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Multi-adjoint concept lattices with heterogeneous conjunctors and hedges

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Abstract This paper is related, on the one hand, to the framework of multi-adjoint concept lattices with heterogeneous conjunctors and, on the other hand, to the use of intensifying hedges as truth-stressers. Specifically, we continue on the line of recent works by Belohlavek and Vychodil, which use intensifying hedges as a tool to reduce the size of a concept lattice. In this paper we use hedges as a reduction tool in the general framework of multi-adjoint concept lattices with heterogeneous conjunctors.

Keywords Galois connection · Residuated algebra · Formal concept analysis · Multi-adjoint concept lattices

1 Introduction

Formal Concept Analysis (FCA) is a very active topic for several research groups throughout the world [1,5,10,13,20,22,28,33]. In this work, the authors aim to merge recent advances obtained in this area: on the one hand, the use of hedges as operators which allow one to modulate the size of fuzzy concept lattices [6] and, on the other hand, the consideration of heterogeneous conjunctors in the general approach to fuzzy FCA so-called multi-adjoint framework [25].

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The hedges were introduced to FCA in a fuzzy setting as a way to address one of the most recognized problems in FCA—reduction of the size of concept lattice. Linguistic hedges represent important language connectives whose study in the context of fuzzy logic has been initiated by Zadeh, see [34]. Our notion of a linguistic hedge as unary logical connective is close to that of Hájek [14,15]. In particular, we use the notion of a truth-stressing hedge of [5], called an idempotent truth-stresser in [4]. Linguistic hedges such as "very" or "extremely" are inserted in the description of the concept-forming operators and become parameters for FCA that control the number of formal concepts extracted from data. From a broader perspective, linguistic hedges represent a feasible way to parameterize methods for knowledge extraction from data that enables one to emphasize or suppress extracted patterns while keeping their interpretation [6]. The study of reduction in concept lattices is an interesting research topic because the size of a concept lattice can be exponential on the size of the context. The idea of reducing concept lattices using hedges was used in [5], a generalization which enables one to use a different hedge for each attribute was proposed in [6]. Hedges for isotone fuzzy concept-forming operators were described in [19].

On the other hand, the multi-adjoint approach was originally developed within the field of generalized fuzzy logic programming [29] introducing a much more flexible syntax and enabling the use of several optimization techniques used in areas such as functional programming [16], and other approaches ranging from the implementation of query languages [2] to more theoretical ones [30] and, finally, applied in the research area of FCA. The main feature that the multi-adjoint approach provides to FCA is related to its ability to encode preferences, as this was also one of its virtues in the logic programming framework (see the last section). The great flexibility provided by this approach makes it a suitable framework to work with, especially taking into account its numerous possible instantiations. In [28], the multi-adjoint philosophy was applied to the fuzzy extension of FCA, relating some of the existing approaches [27]. Since then, it has been recognized as an important and unifying framework for generalization in this area [7,17,31,32], and a number of possible instantiations of this paradigm have been introduced in FCA (the multi-adjoint t-concepts [24], dual multi-adjoint concept lattices [26], multi-adjoint property- or object-oriented concept lattices [23], multi-adjoint concept lattices with heterogeneous conjunctors [25]), and related areas such as solving relation equations [9], or rough set theory [8].

The recently introduced multi-adjoint approach with heterogeneous conjunctors has an interesting feature in that some quasi-closure operators arise which, although do not directly allow one to prove the complete lattice structure of the resulting set of concepts as usual, i.e. in terms of a Galois connection, actually do provide means to manually build the operators for suprema and infima of a set of concepts. The core notion in [25] is that of P-connected pair of posets which, in some sense, turns out to be a more abstract notion than a linguistic hedge. As a consequence of this observation, due to Radim Belohlavek, we now focus on the use of the specific properties of hedges in order to import some results related to the size of fuzzy concept lattices to the more general framework of [25].

In this paper, we continue on the line of recent works by Belohlavek and Vychodil, which use intensifying hedges as a tool to reduce the size of a concept lattice. Specifically, we use hedges as a reduction tool in the general framework of multi-adjoint concept lattices with heterogeneous conjunctors.

The structure of the paper is the following: in Section 2 the preliminary definitions are introduced, interested readers will obtain further comment on the intuitions underlying the definitions in the original papers [6, 25]; the main results are presented in Section 3. In the last section we recover an example which illustrates how easy is to encode a preferred set of attributes, and then hedges are used to reduce the size of the associated concept lattices.

2 Preliminary definitions and results

In this section, we introduce the basic definitions and preliminary results which will be used later in the core of this work.

Definition 1 Let $(L, \preceq, \top, \bot)$ be a complete lattice, a truth-stressing hedge in L is a mapping $*\colon L \to L$ satisfying, for each $x, y \in L$,

$$*(\top) = \top, \tag{1}$$
$$*(x) \preceq x, \tag{2}$$
$$x \preceq y \text{ implies } *(x) \preceq *(y), \tag{3}$$
$$*(*(x)) = *(x) \tag{4}$$

$\mathrm{fix}(*)$ denotes the set of fixed points of $*$ in L, i.e. $\mathrm{fix}(*) = \{a \in L \mid *(a) = a\}$.

Later in this work, we will need the following two lemmas:

Lemma 1 *Let $(L, \preceq)$ be a complete lattice, for any mapping $*\colon L \to L$ satisfying (2), (3), and (4) we have, for each $x_i \in L$,*

$$\bigvee_{i \in I} *(x_i) = *(\bigvee_{i \in I} *(x_i)) \quad \text{and} \quad *(\bigwedge_{i \in I} *(x_i)) = *(\bigwedge_{i \in I} x_i). \tag{5}$$

In addition, if we have $x_j = \bigvee_{i \in I} x_i$ for some $j \in I$ then

$$*(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} *(x_i). \tag{6}$$

Similarly, if we have $x_j = \bigwedge_{i \in I} x_i$ for some $j \in I$ then

$$*(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} *(x_i). \tag{7}$$

Proof The proof of (5) can be found in [6]. (6) and (7) follow from (5) and (3). □

Lemma 2 *(a) Let $*\colon L \to L$ be a mapping satisfying (2), (3), and (4). Then $\mathrm{fix}(*)$ is a complete $\vee$-subsemilattice of L.*
(b) Let K be a complete $\vee$-subsemilattice of L then the mapping $*_K : L \to L$ defined by

$$*_K(x) = \bigvee\{y \in K \mid y \leq x\}$$

satisfies (2), (3), and (4). In addition, $*_{\mathrm{fix}(*)} = *$ and $\mathrm{fix}(*_K) = K$.

Proof (a) follows directly from Lemma 1.

(b) Let $*_K(x) = \bigvee\{y \in K \mid y \leq x\}$ for each $a \in L$. We have (2) holds true by properties of suprema. If $x \leq z$ then $*_K(x)$ is supremum of a smaller (by means of subsethood) set than $*_K(z)$; thus $*_K(x) \leq *_K(z)$, i.e. (3) holds true. (4) is obvious.

Now, we have $*_{\mathrm{fix}(*)}(x) = \bigvee\{y \mid y \in \mathrm{fix}(*) \ : \ y \leq x\} = \bigvee\{*(y) \mid *(y) \in L \ : \ *(y) \leq x\} = \bigvee\{*(y) \mid *(y) \in L \ : \ *(y) \leq *(x)\} = *(x)$.

