The notion of weak-contradiction: definition and measures

Humberto Bustince Nicolás Madrid Manuel Ojeda-Aciego

***Abstract*—In this work we present a way to represent contradiction between fuzzy sets. This representation is given in terms of the notion of f-weak contradiction. Unlike other approaches, we do not define contradiction just by using one of the relations of f-weak-contradiction, but by considering the whole set of relations. This consideration avoids the need to fix an operator beforehand in order to take into account all the information between two fuzzy sets. As a result, we characterize the contradiction between fuzzy sets and define a family of measures of contradiction satisfying four interesting properties: symmetry, antitonicity, if the intersection is empty then the measure is one; and if there is an element in the intersection with degree of membership 1 then the measure is zero.**

***Index Terms*—N-Contradiction, Fuzzy Sets, Measure of Contradiction.**

I. Introduction

The notion of contradiction plays an important role in almost every field of human knowledge, since contradiction usually underlies the development of Sciences: in Physics, apparent contradictions between observed and expected consequences of one experiment evolves in new theories [21]; in Mathematics, the notion of contradiction is also in the basis of the so-called *reductio ad absurdum* method for proving statements.

Contradictions appear in Computer Science as well. Actually, their occurrences seem to be unavoidable in many frameworks, and therefore, formal approaches to deal with them are certainly necessary. In this line of thought, we can find several practical approaches. For instance, [14] describes ways to recognize contradictions in text mining; [6] analyses sets of inputs causing contradictions in medical databases; [7] defines a logic calculus in a database formed by contradictory sources of information; in decision making approaches [3], [11], where the notion of incomparability plays a crucial role in preference structures.

In a purely theoretical treatment of contradiction, we can find out also different approaches. For instance, [24] defines a relation to represent the contradiction between fuzzy sets; this approach is extended to Atanassov's intuitionistic fuzzy setting in [10]; [9] provides an axiomatic definition of *measures of contradiction* on fuzzy sets; and, if we establish a link between contradiction and inconsistency, in [13], [23], [18] the reader can find several *measures of inconsistency* under different logics.

Contradictions between two statements are usually related to negations [22], mainly in the sense that, if one is true then other cannot be true. In some cases the negation appears explicitly (as in the contradiction concerning with the statements *"x is small"* and *"x is not small"*) and in others implicitly (as in *"x is tiny"* and *"x is huge"*). Therefore, it is not strange that most of the approaches are built upon negation operators. In fact, our approach is based on one of those; specifically on the notion of *N-contradiction* given in [24]. However, we show that considering negation operators in such an approach, entails an unexpected behavior with the idea of symmetry between contradictory statements (see Section II). For this reason, our first goal in this work is to present and motivate an extension of the notion of N-contradiction, which is called *f-weak-contradiction*. Hence, we interpret the contradiction between two fuzzy-sets simply as: the greater the membership value of one, the lesser the membership value of the other.

But our main contribution is not just the definition of the *weak-contradiction*. We focus on a deeper interpretation and treatment of contradiction. In most approaches, contradiction is defined as a crisp relation, in the sense that two statements either are or are not contradictory. We believe that contradiction is not a crisp notion, since it involves degrees. For instance, in some cases it is really strong, as in the case of considering the statements *"to be a dog"* and *"to be a tree"*, but in others is slight, as between the statements *"to have short stature"* and *"to be a professional basketball player"*. Fuzziness is introduced within our approach by considering the whole set of f-weak contradiction relations instead of fixing one. Hence we do not represent the notion of contradiction as a binary relation, but as a set of binary relations. In this way, the contradiction between two fuzzy sets is determined by taking into account just the information provided by the fuzzy sets involved; i.e. without the need of taking any preliminary assumption.

The structure of the paper is as follows: In Section II we motivate and present the notion of f-weak contradiction as we are going to understand it in this paper. Section III is devoted to the idea of measure of contradiction, whereas in Section IV we discuss the definition of such measures in terms of negations. We finish with some conclusions and comments on future works, as well as the references.

H. Bustince is with the Department of Automatic and Computation, Public University of Navarra, Pamplona 31006, Spain (email: bustince@unavarra.es); N. Madrid is with the Center of Excellence IT4Innovations, Division University Ostrava, Institute for Research and Applications of Fuzzy Modeling. 30. dubna 22, 701 03 Ostrava 1, Czech Republic (email: nicolas.madrid@osu.cz) and M. Ojeda-Aciego is with the Dept. Matemática Aplicada, ETSI Informática, Univ. de Málaga, Blv. Louis Pasteur 35, 29071 Málaga, Spain. (email: aciego@ctima.uma.es).

