 15 Given $\varpi\colon X \longrightarrow L^Y$, a mapping $\varpi_+\colon L^X \longrightarrow L^Y$ for all $f \in L^X$ is defined by

$$\varpi_+(f)(y) = \bigvee_{x \in X} \big(f(x) \otimes \varpi(x)(y)\big). \tag{7}$$

Definition 16 Consider two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i\rangle, (i = 1,2)$, then the pair $\varphi = (\varphi_L, \varphi_R)$ is said to be a **correspondence** from C_1 to C_2 if φ_L and φ_R are L-multifunctions, respectively, from B_1 to B_2 and from A_2 to A_1 (ie, $\varphi_L : B_1 \longrightarrow L^{B_2}$ and $\varphi_R : A_2 \longrightarrow L^{A_1}$).

The L-correspondence φ is said to be a **weak L-Chu correspondence** if the following equality holds for all $o_1 \in B_1$ and $a_2 \in A_2$:

$$\bigwedge_{a_1 \in A_1} (\varphi_R(a_2)(a_1) \to r_1(o_1, a_1)) = \bigwedge_{o_2 \in B_2} (\varphi_L(o_1)(o_2) \to r_2(o_2, a_2)) \tag{8}$$

A weak Chu correspondence φ is an **L-Chu correspondence** if $\varphi_L(o_1)$ is an L-set of objects closed in C_2 and $\varphi_R(a_2)$ is an L-set of attributes closed in C_1 for all $o_1 \in B_1$ and $a_2 \in A_2$. We will denote the set of all L-Chu correspondences from C_1 to C_2 by L-ChuCors(C_1, C_2).

Table 1

φ_1

$(-)_L$:

	o_{21}	o_{22}	o_{23}
o_{11}	1	1	0.5
o_{12}	1	1	0.5

$(-)_R$:

	a_{11}	a_{12}	a_{13}	a_{14}
a_{21}	1	1	1	0.5
a_{22}	1	0.5	0.5	0
a_{23}	1	0.5	0.5	0

φ_2

$(-)_L$:

	o_{21}	o_{22}	o_{23}
o_{11}	1	1	0.5
o_{12}	1	0.5	0

$(-)_R$:

	a_{11}	a_{12}	a_{13}	a_{14}
a_{21}	1	0.5	1	0.5
a_{22}	1	0.5	0.5	0
a_{23}	1	0.5	0.5	0

φ_3

$(-)_L$:

	o_{21}	o_{22}	o_{23}
o_{11}	1	0.5	0
o_{12}	1	1	0.5

$(-)_R$:

	a_{11}	a_{12}	a_{13}	a_{14}
a_{21}	1	1	0.5	0
a_{22}	1	0.5	0.5	0
a_{23}	1	0.5	0.5	0

φ_4

$(-)_L$:

	o_{21}	o_{22}	o_{23}
o_{11}	1	0.5	0
o_{12}	1	0.5	0

$(-)_R$:

	a_{11}	a_{12}	a_{13}	a_{14}
a_{21}	1	0.5	0.5	0
a_{22}	1	0.5	0.5	0
a_{23}	1	0.5	0.5	0

φ_5

$(-)_L$:

	o_{21}	o_{22}	o_{23}
o_{11}	1	1	1
o_{12}	1	1	0.5

$(-)_R$:

	a_{11}	a_{12}	a_{13}	a_{14}
a_{21}	1	1	1	1
a_{22}	1	0.5	0.5	0
a_{23}	1	0.5	1	0.5

Example 3 All L-Chu correspondences between the L-contexts C_1 and C_2 used in Example 1 can be seen in Table 1.

In the left column of the table one can see all the left parts φ_L, which are L-multifunctions that assign some extent of C_2 to every object of C_1. In the right column of the table the corresponding right parts of the L-Chu correspondences φ_R are shown; they assign some intent of C_1 to every attribute of C_2 in such a way that equality (8) from Definition 16 holds. □

3 Introducing the relevant categories

3.1 The category L-ChuCors

The category of L-fuzzy formal contexts and L-Chu correspondences between them is formally defined below:

- **objects** L-fuzzy formal contexts
- **arrows** L-Chu correspondences
- **identity arrow** $\iota : C \longrightarrow C$ of L-context $C = \langle B, A, r\rangle$
  - $\iota_L(o) = \downarrow\uparrow(\chi_o)$, for all $o \in B$
  - $\iota_R(a) = \uparrow\downarrow(\chi_a)$, for all $a \in A$

  where $\chi_x(x) = 1$ and $\chi_x(y) = 0$ for any $y \neq x$
- **composition** $\varphi_2 \circ \varphi_1 : C_1 \longrightarrow C_3$ **of arrows** $\varphi_1 : C_1 \longrightarrow C_2$, $\varphi_2 : C_2 \longrightarrow C_3$ ($C_i = \langle B_i, A_i, r_i\rangle$, $i \in \{1, 2\}$)

– $(\varphi_2 \circ \varphi_1)_L : B_1 \longrightarrow L^{B_3}$ defined as

$$(\varphi_2 \circ \varphi_1)_L(o_1) = \downarrow_3\uparrow_3 \big(\varphi_{2L+}(\varphi_{1L}(o_1))\big) \tag{9}$$

where

$$\varphi_{2L+}(\varphi_{1L}(o_1))(o_3) = \bigvee_{o_2 \in B_2} \varphi_{1L}(o_1)(o_2) \otimes \varphi_{2L}(o_2)(o_3)$$

– and $(\varphi_2 \circ \varphi_1)_R : A_3 \longrightarrow L^{A_1}$ defined as

$$(\varphi_2 \circ \varphi_1)_R(a_3) = \uparrow_1\downarrow_1 \big(\varphi_{1R+}(\varphi_{2R}(a_3))\big) \tag{10}$$

where

$$\varphi_{1R+}(\varphi_{2R}(a_3))(a_1) = \bigvee_{a_2 \in A_2} \varphi_{2R}(a_3)(a_2) \otimes \varphi_{1R}(a_2)(a_1)$$

All details about the definition of the category L-ChuCors could be found in [27].