Finally, $x \in \mathrm{fix}(*_K)$ means $x = *_K(x) = \bigvee\{y \in K \mid y \leq x\}$; now, as K is a complete $\vee$-subsemilattice, we have that $x \in K$. It is straightforward to check that any $x \in K$ is a fixed point of $*_K$. □

By Lemma 2, given a truth-stressing hedge $*$, the set fix($*$) is a complete $\vee$-subsemilattice. Now we will introduce the basic notions of multi-adjoint concept lattices with heterogeneous conjunctors, in order to show how both frameworks, hedges and heterogeneous conjunctors, can be merged.

Firstly, let us introduce a bit of terminology: in the rest of this work we will call a mapping $*\colon L \to L$ satisfying (2), (3), and (4) an **intensifying hedge**, following the terminology introduced in [3]. In terms of interior structures $(L, \preceq)$, a mapping satisfying (2)–(4) is an interior operator on the lattice of truth degrees.

The two main notions on which multi-adjoint concept lattices with heterogeneous conjunctors is defined are given below: the P-connection between posets, and the adjoint triples.

Definition 2 Given posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P, \leq)$, we say that P_1 and P_2 are *P-connected* if there exist non-decreasing mappings $\psi_1\colon P_1 \to P$, $\phi_1\colon P \to P_1$, $\psi_2\colon P_2 \to P$ and $\phi_2\colon P \to P_2$ verifying that $\phi_1(\psi_1(x)) = x$, and $\phi_2(\psi_2(y)) = y$, for all $x \in P_1$, $y \in P_2$.

Definition 3 Let $(P_1, \leq_1), (P_2, \leq_2), (P_3, \leq_3)$ be posets, and consider mappings $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$, then $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if: $x \leq_1 z \swarrow y$ iff $x \,\&\, y \leq_3 z$ iff $y \leq_2 z \nwarrow x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$.

From Lemma 2 we immediately obtain the following result:

Corollary 1 *Consider lattices $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L, \leq)$, and assume that L_1 and L_2 are* L-connected*, then:*

(a) If $\psi_1 \circ \phi_1$ is contractive (i.e. satisfies (2)*) then L_1 is isomorphic to a complete $\vee$-subsemilattice of L.*
(b) If $*\colon L_1 \to L_1$ is an intensifying hedge (i.e. satisfies properties (2)*,* (3)*, and* (4)*) then the composition $\psi_1 \circ * \circ \phi_1\colon L \to L$ is an intensifying hedge in* $\mathrm{fix}(\psi_1 \circ \phi_1)$.

We will also need the following lemma.

Lemma 3 *Let $(L, \preceq), (L_1, \preceq_1), (L_2, \preceq_2)$ be complete lattices and let $(\&, \swarrow, \nwarrow)$ be an adjoint triple. For $a, a_i \in L_1$, $b, b_i \in L_2$, with i in a set of indexes I, we have*

$$\textstyle\bigvee_{i\in I}(a_i \,\&\, b) = (\bigvee_{1 i\in I} a) \,\&\, b \quad \textit{and} \quad \bigvee_{i\in I}(a \,\&\, b_i) = a \,\&(\bigvee_{2 i\in I} b_i) \tag{8}$$

Now, we introduce basic definitions of multi-adjoint context and multi-adjoint concept lattices [25].

Definition 4 A multi-adjoint frame is a tuple $(L_1, L_2, P, \&_1, \swarrow_1, \nwarrow_1, \ldots, \&_n, \swarrow_n, \nwarrow_n)$ where L_is are complete lattices and P is a poset, such that $(\&_i, \swarrow_i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P for all $i = 1, \ldots, n$.

Definition 5 Let $(L_1, L_2, P, \&_1, \ldots, \&_n)$ be a multi-adjoint frame, a *multi-adjoint context* is a tuple (A, B, R, σ) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon B \to \{1, \ldots, n\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.

Given a complete lattice $(L, \preceq)$ such that L_1 and L_2 are L-connected, a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$, and a context (A, B, R, σ), we can define the mappings ${}^{\uparrow_{c\sigma}}\colon L^B \to L^A$ and ${}^{\downarrow^{c\sigma}}\colon L^A \to L^B$ defined for all $g \in L^B$ and $f \in L^A$ as follows:

$$g^{\uparrow_{c\sigma}}(a) = \psi_1(\inf\{R(a,b) \swarrow^{\sigma(b)} \phi_2(g(b)) \mid b \in B\}) \tag{9}$$

$$f^{\downarrow^{c\sigma}}(b) = \psi_2(\inf\{R(a,b) \nwarrow_{\sigma(b)} \phi_1(f(a)) \mid a \in A\}) \tag{10}$$

The notion of concept is defined as usual. A *concept* is a pair $\langle g, f\rangle$ satisfying $g \in L^B$, $f \in L^A$ and that $g^{\uparrow_{c\sigma}} = f$ and $f^{\downarrow^{c\sigma}} = g$.

Definition 6 Given complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(L, \preceq)$, where L_1 and L_2 are L-connected, the set of *multi-adjoint L-connected concepts* associated with a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$ and context (A, B, R, σ) is given by $\mathfrak{M}_L = \{\langle g, f\rangle \mid \langle g, f\rangle \text{ is a concept}\}$.

The main theorem of concept lattices in [25] proves that $\mathfrak{M}_L$ has the structure of a complete lattice:

Theorem 1 **([25])** *Let $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, and $(L, \preceq)$ be three complete lattices, where L_1 and L_2 are L-connected, a multi-adjoint context (A, B, R, σ), and a multi-adjoint frame $(L_1, L_2, L, \&_1, \ldots, \&_n)$, the multi-adjoint L-connected concept lattice $\mathfrak{M}_L$ is actually a complete lattice with the meet and join operators $\curlywedge, \curlyvee\colon \mathfrak{M}_L \times \mathfrak{M}_L \to \mathfrak{M}_L$ defined below, for all $\langle g_1, f_1\rangle, \langle g_2, f_2\rangle \in \mathfrak{M}_L$,*

$$\langle g_1, f_1\rangle \curlywedge \langle g_2, f_2\rangle = \langle \psi_2 \circ \phi_2(g_1 \wedge g_2), (f_1 \vee f_2)^{\downarrow^c \uparrow_c}\rangle,$$
$$\langle g_1, f_1\rangle \curlyvee \langle g_2, f_2\rangle = \langle (g_1 \vee g_2)^{\uparrow_c \downarrow^c}, \psi_1 \circ \phi_1(f_1 \wedge f_2)\rangle.$$

The order $\preceq$ which corresponds to $\curlywedge$ and $\curlyvee$ is defined as

$$\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle \quad \text{iff} \quad \phi_2(g_1) \leq \phi_2(g_2) \quad (\text{iff } \phi_1(f_2) \leq \phi_1(f_1)).$$

In what follows $\mathfrak{M}$ denotes multi-adjoint L-connected concept lattice of given context (A, B, R, σ). We will also omit subscript $\sigma(b)$ and write just $\swarrow$ instead of $\swarrow^{\sigma(b)}$.