II. On the notion of f-weak-contradiction

The family of measures of contradiction defined in this paper are based on the idea of f-weak-contradiction. That notion was presented originally in [2] as a slight generalization of the notion of N-contradiction given by Trillas et al. in [24].

Definition 1: Let A and B be two fuzzy sets defined over a nonempty universe $\mathcal{U}$ and let $f\colon [0,1] \to [0,1]$ be an antitonic mapping such that $f(0) = 1$. We say that A *is f-weak-contradictory w.r.t.* B if and only if $A(x) \leq f(B(x))$ holds for all $x \in \mathcal{U}$.

The idea underlying the notion of f-weak-contradiction is to manage contradictory information provided by two fuzzy sets via a mapping f. Fixed f and a value of $B(x)$ (for some $x \in \mathcal{U}$), the f-weak-contradiction determines an upper bound on the value of $A(x)$. Note that, as f is antitonic, the greater the value of $B(x)$, the smaller the upper bound, and then also the smaller the value of $A(x)$. Moreover, as $f(0) = 1$, we have that if $B(x) = 0$, then there is no restriction on the value of $A(x)$. Note that the restriction depends strongly on the chosen mapping; therefore, different mappings f somehow determine "different kinds of contradictions." Notice that every negation N allows to define N-weak contradiction, too.

In the rest of this section, we provide some motivations for this generalization of the notion of N-contradiction, and introduce some properties of f-weak-contradiction which facilitate the understanding of how such a notion represents the contradiction, besides of being used in the rest of the paper.

A. Motivating f-weak contradiction.

As we have said, the difference of our approach w.r.t. [24] is simple: we impose weaker conditions on mappings f used to determine the kind of contradiction. Specifically, Trillas et al. considered involutive negations (i.e antitonic mappings $N\colon [0,1] \to [0,1]$ such that $N(0) = 1$, $N(1) = 0$ and verifying $N(N(x)) = x$ for all $x \in [0,1]$), whereas our approach considers antitonic mappings $f\colon [0,1] \to [0,1]$ such that $f(0) = 1$.

The following facts motivate our choice to weaken these requirements:

- **Removing the symmetry imposed by involutions.** The consideration of involutive negations involves an "excessive" symmetry on the representation of contradiction. That is because if A and B are two fuzzy sets such that A is N-contradictory w.r.t. B (with N involutive), then necessarily B is N-contradictory w.r.t. A as well. This feature entails that for all element in the universe, if a degree α of A contradicts a degree β of B, then necessarily a degree β of A contradicts a degree α of B. In our opinion, in general, the notion of contradiction need not have this kind of symmetry. Consider the following situation:
  - The good behavior of one TV set can be measured according to two parameters: the quality of the image and the quality of the audio. Specifically, the more quality of audio and image, the better behavior of the TV set.
  - Under the above consideration, it is obvious that *a high degree of good behavior in TV* contradicts *a low degree of audio quality*.
  - However, *a low degree of good behavior in TV* does not necessarily contradict *a high degree of quality in audio* (since the bad behavior in TV could be due to a bad image quality).

  Therefore, this example shows that although the degree α of A contradicts a degree β of B, the degree β in A does not necessarily contradict a degree α of B (i.e. they can coexist). The following simple example shows the same behavior in a formal framework.

  Example 1: Two gas tanks A and B in a factory, by some requirement, should have the same pressure; however, we can only control the pressure of tank A which, in addition, can be modified only by controlling the injector. Moreover, by safety reasons, there exists a control mechanism which limits the pressure gain in A when its pressure is over certain threshold value. Assume that $P_A(t)$ and $P_B(t)$ are two fuzzy sets denoting the pressure in A and B, respectively, at time t, then the pressure injected in A could be given by a fuzzy set of the following form:

  $$I_A(t) = \min\{P_B(t), 2 - 2 \cdot P_A(t)\}$$

  Then, if we assume that the control system is working well, the values $P_A(t) = 0.9$ and $I_A(t) = 0.5$ represent a clear contradiction, since those values cannot be given by the control system (i.e. be given by the formula which defines the value of I_A). However, the values $P_A(t) = 0.5$ and $I_A(t) = 0.9$ do not represent any contradiction since if $P_B(t) = 0.9$, the formula above explains the value of $I_A(t)$. As a result, in this context, the symmetry imposed by the use of involutive negations need not hold. □

  We considered that the situations above, albeit extremely simplified, are enough to motivate, at least, the consideration of arbitrary negations in the definition of N-contradiction, since the requirement of the involution (i.e. $n^2 = id$) is the responsible of this perplexing symmetry.
- **Mappings f as variables.** Maybe the most important difference with respect to the approaches [5], [8] lies on the conceptual understanding of how to measure contradictions. In such approaches, a specific f-contradiction is fixed *a priori*. Subsequently, the measures of contradiction are defined with respect to the fixed f-contradiction. Contrariwise, we believe that the best way to measure the contradiction is by fixing the mapping f *a posteriori*. Specifically, given two fuzzy sets A and B, we study which is the best f-weak-contradiction to represent the contradiction between A and B. Thus, contradiction is determined by the information provided by the fuzzy sets themselves, without any *a priori* assumption. In other words, whereas in [5], [8] the notion of contradiction is modeled by *a specific* N-contradiction, in our approach the contradiction is represented by *the whole set* of different f-weak-contradictions.