3.2 Category L-CLOS

Here we define another category

Objects are completely lattice L-ordered sets (L-CLOS) i.e. our objects will be represented as $\mathcal{V} = \langle\langle V, \approx\rangle, \preceq\rangle$

Arrows are L-isotone Galois connections between two L-CLOS i.e. $\langle s, z\rangle$ between $\mathcal{V}_1$ and $\mathcal{V}_2$, such that:

1. $s : V_1 \longrightarrow V_2$,
2. $z : V_2 \longrightarrow V_1$,
3. $(s(v_1) \preceq_2 v_2) = (v_1 \preceq_1 z(v_2))$ for all $(v_1, v_2) \in V_1 \times V_2$.

Identity arrow of $\langle\langle V, \approx\rangle, \preceq\rangle$ is a pair of identity morphisms $\langle id_V, id_V\rangle$

Composition of arrows is based on composition of mappings: consider two arrows $\langle s_i, z_i\rangle : \mathcal{V}_i \longrightarrow \mathcal{V}_{i+1}$, where $i \in \{1, 2\}$. Composition is defined as follows:

$$\langle s_2, z_2\rangle \circ \langle s_1, z_1\rangle = \langle s_2 \circ s_1, z_1 \circ z_2\rangle.$$

Thus, given a pair of two arbitrary elements $(v_1, v_3) \in V_1 \times V_3$ then:

$$\begin{aligned}
\big((s_2 \circ s_1)(v_1) \preceq_3 v_3\big) &= \big(s_2(s_1(v_1)) \preceq_3 v_3\big) \\
&= \big(s_1(v_1) \preceq_2 z_2(v_3)\big) \\
&= \big(v_1 \preceq_1 z_1(z_2(v_3))\big) \\
&= \big(v_1 \preceq_1 (z_1 \circ z_2)(v_3)\big)
\end{aligned}$$

Associativity of composition follows trivially because of the associativity of composition of mappings between sets.

4 The categories L-ChuCors and L-CLOS are equivalent

As stated in [4], mathematically significant properties of objects are those that are invariant under isomorphisms and, in category theory, equivalence of categories is the most convenient notion of "isomorphism" (used here with an informal meaning) between categories.

In this section, we reach the main goal of this paper: to prove the equivalence of the categories L-ChuCors and L-CLOS. As a result, we obtain that the generalized approaches based on L-Chu correspondences and those on completely L-lattice ordered sets are mutually interchangeable.

The equivalence between both categories will be proved by defining a suitable functor Γ which links L-ChuCors to L-CLOS. The behavior of Γ for objects is straightforward: to any L-context C the functor Γ assigns its corresponding concept L-CLOS, namely L-FCL(C), . The formal definition is given below:

1. $\Gamma(C) = \langle\langle L\text{-FCL}(C), \approx\rangle, \preceq\rangle$
2. For any L-Chu correspondence $\varphi \in L$-ChuCors(C_1, C_2), the result of $\Gamma(\varphi)$ will be a pair of mappings $\langle\varphi_\vee, \varphi_\wedge\rangle$ defined as follows:

$$\varphi_\vee(\langle f_1, g_1\rangle) = \left\langle \downarrow_2\uparrow_2 \left(\varphi_{L+}(f_1)\right), \uparrow_2 \left(\varphi_{L+}(f_1)\right)\right\rangle \qquad (11)$$

$$\varphi_\wedge(\langle f_2, g_2\rangle) = \left\langle \downarrow_1 \left(\varphi_{R+}(g_2)\right), \uparrow_1\downarrow_1 \left(\varphi_{R+}(g_2)\right)\right\rangle \qquad (12)$$

where $\langle f_i, g_i\rangle \in L$-FCL$(C_i)$ for $i \in \{1, 2\}$.

Example 4 Continuing with our running example, Table 2 contains all pairs of mappings $\langle\varphi_\vee, \varphi_\wedge\rangle$ between L-concept lattices of C_1 and C_2 that are assigned by the mapping Γ to all L-Chu correspondences φ between L-contexts C_1 and C_2 from Example 3. □

Table 2

	p_1	p_2	p_3	p_4	p_5		q_1	q_2	q_3
$\varphi_{1\vee}(-)$	q_2	q_2	q_2	q_3	q_3	$\varphi_{1\wedge(-)}$	p_1	p_1	p_4
$\varphi_{2\vee}(-)$	q_2	q_2	q_3	q_3	q_3	$\varphi_{2\wedge(-)}$	p_1	p_1	p_3
$\varphi_{3\vee}(-)$	q_2	q_3	q_2	q_3	q_3	$\varphi_{3\wedge(-)}$	p_1	p_1	p_2
$\varphi_{4\vee}(-)$	q_3	q_3	q_3	q_3	q_3	$\varphi_{4\wedge(-)}$	p_1	p_1	p_1
$\varphi_{5\vee}(-)$	q_1	q_1	q_2	q_2	q_3	$\varphi_{5\wedge(-)}$	p_1	p_2	p_3

The two following lemmas are needed in order to prove that the morphism part of the functor Γ is well defined, that is, the pair $\langle\varphi_\vee, \varphi_\wedge\rangle$ is an L-isotone Galois connection between $\Gamma(C_1)$ and $\Gamma(C_2)$.