3 Size reducing results for multi-adjoint concept lattices based on heterogeneous conjunctors

The size of the concept lattice $\mathfrak{M}$ can be reduced either by a suitable selection of a complete $\vee$-subsemilattice of L_1 (and/or L_2) and the use of a restriction of $\&$. The following proposition says that the selection of complete $\vee$-subsemilattices of L_1 (resp. L_2) yields a reduction of size of concept lattice and, moreover, preserves extents (resp. intents) of the original concept lattice.

Proposition 1 *Let* $\mathbf{A} = (L_1, L_2, P, \&_1, \dots, \&_n)$, $\mathbf{A}' = (K_1, L_2, P, \&_1', \dots, \&_n')$ *be multi-adjoint frames, s.t.* K_1 *is a complete* $\vee$*-subsemilattice of* L_1*, and* $\&_1', \dots, \&_n'$ *are restrictions of* $\&_1, \dots, \&_n$ *to* $K_1 \times L_2$ *and* $\psi_1' = \psi_1$*,* $\psi_2' = \psi_2$*,* $\phi_2' = \phi_2$*,* $\phi_1' = *_{K_1} \circ \phi_1$*, where* $*_{K_1}$ *is the hedge associated with* K_1 *as introduced in Lemma 2. Then,* $\mathrm{Ext}(\mathfrak{M}_{\mathbf{A}'}) \subseteq \mathrm{Ext}(\mathfrak{M}_{\mathbf{A}})$ *where* $\mathrm{Ext}(\mathfrak{M})$ *denotes the set of extents in* $\mathfrak{M}$.

Proof (sketch) We have $z \nwarrow' x = z \nwarrow x$, for each $x \in K_1, z \in P$, whence $f^{\downarrow'} = f^{\downarrow}$, for each $f\colon A \to \psi_1(K_1)$ where $\psi_1(K_1) \subseteq L$ is the image of ψ_1 (note that $\swarrow'$ is well-defined since K_1 is complete $\vee$-subsemilattice) and thus by Proposition 16 in [25] $\mathrm{Ext}(\mathfrak{M}_{\mathbf{A}'}) \subseteq \mathrm{Ext}(\mathfrak{M}_{\mathbf{A}})$. $\square$

Remark 1 (a) One can state a dual proposition to Proposition 1 for intents. Let $\mathbf{A} = (L_1, L_2, P, \&_1, \dots, \&_n)$, $\mathbf{A}'' = (L_1, K_2, P, \&_1'', \dots, \&_n'')$ be multi-adjoint frames such that K_2 is a complete $\vee$-subsemilattice of L_2, and $\&_1'', \dots, \&_n''$ are restrictions of $\&_1, \dots, \&_n$ to $L_1 \times K_2$ and $\phi_2'' = *_{K_2} \circ \phi_2$.

(b) We have $z \nwarrow' x = z \nwarrow x$; For the right residuum of $\&'$ we have

$$\begin{aligned} z \swarrow' y &= \bigvee\nolimits_2 \{x \in K_1 \mid x \,\&\, y \le z\} = \bigvee\nolimits_2 \{*_{K_1}(x) \in L_1 \mid *_{K_1}(x) \,\&\, y \le z\} = \\ &= \bigvee\nolimits_2 \{*_{K_1}(x) \in L_1 \mid x \,\&\, y \le z\} = \{*_{K_1}(x) \in L_1 \mid x \,\&\, y \le z\} \\ &= \bigvee\nolimits_2 *_{K_1} (\{x \in L_1 \mid x \,\&\, y \le z\}) = *_{K_1}(z \swarrow y). \end{aligned}$$

(c) Even in a crisp case we can find a counterexample to show that we do not generally have $\mathrm{Int}(\mathfrak{M}_{\mathbf{A}'}) \subseteq \mathrm{Int}(\mathfrak{M}_{\mathbf{A}})$ even in a crisp case. Use $L_1, L_2, L_3 = \langle\{0,1\}, \le\rangle$, $\& = \min$, $K_1 = \{0\}$, $\phi_1, \phi_2, \psi_1, \psi_2$ being the identity, and full incidence relation R.

Corollary 2 *(a) Considering* $\mathfrak{M}_{\mathbf{A}'}$*,* $\mathfrak{M}_{\mathbf{A}}$ *from Proposition 1 we have* $\langle \mathrm{Ext}(\mathfrak{M}_{\mathbf{A}'}), \preceq_{\mathrm{Ext}}\rangle$ *is complete* $\bigwedge$*-subsemilattice of* $\langle \mathrm{Ext}(\mathfrak{M}_{\mathbf{A}}), \preceq_{\mathrm{Ext}}\rangle$*, where* $\preceq_{\mathrm{Ext}}$ *is the extent component of* $\preceq$ *(i.e.* $g_1 \preceq g_2$ *iff* $\phi_2(g_1) \le \phi_2(g_2)$*).*

(b) Collection of extent parts $\mathrm{Ext}(\mathfrak{M}_{\mathbf{A}'})$ *of the reduced concept lattices* $\mathfrak{M}_{\mathbf{A}'}$ *with the order relation 'being a complete* $\bigwedge$*-subsemilattice' forms a complete lattice.*

Proof Follows from Proposition 1 and Theorem 1.

Example 1 This example demonstrates the reduction of the concept lattice of the following context.

	1	2	3
1	0	0.25	0.75
2	1	0.5	1
3	0	1	0.25

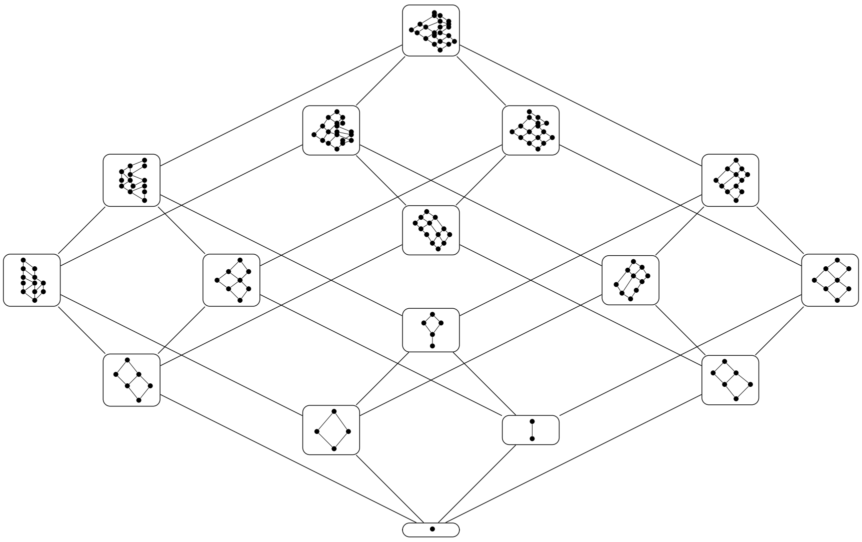


Fig. 1 Demonstration of a reduction of concept lattice for all complete $\vee$-subsemilattices K_1 of the 5-element chain in means of Proposition 1. Nodes contain reduced concept lattices obtained by application of different K_1s, edges represent hierarchy described in Corollary 2 (b).

We use the following setting: L_1, L_2, L_3 are 5-element chains $\{0, 0.25, 0.5, 0.75, 1\}$, &s are the Łukasiewicz conjunctions, and $\phi_1, \phi_2, \psi_1, \psi_2$ are the identities. By use of each complete $\vee$-subsemilattice K_1 instead of L_1 in a way described in Proposition 1, we obtain 16 concept lattices hierarchically ordered with a complete lattice structure (see Fig. 1).