Lemma 4 *Let $C = \langle B, A, r\rangle$ be an L-context. For any L-set $f \in L^B$ and any concept $\langle h, g\rangle$ holds:*

$$\bigwedge_{o\in B} (\downarrow\uparrow (f)(o) \to h(o)) = \bigwedge_{o\in B} (f(o) \to h(o))$$

Proof

$$\bigwedge_{o\in B_2}(\downarrow_2\uparrow_2(f)(o)\rightarrow h(o)) = \bigwedge_{o\in B_2}(\downarrow_2\uparrow_2(f)(o)\rightarrow\downarrow_2\uparrow_2(h)(o))$$

because of lemma 1

$$= \bigwedge_{a\in A_2}(\uparrow_2(h)(a)\rightarrow\uparrow_2(f)(a))$$

$$\overset{(1)}{=} \bigwedge_{a\in A_2}\left(\uparrow_2(h)(a)\rightarrow\bigwedge_{o\in B_2}(f(o)\rightarrow r_2(o,a))\right)$$

$$\overset{(3)}{=} \bigwedge_{o\in B_2}\bigwedge_{a\in A_2}(\uparrow_2(h)(a)\rightarrow(f(o)\rightarrow r_2(o,a)))$$

$$\overset{(2)}{=} \bigwedge_{o\in B_2}\bigwedge_{a\in A_2}(f(o)\rightarrow(\uparrow_2(h)(a)\rightarrow r_2(o,a)))$$

$$\overset{(3)}{=} \bigwedge_{o\in B_2}\left(f(o)\rightarrow\bigwedge_{a\in A_2}(\uparrow_2(h)(a)\rightarrow r_2(o,a))\right)$$

$$\overset{(1)}{=} \bigwedge_{o\in B_2}(f(o)\rightarrow\downarrow_2\uparrow_2(h)(o))$$

h is closed L-set

$$= \bigwedge_{o\in B_2}(f(o)\rightarrow h(o))\square$$

Lemma 5 *Let $C_i = \langle B_i, A_i, r_i\rangle$ for $i\in\{1,2\}$ be two arbitrary L-contexts. Let ω be an L-multifunction between B_1 and B_2 ($\omega : B_1 \longrightarrow L^{B_2}$) and f be an arbitrary L-set from L^{B_1}. Then*

$$\uparrow_2(\omega_+(f))(a) = \bigwedge_{b\in B_1}(f(b)\rightarrow\uparrow_2(\omega(b))(a)).$$

Proof

$$\uparrow_2(\omega_+(f))(a) \overset{(1)}{=} \bigwedge_{o\in B_2}(\omega_+(f)(o)\rightarrow r_2(o,a))$$

$$\overset{(7)}{=} \bigwedge_{o\in B_2}\left(\bigvee_{b\in B_1}(f(b)\otimes\omega(b)(o))\rightarrow r_2(o,a)\right)$$

$$\overset{(4)}{=} \bigwedge_{o\in B_2}\bigwedge_{b\in B_1}((f(b)\otimes\omega(b)(o))\rightarrow r_2(o,a))$$

$$\overset{(2)}{=} \bigwedge_{o\in B_2}\bigwedge_{b\in B_1}(f(b)\rightarrow(\omega(b)(o))\rightarrow r_2(o,a))$$

$$\overset{(3)}{=} \bigwedge_{b\in B_1}\left(f(b)\to\bigwedge_{o\in B_2}(\omega(b)(o))\to r_2(o,a)\right)$$

$$\overset{(1)}{=} \bigwedge_{b\in B_1}(f(b)\to\uparrow_2(\omega(b))(a))$$

□

With the help of the previous lemmas we can now prove that the morphism part of Γ is well-defined.

Lemma 6 *$\Gamma(\varphi)\in L$-CLLOS$(\Gamma(C_1),\Gamma(C_2))$ for any $\varphi\in L$-ChuCors(C_1,C_2).*

Proof Firstly, L-isotonicity of the pair of mappings $\langle\varphi_\vee,\varphi_\wedge\rangle$ will be shown. Let us consider $\langle f,\uparrow_1(f)\rangle$ and $\langle h,\uparrow_1(h)\rangle$ be two L-concepts of context $C_1=\langle B_1,A_1,r_1\rangle$

$$\varphi_\vee(\langle f,\uparrow_1(f)\rangle)\preceq_2\varphi_\vee(\langle h,\uparrow_1(h)\rangle)$$

$$\overset{(5)}{=}\bigwedge_{o\in B_2}(\downarrow_2\uparrow_2(\varphi_{L+}(f))(o)\to\downarrow_2\uparrow_2(\varphi_{L+}(h))(o))$$

because of Lemma 4

$$=\bigwedge_{o\in B_2}(\varphi_{L+}(f)(o)\to\downarrow_2\uparrow_2(\varphi_{L+}(h))(o))$$

$$\overset{(7)}{=}\bigwedge_{o\in B_2}\left(\bigvee_{b\in B_1}(\varphi_L(b)(o)\otimes f(b))\to\downarrow_2\uparrow_2(\varphi_{L+}(h))(o)\right)$$

$$\overset{(4)}{=}\bigwedge_{o\in B_2}\bigwedge_{b\in B_1}((\varphi_L(b)(o)\otimes f())\to\downarrow_2\uparrow_2(\varphi_{L+}(h))(o))$$

$$\overset{(2)}{=}\bigwedge_{o\in B_2}\bigwedge_{b\in B_1}(f(b)\to(\varphi_L(b)(o)\to\downarrow_2\uparrow_2(\varphi_{L+}(h))(o)))$$

$$\overset{(3)}{=}\bigwedge_{b\in B_1}(f(b)\to\bigwedge_{o\in B_2}(\varphi_L(b)(o)\to\downarrow_2\uparrow_2(\varphi_{L+}(h))(o)))$$

because of Lemma 1

$$=\bigwedge_{b\in B_1}(f(b)\to\bigwedge_{a\in A_2}(\uparrow_2(\varphi_{L+}(h))(a)\to\uparrow_2(\varphi_L(b))(a)))$$

because of Lemma 5

$$=\bigwedge_{b\in B_1}\left(f(b)\to\bigwedge_{a\in A_2}\left(\bigwedge_{j\in B_1}\big(h(j)\to\uparrow_2(\varphi_L(j))(a)\big)\to\uparrow_2(\varphi_L(b))(a)\right)\right)$$

denoting $\uparrow_2(\varphi_L(b))(a)=\beta(b,a)$

$$=\bigwedge_{b\in B_1}\left(f(b)\to\bigwedge_{a\in A_2}\left(\bigwedge_{j\in B_1}\big(h(j)\to\beta(j,a)\big)\to\beta(b,a)\right)\right)$$

$$\stackrel{(1)}{=} \bigwedge_{b\in B_1} \big(f(b) \to \downarrow_\beta\uparrow_\beta (h)(b)\big)$$

by the property of closure

$$\geq \bigwedge_{b\in B_1} (f(b) \to h(b))$$

$$=(\langle f, \uparrow_1 (f)\rangle \preceq_1 \langle h, \uparrow_1 (h)\rangle)$$

Hence $\varphi_\vee$ is L-isotone. Similarly for $\varphi_\wedge$.

Consider two arbitrary L-concepts $\langle f_i, g_i\rangle$ of $\langle\langle L\text{-FCL}(C_i), \approx_i\rangle, \preceq_i\rangle$ for $i \in \{1, 2\}$ where $C_i = \langle B_i, A_i, r_i\rangle$.

$$(\varphi_\vee\big(\langle f_1, g_1\rangle\big) \preceq_2 \langle f_2, g_2\rangle) \stackrel{(11)}{=} \big(\big\langle \downarrow_2\uparrow_2 \big(\varphi_{L^+}(f_1)\big), \uparrow_2 \big(\varphi_{L^+}(f_1)\big)\big\rangle \preceq_2 \langle f_2, g_2\rangle\big)$$

$$\stackrel{(5)}{=} \bigwedge_{a_2\in A_2} \big(g_2(a_2) \to \uparrow_2 \big(\varphi_{L^+}(f_1)\big)(a_2)\big)$$

$$\stackrel{(1)}{=} \bigwedge_{a_2\in A_2} \left(g_2(a_2) \to \bigwedge_{o_2\in B_2} (\varphi_{L^+}(f_1)(o_2) \to r_2(o_2, a_2))\right)$$

$$\stackrel{(7)}{=} \bigwedge_{a_2\in A_2} \left(g_2(a_2) \to \bigwedge_{o_2\in B_2} \left(\bigvee_{o_1\in B_1} \big(\varphi_L(o_1)(o_2) \otimes f_1(o_1)\big) \to r_2(o_2, a_2)\right)\right)$$

$$\stackrel{(4)}{=} \bigwedge_{a_2\in A_2} \left(g_2(a_2) \to \bigwedge_{o_2\in B_2} \bigwedge_{o_1\in B_1} \big(\big(\varphi_L(o_1)(o_2) \otimes f_1(o_1)\big) \to r_2(o_2, a_2)\big)\right)$$

$$\stackrel{(3)}{=} \bigwedge_{a_2\in A_2} \bigwedge_{o_2\in B_2} \bigwedge_{o_1\in B_1} \big(g_2(a_2) \to \big(\big(\varphi_L(o_1)(o_2) \otimes f_1(o_1)\big) \to r_2(o_2, a_2)\big)\big)$$

$$\stackrel{(2)}{=} \bigwedge_{a_2\in A_2} \bigwedge_{o_2\in B_2} \bigwedge_{o_1\in B_1} \big(g_2(a_2) \to \big(\varphi_L(o_1)(o_2) \to \big(f_1(o_1) \to r_2(o_2, a_2)\big)\big)\big)$$

$$\stackrel{(3)}{=} \bigwedge_{a_2\in A_2} \bigwedge_{o_1\in B_1} \left(g_2(a_2) \to \left(f_1(o_1) \to \bigwedge_{o_2\in B_2} (\varphi_L(o_1)(o_2) \to r_2(o_2, a_2))\right)\right)$$

$$\stackrel{(8)}{=} \bigwedge_{a_2\in A_2} \bigwedge_{o_1\in B_1} \left(f_1(o_1) \to \left(g_2(a_2) \to \bigwedge_{a_1\in A_1} (\varphi_R(a_2)(a_1) \to r_1(o_1, a_1))\right)\right)$$

by a similar chain of computation we obtain

$$= \bigwedge_{o_1\in B_1} \left(f_1(o_1) \to \bigwedge_{a_1\in A_1} \left(\bigvee_{a_2\in A_2} \big(\varphi_R(a_2)(a_1) \otimes g_2(a_2)\big) \to r_1(o_1, a_1)\right)\right)$$

$$\stackrel{(7)}{=} \bigwedge_{o_1\in B_1} \big(f_1(o_1) \to \downarrow_1 \big(\varphi_{R^+}(g_2)\big)(o_1)\big)$$

$$\stackrel{(5)}{=} \big(\langle f_1, g_1\rangle \preceq_1 \big\langle \downarrow_1 \big(\varphi_{R^+}(g_2)\big), \uparrow_1\downarrow_1 \big(\varphi_{R^+}(g_2)\big)\big\rangle\big)$$

$$\stackrel{(12)}{=} \big(\langle f_1, g_1\rangle \preceq_1 \varphi_\wedge\big(\langle f_2, g_2\rangle\big)\big)$$

So $\langle\varphi_\vee, \varphi_\wedge\rangle$ is an L-isotone Galois connection between the completely lattice L-ordered sets $\langle\langle L\text{-FCL}(C_1), \approx_1\rangle \preceq_1\rangle$ and $\langle\langle L\text{-FCL}(C_2), \approx_2\rangle \preceq_2\rangle$. □

Table 3

φ_1

	$\varphi_{1\wedge}(q_1)$	$\varphi_{1\wedge}(q_2)$	$\varphi_{1\wedge}(q_3)$
$\preceq_1$	p_1	p_1	p_4
p_1	1	1	0.5
p_2	1	1	0.5
p_3	1	1	0.5
p_4	1	1	1
p_5	1	1	1

	$\preceq_2$	q_1	q_2	q_3
$\varphi_{1\vee}(p_1)$	q_2	1	1	0.5
$\varphi_{1\vee}(p_2)$	q_2	1	1	0.5
$\varphi_{1\vee}(p_3)$	q_2	1	1	0.5
$\varphi_{1\vee}(p_4)$	q_3	1	1	1
$\varphi_{1\vee}(p_5)$	q_3	1	1	1