The following proposition says that by selection of complete $\vee$-subsemilattices of both L_1 and L_2 we obtain a reduction of the size as well. However, the preservation of extents (or intents) is lost, see Remark 1(c).

Remark 2 Now, consider a little more general setting of concept-forming operators as in the Proposition 1 where we have different K_1 for each attribute.[1] Note that in this more general setting the Proposition 1 still holds true, i.e. we still obtain extent reduct of the original multi-adjoint concept lattice.

Furthermore, note that putting $K_1 = \{\perp\}$ has the same effect as removal of the corresponding column. In that way, Proposition 1 generalizes [11, Propositions 30 and 31].

Finally, Note that in the crisp setting the method of reduction becomes equivalent to selection of attributes.

Proposition 2 *Let* $\mathbf{A} = (L_1, L_2, L, \&_1, \ldots, \&_n), \mathbf{A}' = (K_1, K_2, L, \&'_1, \ldots, \&'_n)$ *be multi-adjoint frames, s.t.* K_1 *is a complete* $\vee$*-subsemilattice of* L_1*,* K_2 *is a complete* $\vee$*-subsemilattice of* L_2*, and* $\&'_1, \ldots, \&'_n$ *are restrictions of* $\&_1, \ldots, \&_n$ *to* $K_1 \times K_2$*, and* $\phi'_1 = *_{K_1} \circ \phi_1, \phi'_2 = *_{K_2} \circ \phi_2$*. Then we have* $|\mathfrak{M}_{\mathbf{A}'}| \leq |\mathfrak{M}_{\mathbf{A}}|$*.*

[1] To make this formally correct, we would require to extend the framework of multi-adjoin frames to allow individual lattice for each attribute and/or object. To keep the framework simpler we do not make this extension and limit ourselves to this verbal description.

Proof (sketch) By applying Proposition 1 and Remark 1 we obtain the result. $\square$

Remark 3 Continuing the argument made in Remark 2, Proposition 2 generalizes [11, Proposition 32].

In the next result we show how to generate new adjoint triples using hedges.

Lemma 4 *Assume* $(\&, \swarrow, \nwarrow)$ *is an adjoint triple with respect to* L_1*,* L_2*,* P*, and* $*_1 \colon L_1 \to L_1$*,* $*_2 \colon L_2 \to L_2$ *are hedges, then* $x \,\&^*\, y = *_1(x) \,\&\, *_2(y)$ *has two residuated implications* $\swarrow^*, \nwarrow_*$ *which form a new adjoint triple with respect to* L_1*,* L_2*,* P*, if and only if the following equalities hold:*

$$*_1(z \swarrow *_2(y)) = *_1(\bigvee\{x \mid x \,\&^*\, y \le z\}) \tag{11}$$

$$*_2(z \nwarrow *_1(x)) = *_2(\bigvee\{y \mid x \,\&^*\, y \le z\}) \tag{12}$$

Proof "⇒": Let $(\&^*, \swarrow^*, \nwarrow_*)$ be an adjoint triple. We have

$$x \,\&^*\, y \le z \quad \text{iff} \quad y \preceq_2 z \nwarrow_* x$$

by definition. In particular, we obtain

$$*_1(x) \,\&^*\, *_2(y) \le z \quad \text{iff} \quad *_2\,(y) \preceq_2 z \nwarrow_* *_1(x)$$

and $*_1(x) \,\&^*\, *_2(y) = *_1(*_1(x)) \,\&\, *_2(*_2(y)) = *_1(x) \,\&\, *_2(y) = x \,\&^*\, y$. Hence, we have

$$x \,\&^*\, y \le z \quad \text{iff} \quad *_2\,(y) \preceq_2 z \nwarrow_* *_1(x)$$

From (3) and (4) we obtain that

$$*_2(y) \preceq_2 z \nwarrow_* *_1(x) \quad \text{implies} \quad *_2\,(y) \preceq_2 *_2(z \nwarrow_* *_1(x))$$

and due to (2) we have

$$*_2(y) \preceq_2 *_2(z \nwarrow_* *_1(x)) \quad \text{implies} \quad *_2\,(y) \preceq_2 z \nwarrow_* *_1(x)$$

Therefore, we have

$$x \,\&^*\, y \le z \quad \text{iff} \quad *_2\,(y) \preceq_2 *_2(z \nwarrow_* *_1(x)). \tag{13}$$

Analogously, we obtain

$$*_1(x) \,\&\, *_2(y) \le z \quad \text{iff} \quad *_2\,(y) \preceq_2 *_2(z \nwarrow *_1(x)) \tag{14}$$

By setting $y = (z \nwarrow *_1(x))$, in Equation (13), and $y = (z \nwarrow_* *_1(x))$, in Equation (14), we obtain equivalent inequalities $*_2(z \nwarrow *_1(x)) \preceq *_2(z \nwarrow_* *_1(x))$, $*_2(z \nwarrow *_1(x)) \succeq *_2(z \nwarrow_* *_1(x))$ respectively. Thus we have

$$*_2(z \nwarrow *_1(x)) = *_2(z \nwarrow_* *_1(x)).$$

Which is equal to (12). The first equation (11) can be obtained dually.

"⇐": Assume that (12) holds true. By properties of adjointness, to show that $\&^*$ generates an adjoint triple we need to show that

$$R = \{y \mid *_1(x) \,\&\, *_2(y) \le z\}$$

has a greatest element.

In the previous part, we proved that

$$*_1(x) \mathbin{\&} *_2(y) \leq z \quad \text{iff} \quad *_2(y) \preceq_2 *_2(z \nwarrow *_1(x)) \tag{15}$$

hence $R = \{y \mid *_2(y) \preceq *_2(z \nwarrow *_1(x))\}$. Now, if R has no greatest element, i.e. $\bigvee R \notin R$, then we have $*_2(\bigvee R) \not\preceq *_2(z \nwarrow *_1(x))$ which is a contradiction with the assumption. By the contradiction we proved that R has a greatest element. $\square$

Proposition 3 *Let* $\mathbf{A} = (L_1, L_2, P, \&_1, \ldots, \&_n)$ *be a multi-adjoint frame, let* $*_1, *_2$ *be hedges on* L_1 *and* L_2*, respectively. Let* $\mathbf{A}' = (\mathrm{fix}(*_1), \mathrm{fix}(*_2), P, \&_1', \ldots, \&_n')$ *s.t.* $\&_1', \ldots, \&_n'$ *are restrictions of* $\&_1, \ldots, \&_n$ *to* $\mathrm{fix}(*_1) \times \mathrm{fix}(*_2)$*, and* $\phi_1' = *_1 \circ \phi_1, \phi_2' = *_2 \circ \phi_2$*. Let* $\mathbf{A}^* = (L_1, L_2, P, \&_1^*, \ldots, \&_n^*)$ *be a multi-adjoint frame where* $\&_i^*$ *is defined by* $a \mathbin{\&_i^*} b = *_1(a) \mathbin{\&_i} *_2(b)$*, for all* $i \in \{1, \ldots, n\}$*, and the conditions in Lemma 4 are satisfied. Then* $(\mathfrak{M}_{A'}, \preceq')$ *and* $(\mathfrak{M}_{A^*}, \preceq^*)$ *are isomorphic.*

Proof Let $\mathbb{K} = (A, B, R, \sigma)$ be a formal context, denote by ${}^{\uparrow}, {}^{\downarrow}$ concept-forming operators induced by $\mathbb{K}$ and $\mathbf{A}'$ and denote by ${}^{\Uparrow}, {}^{\Downarrow}$ concept-forming operators induced by $\mathbb{K}$ and $\mathbf{A}^*$. Furthermore, denote compositions $\psi_1 \circ *_1 \circ \phi_1$ and $\psi_2 \circ *_2 \circ \phi_2$ by $\bullet_1$ and $\bullet_2$ respectively.

For each mapping $g : B \to L$ we have

$$\begin{aligned}
\bullet_1(g^{\Uparrow}(a)) &= \bullet_1(\psi_1 \textstyle\bigwedge_1 (R(a,b) \swarrow^* \phi_2(g(b)))) \\
&= \psi_1 *_1 (\phi_1 \psi_1 \textstyle\bigwedge_1 (R(a,b) \swarrow^* (\phi_2(g(b))))) \\
&= \psi_1 \textstyle\bigwedge_1 *_1 (\textstyle\bigvee_1 \{x \mid *_1(x) \mathbin{\&} *_2(\phi_2(g(b))) \leq R(a,b)\}) \\
&\overset{(\Delta)}{=} \psi_1 \textstyle\bigwedge_1 (\textstyle\bigvee_1 \{*_1(x) \mid *_1(x) \mathbin{\&} *_2(\phi_2(g(b))) \leq R(a,b)\}) \\
&= \psi_1 \textstyle\bigwedge_1 (\textstyle\bigvee_1 \{x \in \mathrm{fix}(*_1) \mid x \mathbin{\&} *_2(\phi_2(g(b))) \leq R(a,b)\}) \\
&= \psi_1 \textstyle\bigwedge_1 (R(a,b) \swarrow' *_2(\phi_2(g(b)))) \\
&= \psi_1 \textstyle\bigwedge_1 (R(a,b) \swarrow' \phi_2 \psi_2 *_2 (\phi_2 g(b))) \\
&= \psi_1 \textstyle\bigwedge_1 (R(a,b) \swarrow' \phi_2 \bullet_2 (g(b))) \\
&= (\bullet_2 \circ g)^{\uparrow}(a)
\end{aligned}$$

where (Δ) is due to Lemma 1 (6) and the fact that $\&$ generates adjoint triple and thus $\bigvee_1 \{x \mid *_1(x) \mathbin{\&} *_2(\phi_2(g(b))) \leq R(a,b)\})$ has a greatest element. Dually, we have $\bullet_2 \circ (f^{\Downarrow}) = (\bullet_1 \circ f)^{\downarrow}$ for each mapping $f : A \to L$. From that we have

$$g^{\uparrow} = \bullet_1 \circ (g^{\Uparrow}) \quad \text{and} \quad f^{\downarrow} = \bullet_2 \circ (f^{\Downarrow})$$

for each $g : B \to \mathrm{fix}(\bullet_2)$, $f : A \to \mathrm{fix}(\bullet_1)$. As a result of the previous equalities, we have that $\bullet_2$ is a surjective mapping $\mathrm{Ext}(\mathfrak{M}_{A^*}) \to \mathrm{Ext}(\mathfrak{M}_{A'})$ and $\bullet_1$ is a surjective mapping $\mathrm{Int}(\mathfrak{M}_{A^*}) \to \mathrm{Int}(\mathfrak{M}_{A'})$. In addition, for $g \in \mathrm{Ext}(\mathfrak{M}_{A^*})$ we have

$$\begin{aligned}
\bullet_2(g)^{\Uparrow}(a) &= \psi_1 \bigwedge\nolimits_1 R(a,b) \swarrow^{*} \phi_2\psi_2 *_2 \phi_2(g(b)) \\
&= \psi_1 \bigwedge\nolimits_1 \bigvee\nolimits_2 \{x \mid *_1(x) \mathbin{\&} *_2 *_2 (\phi_2(g(b))) \leq R(a,b)\} \\
&= \psi_1 \bigwedge\nolimits_1 \bigvee\nolimits_2 \{x \mid *_1(x) \mathbin{\&} *_2(\phi_2(g(b))) \leq R(a,b)\} \\
&= \psi_1 \bigwedge\nolimits_1 R(a,b) \swarrow^{*} \phi_2(g(b))) \\
&= g^{\Uparrow}(a)
\end{aligned}$$

and dually $\bullet_1(f)^{\Downarrow} = f^{\Downarrow}$. Putting it together, we have $g = g^{\Uparrow\Downarrow} = \bullet_1(g^{\Uparrow})^{\Downarrow} = \bullet_2(g)^{\uparrow\Downarrow}$ showing that $\uparrow\Downarrow$ is injective; whence $\bullet_1, \bullet_2$ are bijections.

To show that $\bullet_1, \bullet_2$ are order-preserving let $\langle g_1, f_1\rangle, \langle g_2, f_2\rangle \in \mathfrak{M}_{A^*}$. An extent of $\langle g_1, f_1\rangle \wedge \langle g_2, f_2\rangle$ is equal to $\psi_2\phi_2(g_1 \wedge g_2)$ by the main Theorem in [25].

For $g_1, g_2 \in \mathrm{Ext}(\mathfrak{M}_{A^*})$ we have

$$\begin{aligned}
\bullet_2\psi_2\phi_2(g_1 \wedge g_2) &= \psi_2 *_2 \phi_2\psi_2\phi_2(g_1 \wedge g_2) \\
&= \psi_2 *_2 \phi_2(g_1 \wedge g_2) \\
&= \psi_2 *_2 \phi_2 \bullet_2 (g_1 \wedge g_2) \\
&\overset{(\Delta)}{=} \psi_2\phi_2'(\bullet_2(g_1) \wedge \bullet_2(g_2))
\end{aligned}$$

Equality (Δ) is due to Corollary 1(b) since g_1, g_2 are fixpoints of $\psi_2 \circ \phi_2$.