φ_2

	$\varphi_{2\wedge}(q_1)$	$\varphi_{2\wedge}(q_2)$	$\varphi_{2\wedge}(q_3)$
$\preceq_1$	p_1	p_1	p_3
p_1	1	1	0.5
p_2	1	1	0.5
p_3	1	1	1
p_4	1	1	1
p_5	1	1	1

	$\preceq_2$	q_1	q_2	q_3
$\varphi_{2\vee}(p_1)$	q_2	1	1	0.5
$\varphi_{2\vee}(p_2)$	q_2	1	1	0.5
$\varphi_{2\vee}(p_3)$	q_3	1	1	1
$\varphi_{2\vee}(p_4)$	q_3	1	1	1
$\varphi_{2\vee}(p_5)$	q_3	1	1	1

φ_3

	$\varphi_{3\wedge}(q_1)$	$\varphi_{3\wedge}(q_2)$	$\varphi_{3\wedge}(q_3)$
$\preceq_1$	p_1	p_1	p_2
p_1	1	1	0.5
p_2	1	1	1
p_3	1	1	0.5
p_4	1	1	1
p_5	1	1	1

	$\preceq_2$	q_1	q_2	q_3
$\varphi_{3\vee}(p_1)$	q_2	1	1	0.5
$\varphi_{3\vee}(p_2)$	q_3	1	1	1
$\varphi_{3\vee}(p_3)$	q_2	1	1	0.5
$\varphi_{3\vee}(p_4)$	q_3	1	1	1
$\varphi_{3\vee}(p_5)$	q_3	1	1	1

φ_4

	$\varphi_{4\wedge}(q_1)$	$\varphi_{4\wedge}(q_2)$	$\varphi_{4\wedge}(q_3)$
$\preceq_1$	p_1	p_1	p_1
p_1	1	1	1
p_2	1	1	1
p_3	1	1	1
p_4	1	1	1
p_5	1	1	1

	$\preceq_2$	q_1	q_2	q_3
$\varphi_{4\vee}(p_1)$	q_3	1	1	1
$\varphi_{4\vee}(p_2)$	q_3	1	1	1
$\varphi_{4\vee}(p_3)$	q_3	1	1	1
$\varphi_{4\vee}(p_4)$	q_3	1	1	1
$\varphi_{4\vee}(p_5)$	q_3	1	1	1

φ_5

	$\varphi_{5\wedge}(q_1)$	$\varphi_{5\wedge}(q_2)$	$\varphi_{5\wedge}(q_3)$
$\preceq_1$	p_1	p_3	p_5
p_1	1	0.5	0
p_2	1	0.5	0
p_3	1	1	0.5
p_4	1	1	0.5
p_5	1	1	1

	$\preceq_2$	q_1	q_2	q_3
$\varphi_{5\vee}(p_1)$	q_1	1	0.5	0
$\varphi_{5\vee}(p_2)$	q_1	1	0.5	0
$\varphi_{5\vee}(p_3)$	q_2	1	1	0.5
$\varphi_{5\vee}(p_4)$	q_2	1	1	0.5
$\varphi_{5\vee}(p_5)$	q_3	1	1	1

Example 5 In Table 3 one can see that all pairs $\langle \varphi_\vee, \varphi_\wedge \rangle$ in Example 4 with the L-ordering given in Example 2 satisfy Definition 13 and, therefore, all are L-isotone Galois connections. ◻

The following result checks that Γ preserves identity morphisms.

Lemma 7 *For the identity arrow $\iota \in L$-ChuCors(C, C) of any L-context $C = \langle B, A, r \rangle$, $\Gamma(\iota)$ is the identity arrow from L-CLOS$(\Gamma(C), \Gamma(C))$.*

Proof Consider any L-concept $\langle f, g \rangle$ from L-FCL(C).

$$\uparrow \big(\iota_{L^+}(f)\big)(a) = \bigwedge_{o \in B} \big(\iota_{L^+}(f)(o) \to r(o, a)\big)$$

$$\overset{(7)}{=} \bigwedge_{o\in B} \left(\bigvee_{b\in B} \big(\iota_L(b)(o) \otimes f(b)\big) \to r(o,a) \right)$$

$$\overset{(4)}{=} \bigwedge_{o\in B} \bigwedge_{b\in B} \big(\big(\iota_L(b)(o) \otimes f(b)\big) \to r(o,a)\big)$$

$$\overset{(2)(3)}{=} \bigwedge_{o\in B} \left(f(b) \to \bigwedge_{b\in B} \big(\iota_L(b)(o) \to r(o,a)\big) \right)$$

$$\overset{(1)}{=} \bigwedge_{b\in B} \big(f(b) \to \uparrow (\chi_b)(a)\big)$$

$$= \bigwedge_{b\in B} \big(f(b) \to \uparrow\downarrow\uparrow (\chi_b)(a)\big)$$

$$= \bigwedge_{b\in B} \big(f(b) \to r(b,a)\big) = \uparrow (f)(a)$$

Therefore, we have $\iota_\vee(\langle f,g\rangle) = \langle f,g\rangle$. The proof for $\iota_\wedge$ is similar. □

We continue with a technical lemma which proves an equality needed in the proof that Γ preserves composition.

Lemma 8 *Consider two arbitrary* $\varphi_i \in L\text{-ChuCors}(C_i, C_{i+1})$ *for* $i \in \{1,2\}$ *and any element* $o_1 \in B_1$ *and* $g_3 \in L^{A_3}$*. Then*

$$\downarrow_1 \big(\varphi_{1R^+}\big(\varphi_{2R^+}(g_3)\big)\big)(o_1) = \downarrow_1 \big(\varphi_{1R^+}\big(\uparrow_2\downarrow_2 \big(\varphi_{2R^+}(g_3)\big)\big)\big)(o_1).$$

Proof

$\downarrow_1 \big(\varphi_{1R^+}\big(\varphi_{2R^+}(g_3)\big)\big)(o_1) =$

$$\overset{(1)}{=} \bigwedge_{a_1\in A_1} \big(\varphi_{1R^+}\big(\varphi_{2R^+}(g_3)\big)(a_1) \to r_1(o_1,a_1)\big)$$

$$\overset{(7)}{=} \bigwedge_{a_1\in A_1} \left(\bigvee_{a_2\in A_2} \big(\varphi_{1R}(a_2)(a_1) \otimes \varphi_{2R^+}(g_3)(a_2)\big) \to r_1(o_1,a_1) \right)$$

$$\overset{(4)(2)(3)}{=} \bigwedge_{a_2\in A_2} \left(\varphi_{2R^+}(g_3)(a_2) \to \bigwedge_{a_1\in A_1} \big(\varphi_{1R}(a_2)(a_1) \to r_1(o_1,a_1)\big) \right)$$

$$\overset{(8)}{=} \bigwedge_{a_2\in A_2} \left(\varphi_{2R^+}(g_3)(a_2) \to \bigwedge_{o_2\in B_2} \big(\varphi_{1L}(o_1)(o_2) \to r_2(o_2,a_2)\big) \right)$$

$$\overset{(2)(3)}{=} \bigwedge_{o_2\in B_2} \left(\varphi_{1L}(o_1)(o_2) \to \bigwedge_{a_2\in A_2} \big(\varphi_{2R^+}(g_3)(a_2) \to r_2(o_2,a_2)\big) \right)$$

$$\overset{(1)}{=} \bigwedge_{o_2\in B_2} \big(\varphi_{1L}(o_1)(o_2) \to \downarrow_2 \big(\varphi_{2R^+}(g_3)\big)(o_2)\big)$$

by the closure property

$$= \bigwedge_{o_2\in B_2} \big(\varphi_{1L}(o_1)(o_2) \to \downarrow_2\uparrow_2\downarrow_2 \big(\varphi_{2R^+}(g_3)\big)(o_2)\big)$$

Finally, by applying the same chain of modifications in opposite way we will obtain

$$\downarrow_1 \big(\varphi_{1R^+}\big(\uparrow_2\downarrow_2 (\varphi_{2R^+}(g_3))\big)\big)(o_1)$$

□

Lemma 9 *Γ is closed under arrow composition.*

Proof Consider $\varphi_i \in L\text{-ChuCors}(C_i, C_{i+1})$ for $i \in \{1,2\}$. Let $\langle f_i, g_i\rangle \in L\text{-FCL}(C_i)$ be an arbitrary L-context for all $i \in \{1,3\}$. Recall that

1. $\Gamma(\varphi_2 \circ \varphi_1) = \big\langle (\varphi_2 \circ \varphi_1)_\vee, (\varphi_2 \circ \varphi_1)_\wedge \big\rangle$
2. $\Gamma(\varphi_2) \circ \Gamma(\varphi_1) = \big\langle \varphi_{2\vee} \circ \varphi_{1\vee}, \varphi_{1\wedge} \circ \varphi_{2\wedge} \big\rangle$

The proof will be based on equality of corresponding elements of the previous pairs: only one part will be proved, the other one is similar.

$$\begin{aligned}
&\downarrow_1 (\varphi_{1R^+}(\varphi_{2R^+}(g_3)))(o_1) = \\
&\overset{(7)}{=} \bigwedge_{a_1\in A_1} \left(\bigvee_{a_2\in A_2} (\varphi_{1R}(a_2)(a_1) \otimes \varphi_{2R^+}(g_3)(a_2)) \to r_1(o_1,a_1) \right) \\
&\overset{(7)}{=} \bigwedge_{a_1\in A_1} \left(\bigvee_{a_2\in A_2} (\varphi_{1R}(a_2)(a_1) \otimes \bigvee_{a_3\in A_3} (\varphi_{2R}(a_3)(a_2) \otimes g_3(a_3))) \to r_1(o_1,a_1) \right) \\
&\overset{(7)}{=} \bigwedge_{a_1\in A_1} \left(\bigvee_{a_3\in A_3} \big(\varphi_{1R^+}(\varphi_{2R}(a_3))(a_1) \otimes g_3(a_3)\big) \to r_1(o_1,a_1) \right) \\
&\overset{(4)(2)(3)}{=} \bigwedge_{a_3\in A_3} \left(g_3(a_3) \to \bigwedge_{a_1\in A_1} (\varphi_{1R^+}(\varphi_{2R}(a_3))(a_1) \to r_1(o_1,a_1)) \right) \\
&\overset{(1)}{=} \bigwedge_{a_3\in A_3} \big(g_3(a_3) \to \downarrow_1 \big(\varphi_{1R^+}(\varphi_{2R}(a_3))\big)(o_1)\big) \\
&= \bigwedge_{a_3\in A_3} \big(g_3(a_3) \to \downarrow_1\uparrow_1\downarrow_1 \big(\varphi_{1R^+}(\varphi_{2R}(a_3))\big)(o_1)\big) \\
&\overset{(10)}{=} \bigwedge_{a_3\in A_3} \big(g_3(a_3) \to \downarrow_1 \big((\varphi_2 \circ \varphi_1)_R(a_3)\big)(o_1)\big) \\
&\overset{(1)}{=} \bigwedge_{a_3\in A_3} \big(g_3(a_3) \to \bigwedge_{a_1\in A_1} \big((\varphi_2 \circ \varphi_1)_R(a_3)(a_1) \to r_1(o_1,a_1)\big)\big) \\
&\overset{(3)(2)}{=} \bigwedge_{a_1\in A_1} \bigwedge_{a_3\in A_3} \big(\big(g_3(a_3) \otimes (\varphi_2 \circ \varphi_1)_R(a_3)(a_1)\big) \to r_1(o_1,a_1)\big) \\
&\overset{(4)}{=} \bigwedge_{a_1\in A_1} \left(\bigvee_{a_3\in A_3} (g_3(a_3) \otimes (\varphi_2 \circ \varphi_1)_R(a_3)(a_1)) \to r_1(o_1,a_1) \right) \\
&\overset{(7)}{=} \downarrow_1 \big((\varphi_2 \circ \varphi_1)_{R^+}(g_3)\big)(o_1)
\end{aligned}$$