Now, let $g_1, g_2 \in \mathrm{Ext}(\mathfrak{M}_{A'})$. We have

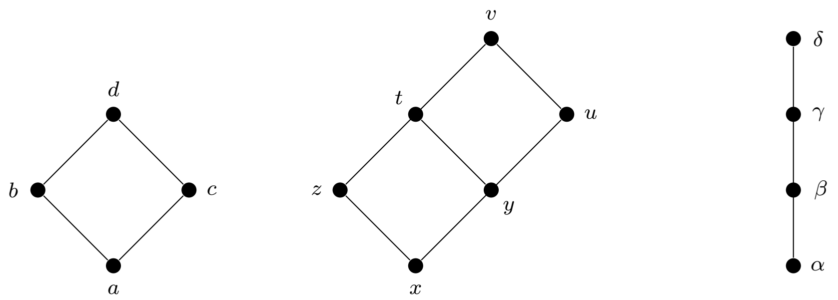
$$\begin{aligned}
(\psi_2\phi_2'(g_1 \wedge g_2))^{\uparrow\Downarrow} &= (\psi_2 *_2 \phi_2(g_1 \wedge g_2))^{\uparrow\Downarrow} \\
&= (\bullet_1(g_1 \wedge g_2))^{\uparrow\Downarrow} \\
&= (\bullet_1(g_1^{\uparrow\downarrow} \wedge g_2^{\uparrow\downarrow}))^{\uparrow\Downarrow} \\
&= (\bullet_1(\bullet_1(g_1^{\uparrow\Downarrow}) \wedge \bullet_1(g_2^{\uparrow\Downarrow}))^{\uparrow\Downarrow} \\
&\overset{(\Delta)}{=} (\bullet_1 \bullet_1 (g_1^{\uparrow\Downarrow} \wedge g_2^{\uparrow\Downarrow}))^{\uparrow\Downarrow} \\
&= (\bullet_1(g_1^{\uparrow\Downarrow} \wedge g_2^{\uparrow\Downarrow}))^{\uparrow\Downarrow} \\
&= (g_1^{\uparrow\Downarrow} \wedge g_2^{\uparrow\Downarrow})^{\Uparrow\Downarrow} \\
&\overset{(\nabla)}{=} \psi_2\phi_2(g_1^{\uparrow\Downarrow} \wedge g_2^{\uparrow\Downarrow})
\end{aligned}$$

Equality (Δ) is due to Corollary 1(b) since g_1, g_2 are fixpoints of $\psi_2 \circ \phi_2$; equality (∇) is due to [25, Lemma 21].

This proves that $\bullet_1, \bullet_2, \uparrow\Downarrow$, and $\downarrow\Uparrow$ are order-preserving. $\square$

Example 2 Consider the multi-adjoint frame depicted in Fig. 2 (structures are the same as in [25, Example 3 (Fig. 2)] where all $\&_i$'s coincide). Figure 3 depicts a formal context with two objects and two attributes, together with their associated multi-adjoint concept lattice.

In this final part, we follow the way in which the hedges are used in [6], i.e. we generalize concept-forming operators using intensifying hedges. Then we show how this is related to the theory described above.



	a	b	c	d
ψ_1	x	t	u	v

	α	β	γ	δ
ψ_2	x	y	t	v

	x	y	z	t	u	v
ϕ_1	a	b	a	b	c	d

	x	y	z	t	u	v
ϕ_2	α	β	γ	γ	δ	δ

&	α	β	γ	δ
a	x	x	x	x
b	x	y	v	v
c	x	y	y	t
d	x	y	v	u

$\swarrow$	α	β	γ	δ
t	d	d	c	c
u	d	d	d	d
v	d	d	d	b
x	d	a	a	a
y	d	d	c	a
z	d	a	a	a

$\nwarrow$	a	b	c	d
t	δ	β	δ	β
u	δ	δ	δ	δ
v	δ	δ	γ	γ
x	δ	α	α	α
y	δ	β	γ	β
z	δ	α	α	α

Fig. 2 L_1 (top left), L (top middle), L_2 (top right), connection operators $\phi_1, \phi_2, \psi_1, \psi_2$ (middle), adjoint triple $(\langle \&, \nwarrow, \swarrow \rangle)$ (bottom).

	1	2
1	u	v
2	v	y

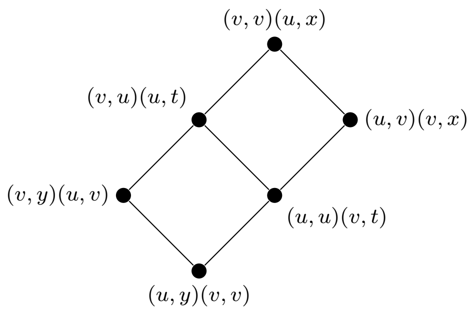


Fig. 3 Multi-adjoint formal context with two objects and two attributes (left) and the multi-adjoint concept lattice associated to the context (right).

Following the way how the truth-stressing hedges are used in [6] we can define the concept-forming operators as

$$g^{\triangle}(a) = \psi_1 {\bigwedge}_{1\, b \in B} R(a, b) \swarrow *_2(\phi_2(g(b)))$$
$$f^{\triangledown}(b) = \psi_2 {\bigwedge}_{2\, a \in A} R(a, b) \nwarrow *_1(\phi_1(f(a)))$$

Note that this is not strictly the same as approach used in Proposition 3 since $\swarrow$ and $\nwarrow$ are residua of the original adjoint operators &, not the altered operators $\&^*$. In fact, generally there is no base operation & such that $(\cdot) \swarrow *_2(\cdot)$ and $(\cdot) \nwarrow *_1(\cdot)$

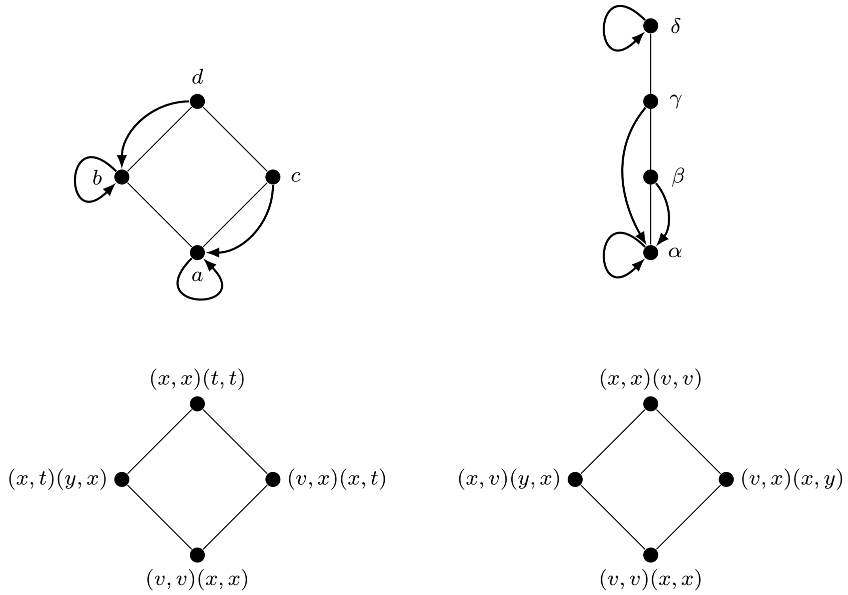


Fig. 4 Intensifying hedge on L_1 (left) and L_2 (right); concept lattices $\mathfrak{M}_{A'}$ (center left), $\mathfrak{M}_{A^*}$ (center right) of the formal context in Fig. 3; labels of nodes of $\mathfrak{M}_{A'}$ and $\mathfrak{M}_{A^*}$ represent characteristic vectors of corresponding extents and intents.

are its residua, since we do not generally have

$$x \leq z \swarrow *_2(y) \text{ iff } y \leq z \nwarrow *_1(x)$$

for each $x \in L_1, y \in L_2, z \in L$.

Proposition 4 *Assume* $(\&, \swarrow, \nwarrow)$ *is an adjoint triple,* $*_1, *_2$ *are intensifying hedges, and* $\swarrow^\diamond, \nwarrow_\diamond$ *being defined as* $z \swarrow^\diamond y = z \swarrow *_2(y)$, *and* $z \nwarrow^\diamond x = z \nwarrow *_1(x)$*; then* $\swarrow^\diamond, \nwarrow_\diamond$ *are part of an adjoint triple with conjunctor* $\&^\diamond$ *if and only if for all* x, y *the following equality holds*

$$x \,\&\, *_2(y) = *_1(x) \,\&\, y$$

and, in this case the previous value is the definition of $\&^\diamond$.