Hence

$$
\begin{aligned}
(\varphi_{1\wedge} \circ \varphi_{2\wedge})(\langle f_3, g_3\rangle) &= \varphi_{1\wedge}(\varphi_{2\wedge}(\langle f_3, g_3\rangle)) \\
&\overset{(12)}{=} \Big\langle \downarrow_1 (\varphi_{1R^+}(\uparrow_2\downarrow_2 (\varphi_{2R^+}(g_3)))), \uparrow_1\downarrow_1 (\varphi_{1R^+}(\uparrow_2\downarrow_2 (\varphi_{2R^+}(g_3))))\Big\rangle \\
&\quad \text{by lemma 8 we have} \\
&= \Big\langle \downarrow_1 (\varphi_{1R^+}(\varphi_{2R^+}(g_3))), \uparrow_1\downarrow_1 (\varphi_{1R^+}(\varphi_{2R^+}(g_3)))\Big\rangle \\
&\overset{(10)}{=} \Big\langle \downarrow_1 ((\varphi_2 \circ \varphi_1)_{R^+}(g_3)), \uparrow_1\downarrow_1 ((\varphi_2 \circ \varphi_1)_{R^+}(g_3))\Big\rangle \\
&\overset{(12)}{=} (\varphi_2 \circ \varphi_1)_\wedge(\langle f_3, g_3\rangle)
\end{aligned}
$$

□

Now, using Lemmas 6, 7 and 9 we directly obtain the following result.

Proposition 1 *Γ is a functor from L-ChuCors to L-CLOS.*

We recall now some necessary notions which will be used in order to prove that the previous functor satisfies the conditions to define a categorical equivalence.

Definition 17

1. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is *faithful* if for all objects A, B of a category $\mathcal{C}$, the map $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is injective.
2. Similarly, F is *full* if $F_{A,B}$ is always surjective.

The proof of the categorical equivalence will be done by using the following characterization:

Theorem 2 (See [4]) *The following conditions on a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ are equivalent:*

- *F is an equivalence of categories.*
- *F is full and faithful and "essentially surjective" on objects: for every $D \in \mathcal{D}$ there is some $C \in \mathcal{C}$ such that $F(C) \cong D$.*

In our cases, for proving fullness and faithfulness of the functor Γ we need to prove surjectivity and injectivity of the mapping

$$\Gamma_{C_1,C_2} : L\text{-ChuCors}(C_1, C_2) \longrightarrow L\text{-CLOS}(\Gamma(C_1), \Gamma(C_2))$$

for any two L-contexts C_1 and C_2. This will be done in the forthcoming lemmas.

Lemma 10 *Γ is full.*

Proof The point of the proof is to show that given any arrow $\langle s, z\rangle$ from the set L-CLOS$(\Gamma(C_1), \Gamma(C_2))$ there exists an L-Chu correspondence $\varphi^{\langle s,z\rangle}$ from the set L-ChuCors(C_1, C_2), for any two L-contexts $C_i = \langle B_i, A_i, r_i\rangle$ for $i = \{1, 2\}$. Let us define the following mappings:

- $\varphi_L^{\langle s,z\rangle}(o_1) = Ext\big(s\big(\langle\downarrow_1\uparrow_1 (\chi_{o_1}), \uparrow_1 (\chi_{o_1})\rangle\big)\big)$
- $\varphi_R^{\langle s,z\rangle}(a_2) = Int\big(z\big(\langle\downarrow_2 (\chi_{a_2}), \uparrow_2\downarrow_2 (\chi_{a_2})\rangle\big)\big)$

$$
\begin{aligned}
\uparrow_2 \big(\varphi_L^{\langle s,z\rangle}(o_1)\big)(a_2) &= \bigwedge_{o_2\in B_2} \big(\varphi_L^{\langle s,z\rangle}(o_1)(o_2) \to r_2(o_2,a_2)\big)\\
&= \bigwedge_{o_2\in B_2} \big(Ext(s(\langle\downarrow_1\uparrow_1 (\chi_{o_1}),\uparrow_1 (\chi_{o_1})\rangle))(o_2) \to \downarrow_2 (\chi_{a_2})(o_2)\big)\\
&= s\big(\langle\downarrow_1\uparrow_1 (\chi_{o_1}),\uparrow_1 (\chi_{o_1})\rangle\big) \preceq_2 \langle\downarrow_2 (\chi_{a_2}),\uparrow_2\downarrow_2 (\chi_{a_2})\rangle\\
&= \langle\downarrow_1\uparrow_1 (\chi_{o_1}),\uparrow_1 (\chi_{o_1})\rangle \preceq_1 z\big(\langle\downarrow_2 (\chi_{a_2}),\uparrow_2\downarrow_2 (\chi_{a_2})\rangle\big)\\
&= \bigwedge_{a_1\in A_1} \big(Int(z(\langle\downarrow_2 (\chi_{a_2}),\uparrow_2\downarrow_2 (\chi_{a_2})\rangle))(a_1) \to \uparrow_1 (\chi_{o_1})(a_1)\big)\\
&= \bigwedge_{a_1\in A_1} \big(\varphi_R^{\langle s,z\rangle}(a_2)(a_1) \to r_1(o_1,a_1)\big)\\
&= \downarrow_1 \big(\varphi_R^{\langle s,z\rangle}(a_2)\big)(o_1)
\end{aligned}
$$

So $\varphi^{\langle s,z\rangle} \in L$-ChuCors$(C_1, C_2)$ and Γ_{C_1,C_2} is surjective, hence Γ is full. □

Lemma 11 *Γ is faithful.*

Proof Now the point is to prove the injectivity of Γ_{C_1,C_2}.