Proof For all x, y, z, on the one hand, we have

$$*_1(x) \,\&\, y \leq z \quad \text{iff} \quad y \preceq_2 z \nwarrow *_1(x) \quad \text{iff} \quad y \preceq_2 z \nwarrow^\diamond x$$

On the other hand, we have

$$x \,\&\, *_2(y) \leq z \quad \text{iff} \quad x \preceq_1 z \swarrow *_2(y) \quad \text{iff} \quad x \preceq_1 z \swarrow_\diamond y$$

Thus we have

$$y \preceq_2 z \swarrow^\diamond x \quad \text{iff} \quad x \preceq_1 z \nwarrow_\diamond y$$

is equivalent to

$$x \,\&\, *_2(y) \leq z \quad \text{iff} \quad *_1\,(x) \,\&\, y \leq z,$$

which is equivalent to $x \,\&\, *_2(y) = *_1(x) \,\&\, y$. $\square$

Given the frame $\mathbf{A} = (L_1, L_2, P, \&_1, \ldots, \&_n)$, we denote

$$\mathfrak{M}^*_L = \{\langle g, f\rangle \mid g \in L^B, f \in L^A, g^\vartriangle = f, g = f^\triangledown\}$$

a set of fixed points of $^\vartriangle, ^\triangledown$. As usual, an ordering $\preceq^*$ can be defined to obtain a complete lattice $(\mathfrak{M}^*_L, \preceq^*)$.

This lattice is isomorphic to $(\mathfrak{M}_{A'}, \preceq')$, where $\mathbf{A}' = (\mathrm{fix}(*_1), \mathrm{fix}(*_2), P, \&'_1, \ldots, \&'_n)$, as we will show below. First of all, we need the following technical result.

Lemma 5 *Given $(\mathfrak{M}^*_L, \preceq^*)$ and $(\mathfrak{M}_{A'}, \preceq')$, the equalities $\bullet_1(g^\vartriangle) = g^\uparrow$ and $\bullet_2(f^\triangledown) = f^\downarrow$ hold.*

Proof Given $a \in A$, the following equalities are obtained:

$$\begin{aligned}
\bullet_1(g^\vartriangle(a)) &= \bullet_1(\psi_1 \textstyle\bigwedge_{b\in B} R(a,b) \swarrow *_2\phi_2(g(b)))\\
&= \psi_1 *_1 (\textstyle\bigwedge_{1\, b\in B} R(a,b) \swarrow *_2(\phi_2(g(b))))\\
&= \psi_1 \textstyle\bigwedge_{1\, b\in B} *_1 (R(a,b) \swarrow *_2(\phi_2(g(b))))\\
&= \psi_1 \textstyle\bigwedge_{1\, b\in B} *_1 \bigvee_1 \{x \mid x \,\&\, *_2(\phi_2(g(b))) \le R(a,b)\}\\
&= \psi_1 \textstyle\bigwedge_{1\, b\in B} \bigvee_1 \{*_1(x) \mid x \,\&\, *_2(\phi_2(g(b))) \le R(a,b)\}\\
&\overset{(\Delta)}{=} \psi_1 \textstyle\bigwedge_{1\, b\in B} \bigvee_1 \{*_1(x) \mid *_1(x) \,\&\, *_2(\phi_2(g(b))) \le R(a,b)\}\\
&= \psi_1 \textstyle\bigwedge_{1\, b\in B} \bigvee_1 \{x \in \mathrm{fix}(*_1) \mid x \,\&\, *_2(\phi_2(g(b))) \le R(a,b)\}\\
&= \psi_1 \textstyle\bigwedge_{1\, b\in B} \bigvee_1 \{x \in \mathrm{fix}(*_1) \mid x \,\&\, \phi'_2(g(b)) \le R(a,b)\}\\
&= \psi_1 \textstyle\bigwedge_{1\, b\in B} R(a,b) \swarrow' \phi'_2(g(b))\\
&= g^\uparrow(a)
\end{aligned}$$

The equality (Δ) holds because each x satisfying $x \,\&\, y \le z$ satisfies $*_2(x) \,\&\, y \le z$ as well; and because for each $*_2(x)$ such that $*_2(x) \,\&\, y \le z$ there is x' (explicitly, $*_2(x)$) with $*_2(x') = *_2(x)$ such that $x' \,\&\, y \le z$. Dually, one can show $\bullet_2(f^\triangledown) = f^\downarrow$. □

The following result proves the announced isomorphism.

Theorem 2 *The concept lattices $(\mathfrak{M}^*_L, \preceq^*)$ and $(\mathfrak{M}_{A'}, \preceq')$ are isomorphic.*

Proof First of all, a concept $\langle g, f\rangle$ in $(\mathfrak{M}^*_L, \preceq^*)$ is considered and we prove that the pair $\langle \bullet_2(g), \bullet_1(f)\rangle$ is a concept in $(\mathfrak{M}_{A'}, \preceq')$, that is $[\bullet_2(g)]^\uparrow = \bullet_1(f), \bullet_2(g) = [\bullet_1(f)]^\downarrow$.

$$\begin{aligned}
[\bullet_2(g)]^\uparrow(a) &\overset{(1)}{=} \bullet_1(\psi_1 \textstyle\bigwedge_{b\in B} R(a,b) \swarrow *_2\phi_2(\bullet_2(g(b))))\\
&= \bullet_1(\psi_1 \textstyle\bigwedge_{b\in B} R(a,b) \swarrow *_2\phi_2\psi_2 *_2 \phi_2(g(b))))\\
&= \bullet_1(\psi_1 \textstyle\bigwedge_{b\in B} R(a,b) \swarrow *_2\phi_2(g(b)))\\
&= \bullet_1(g^\vartriangle)\\
&\overset{(2)}{=} \bullet_1(f)(a)
\end{aligned}$$

where (1) is given from the proof of Lemma 5 and (2) by hypothesis. Analogously, we obtain the other equality $\bullet_2(g) = [\bullet_1(f)]^{\downarrow}$.

Secondly, given a concept $\langle g, f\rangle$ in $(\mathfrak{M}_{A'}, \preceq')$, the pair $\left\langle f^{\triangledown}, g^{\vartriangle}\right\rangle$ needs to be a concept in $(\mathfrak{M}^*_L, \preceq^*)$, that is $g^{\vartriangle\triangledown} = f^{\triangledown}, g^{\vartriangle} = f^{\triangledown\vartriangle}$.

$$\begin{aligned}
g^{\vartriangle\triangledown}(b) &= \psi_2 \bigwedge\nolimits_{a\in A} R(a,b) \nwarrow *_1\phi_1(g^{\vartriangle}(a))\\
&\overset{(1)}{=} \psi_2 \bigwedge\nolimits_{a\in A} R(a,b) \nwarrow \phi_1'\psi_1' *_1 \phi_1(g^{\vartriangle}(a))\\
&\overset{(2)}{=} \psi_2 \bigwedge\nolimits_{a\in A} R(a,b) \nwarrow *_1\phi_1\psi_1 *_1 \phi_1(g^{\vartriangle}(a))\\
&\overset{(3)}{=} \psi_2 \bigwedge\nolimits_{a\in A} R(a,b) \nwarrow *_1\phi_1 \bullet_1 (g^{\vartriangle}(a))\\
&\overset{(4)}{=} \psi_2 \bigwedge\nolimits_{a\in A} R(a,b) \nwarrow *_1\phi_1(g^{\uparrow}(a))\\
&\overset{(5)}{=} \psi_2 \bigwedge\nolimits_{a\in A} R(a,b) \nwarrow *_1\phi_1(f(a))\\
&= f^{\triangledown}(b)
\end{aligned}$$

where (1) is given because $*_1\phi_1(g^{\vartriangle}(a)) \in \text{fix}(*_1)$ and $\phi_1'\psi_1'$ is the identity mapping in $\text{fix}(*_1)$. By definition we obtain (2) and (3). Finally, the equality (4) is obtained from Lemma 5 and (5) by hypothesis. The other equality is obtained analogously.