Consider two L-Chu correspondences φ_1, φ_2 from L-ChuCors(C_1, C_2) such that $\varphi_1 \neq \varphi_2$, and let us fix the pair $(o_1, a_2) \in B_1 \times A_2$, such that

$$\uparrow_2 \big(\varphi_{1L}(o_1)\big)(a_2) = \downarrow_1 \big(\varphi_{1R}(a_2)\big)(o_1) \neq \uparrow_2 \big(\varphi_{2L}(o_1)\big)(a_2) = \downarrow_1 \big(\varphi_{2R}(a_2)\big)(o_1)$$

Let us assume that either $\downarrow_1 \big(\varphi_{1R}(a_2)\big)(o_1) > \uparrow_2 \big(\varphi_{2L}(o_1)\big)(a_2)$ or both values from L are incomparable, that is equivalent to the following:

$$\downarrow_1 \big(\varphi_{1R}(a_2)\big)(o_1) \to \uparrow_2 \big(\varphi_{2L}(o_1)\big)(a_2) < 1$$

Now consider the L-concept $\langle\downarrow_1 (\varphi_{1R}(a_2)), \varphi_{1R}(a_2)\rangle$ and let us compare its images under the mappings $\varphi_{1\vee}$ and $\varphi_{2\vee}$.

$$
\begin{aligned}
&\uparrow_2\big(\varphi_{2L+}\big(\downarrow_1 (\varphi_{1R}(a_2))\big)\big)(a_2)\\
&\overset{(1)}{=} \bigwedge_{o_2\in B_2} \big(\varphi_{2L+}(\downarrow_1 (\varphi_{1R}(a_2)))(o_2) \to r_2(o_2,a_2)\big)\\
&\overset{(7)}{=} \bigwedge_{o_2\in B_2} \left(\bigvee_{b_1\in B_1} \big(\varphi_{2L}(b_1)(o_2)\otimes \downarrow_1 (\varphi_{1R}(a_2))(b_1)\big) \to r_2(o_2,a_2)\right)\\
&\overset{(4)(2)(3)}{=} \bigwedge_{b_1\in B_1} \left(\downarrow_1 (\varphi_{1R}(a_2))(b_1) \to \bigwedge_{o_2\in B_2} \big(\varphi_{2L}(b_1)(o_2) \to r_2(o_2,a_2)\big)\right)\\
&\overset{(1)}{=} \bigwedge_{b_1\in B_1} \big(\downarrow_1 (\varphi_{1R}(a_2))(b_1) \to \uparrow_2 (\varphi_{2L}(b_1))(a_2)\big)\\
&\leq \downarrow_1 (\varphi_{1R}(a_2))(o_1) \to \uparrow_2 (\varphi_{2L}(o_1))(a_2)\\
&\text{because of the restriction given above}\\
&< 1
\end{aligned}
$$

Similarly, we can obtain:

$$\begin{aligned}
&\uparrow_2(\varphi_{1L^+}(\downarrow_1(\varphi_{1R}(a_2))))(a_2) = \\
&\quad = \bigwedge_{b_1 \in B_1} \big(\downarrow_1(\varphi_{1R}(a_2))(b_1) \to \uparrow_2(\varphi_{1L}(b_1))(a_2) \big) \\
&\quad \overset{(8)}{=} \bigwedge_{b_1 \in B_1} \big(\downarrow_1(\varphi_{1R}(a_2))(b_1) \to \downarrow_1(\varphi_{1R}(a_2))(b_1) \big) = 1
\end{aligned}$$

It means that $\varphi_{1\vee}\big(\langle\downarrow_1(\varphi_{1R}(a_2)), \varphi_{1R}(a_2)\rangle\big) \neq \varphi_{2\vee}\big(\langle\downarrow_1(\varphi_{1R}(a_2)), \varphi_{1R}(a_2)\rangle\big)$

Hence $\varphi_{1\vee}(\langle\downarrow_1(\varphi_{1R}(a_2)), \varphi_{1R}(a_2)\rangle) \neq \varphi_{2\vee}(\langle\downarrow_1(\varphi_{1R}(a_2)), \varphi_{1R}(a_2)\rangle)$ and $\varphi_{1\vee} \neq \varphi_{2\vee}$. So Γ_{C_1,C_2} is injective and Γ is faithful. □

Proposition 2 *The functor Γ is an equivalence of categories L-*ChuCors *and L-CLOS.*

Proof Fullness and faithfulness of Γ is given by previous lemmas. Essential surjectivity on objects is ensured by the fact that given any object $\langle\langle V, \approx\rangle, \preceq\rangle$ of L-CLOS there exists an L-context $\langle V, V, \preceq\rangle$, such that $\Gamma\big(\langle V, V, \preceq\rangle\big) \cong \langle\langle V, \approx\rangle, \preceq\rangle$. Hence, we can state that Γ is the functor of equivalence between L-ChuCors and L-CLOS. □

5 Conclusions and future work

After introducing the basic definitions concerning the L-fuzzy extension of formal concept analysis, as well as those concerning L-Chu correspondences, the categories associated to L-formal contexts and L-CLOS are defined and, finally, we provide a constructive proof of the equivalence between the categories of L-formal contexts with L-Chu correspondences as morphisms and that of completely lattice L-ordered sets and their corresponding morphisms. As a result, we obtain that the generalized approaches based on L-Chu correspondences and those on completely L-lattice ordered sets are mutually interchangeable.

Roughly similar results, in essence, have already been obtained, for instance, in [24]. In that paper, a new notion of morphism on formal contexts resulted in a category equivalent to both the category of complete algebraic lattices and Scott continuous functions, and a category of information systems and approximable mappings.

Other researchers have studied as well the relationships between Chu constructions and L-fuzzy FCA. For instance, in [14] FCA is linked to both order-theoretic developments in the theory of Galois connections and to Chu spaces; as a result, not surprisingly from our previous works, they obtain further relationships between formal contexts and topological systems within the category of Chu systems. Recently, Solovyov, in [41], extends the results of [14] to clarify the relationships between Chu spaces, many-valued formal contexts of FCA, lattice-valued interchange systems and Galois connections.

Potential applications are primary motivations for future work, for instance, to consider possible classes of formal L-contexts induced from existing datamining notions, and study its associated categories.

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