Now, we prove that these mappings are mutually inverse. First of all, we need to prove that $\left\langle [\bullet_1(f)]^{\triangledown}, [\bullet_2(g)]^{\vartriangle}\right\rangle = \langle g, f\rangle$, which is equivalent to prove $[\bullet_1(f)]^{\triangledown} = g$. Therefore, for each $b \in B$, we have

$$\begin{aligned}
[\bullet_1(f)]^{\triangledown}(b) &= \psi_2 \bigwedge\nolimits_{a\in A} R(a,b) \nwarrow *_1\phi_1 \bullet_1 (f(a))\\
&= \psi_2 \bigwedge\nolimits_{a\in A} R(a,b) \nwarrow *_1\phi_1\psi_1 *_1 \phi_1(f(a))\\
&= \psi_2 \bigwedge\nolimits_{a\in A} R(a,b) \nwarrow *_1\phi_1(f(a))\\
&= f^{\triangledown}(b)\\
&= g(b)
\end{aligned}$$

Secondly the equality $\left\langle \bullet_2(f^{\triangledown}), \bullet_1(g^{\vartriangle})\right\rangle = \langle g, f\rangle$ needs to be verified, which is equivalent to $\bullet_2(f^{\triangledown}) = g$, and this is straightforward from Lemma 5 and by hypothesis.

Finally, it is not difficult to check that supreme and infima are preserved; hence, we conclude that $(\mathfrak{M}^*_L, \preceq^*)$ and $(\mathfrak{M}_{A'}, \preceq')$ are isomorphic lattices. $\square$

4 A worked example

The following is an adaptation of an example taken from [28] – we refer the reader to that paper for further motivation on the formal context.

The problem is finding a suitable target journal to submit a paper. Assume the following sets of objects and attributes

$$\begin{aligned}
B &= \{\text{AMC}, \text{CAMWA}, \text{FSS}, \text{IEEE-FS}, \text{IJGS}, \text{IJUFKS}, \text{JIFS}\},\\
A &= \{\text{Impact Factor}, \text{Immediacy Index}, \text{Cited Half-Life}, \text{Best Position}\}.
\end{aligned}$$

We will consider a multi-adjoint frame with three different lattices:

$$([0,1]_{20},[0,1]_8,[0,1]_{100},\leq,\leq,\leq)$$

Where $[0,1]_m$ denotes a regular partition of $[0,1]$ into m pieces; & is discretizations of the product.

$\swarrow\colon [0,1]_{100}\times[0,1]_8 \to [0,1]_{20}$ and $\nwarrow\colon [0,1]_{100}\times[0,1]_{20} \to [0,1]_8$ are defined in terms of floor function $\lfloor\_\rfloor$ as:

$$z \swarrow y = \frac{\lfloor 20\cdot\min\{1, z/y\}\rfloor}{20} \qquad\qquad z \nwarrow x = \frac{\lfloor 8\cdot\min\{1, z/x\}\rfloor}{8}$$

Our input formal context is shown in Table 1.

Table 1 Fuzzy relation between the objects and the attributes.

R	AMC	CAMWA	FSS	IEEE-FS	IJGS	IJUFKS	JIFS
I. Factor	0.34	0.21	0.52	0.85	0.43	0.21	0.09
I. Index	0.13	0.09	0.36	0.17	0.1	0.04	0.06
Cited H.L.	0.31	0.71	0.92	0.65	0.89	0.47	0.93
Best P.	0.75	0.5	1	1	0.5	0.25	0.25

The problem of choosing a *suitable* journal depends on the definition of "suitability" we have in mind. For example, in the context (A,B,R,σ) where $\sigma(b) = \&$ for every $b\in B$, by the fuzzy subset $f\colon A\to[0,1]$ below:

$$f(\text{Impact Factor}) = 0.75, \qquad f(\text{Immediacy Index}) = 0.3,$$
$$f(\text{Cited Half-Life}) = 0.55, \qquad f(\text{Best Position}) = 0.5$$

Now, the problem consists in finding a multi-adjoint concept which represents the suitable journal as defined by the fuzzy set f.

$f^{\downarrow}(\text{AMC}) = 0.375$ $\quad f^{\downarrow}(\text{CAMWA}) = 0.25$ $\quad f^{\downarrow}(\text{FSS}) = 0.625$
$f^{\downarrow}(\text{JIFS}) = 0$ $\quad f^{\downarrow}(\text{IJGS}) = 0.25$ $\quad f^{\downarrow}(\text{IJUFKS}) = 0.125$
$f^{\downarrow}(\text{IEEE-FS}) = 0.5$

It turns out that the most suitable journal is FSS.

If we wanted to study entire multi-adjoint concept lattice of (A,B,R,σ), we would have to deal with a structure containing 248 concepts.Concept lattices of this size are not readable for a human user, so let us apply the methods of reduction described in this paper. Instead of $L_1 = [0,1]_{20}, L_2 = [0,1]_8$ we use their sublattices $K_1 = [0,1]_5, K_2 = [0,1]_4$, respectively.

Now, we obtain a multi-adjoint concept lattice with 59 concepts. The fact that FSS and IEEE-FS are the most two suitable journals is preserved but we lost information on which one of them is better:

$f^{\downarrow'}(\text{AMC}) = 0.5$ $\quad f^{\downarrow'}(\text{CAMWA}) = 0.25$ $\quad f^{\downarrow'}(\text{FSS}) = 0.75$
$f^{\downarrow'}(\text{JIFS}) = 0$ $\quad f^{\downarrow'}(\text{IJGS}) = 0.5$ $\quad f^{\downarrow'}(\text{IJUFKS}) = 0$
$f^{\downarrow'}(\text{IEEE-FS}) = 0.75$

This shows a natural behavior of the reduction.

5 Conclusions

We have studied a merge of two separately developed generalizations of fuzzy formal concept analysis – multi-adjoint concept lattices with heterogeneous conjunctors and concept lattices with truth-stressing hedges. We have shown how reduction of size of multi-adjoint concept lattice heterogeneous conjunctors can be obtained using the approach of the hedges. We also specified conditions under which the two approaches can be considered to be equal.

In the future the relation between proposed framework and the standard *projections* of contexts and pattern structures, as described in [12, 18, 21]. Moreover, our future research in this area includes study of geometry of multi-adjoint formal concepts and how this geometry is affected by the use of hedges.

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