  Note that this difference changes completely the conceptualization of the notion of contradiction: under this approach *"the notion of contradiction"* is not just a relation between fuzzy sets, but a set of relations.

Summarizing, and roughly speaking, we need to consider mappings f as variables instead of constants. Note that, under this assumption, the structure of the set of mappings considered to define f-weak-contradictions is now crucial.

- **But what about symmetry?** We recall that in the first motivational item we have removed the symmetry imposed by involutions. However, everybody has in mind that the idea of contradiction is, somehow, symmetric. That idea can be roughly described by: if A is "contradictory" w.r.t. B then B must be, somehow, also "contradictory" w.r.t. A as well; although possibly under different f-weak-contradictions. Unfortunately, considering only negation operators is not enough to guarantee that feature, since there are fuzzy sets A and B such that A is n_1-weak-contradictory w.r.t. B but B is not n_2-weak-contradictory w.r.t. A for any negation n_2; as the example below shows.

  Example 2: It is not difficult to check that, by assuming that the control system in Example 1 works good, I_A is n_1-weak-contradictory w.r.t. P_A, where:

$$n_1(x) = \begin{cases} 1 & \text{if } x \leq \frac{1}{2} \\ 2 - 2 \cdot x & \text{otherwise} \end{cases}$$

  Note that n_1 is effectively a negation operator, since is antitonic and satisfies $n_1(0) = 1$ and $n_1(1) = 0$. Nevertheless, there is no negation n_2 satisfying that P_A is n_2-weak-contradictory w.r.t. I_A. If such a negation would exist, then the values $P_A(t) > 0$ and $I_A(t) = 1$ could not be given by the restriction imposed by the n_2-weak-contradiction. However, the values $I_A(t) = 1, P_B(t) = 1$ and $P_A(t) = \frac{1}{4}$ can be clearly given by the control system. □

- **Something weaker than negations.** By the previous items, it is not feasible to consider just the class of negation operators. Therefore, we have two options: either imposing more restrictions on negation operators (considering a subclass) or reducing them (considering operators weaker than a negation).

  Let us recall that N-contradiction is based on the idea that two logic predicates φ and ψ are considered contradictory if the logic statement *"If ψ is true, then φ is not true"* holds. Now, if this implication is interpreted as a residual implication, and the negation is interpreted with a negation operator n, we obtain that *"If ψ is true then φ is not true"* holds if and only if the inequality $\psi \leq n(\varphi)$ holds.[1]

  As a result, there are no reasons to restrict the use of any negation in modeling contradictions, since *any* negation operator can be linked to the idea underlying the notion of N-contradiction. Therefore, the reasonable approach is to consider something weaker than a negation, since it makes no sense to restrict to a subclass of negations.

- **Towards symmetry in f-weak-contradiction.** Once the need of considering the whole set of negations to deal with contradictions has been motivated, let us study how to recover the symmetry in f-weak-contradiction. Recall that the symmetry consist in the idea: if A is f_1-weak-contradictory w.r.t. B, then B is f_2-weak-contradictory w.r.t. A for some operator f_2. It easy to check that this kind of symmetry does not hold if for an element in the universe $t \in \mathcal{U}$ we have $A(t) = 1$ and $B(t) = \alpha \neq 0$; since in that case $\alpha = B(t) \nleq n(A(t)) = n(1) = 0$ for all negation n. Note that the reason of this behavior is that every negation operator n assigns $n(1) = 0$.

  Therefore, we have chosen to substitute the requirement $f(1) = 0$ (imposed on negation operators) by $f(1) \in [0, 1)$. This weakening still allows us to prove of Proposition 5 (in next section), which relates the weak-contradiction of A w.r.t. B with the weak-contradiction of B w.r.t. A.

  Example 3: Continuing with Examples 1 and 2, we have seen that I_A is n_1-weak-contradictory w.r.t. P_A, with n_1 a negation operator. Now, if we consider the operator

$$f_2(x) = \begin{cases} \frac{3}{2} - \frac{1}{2} \cdot x & \text{if } x \leq \frac{1}{2} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

  it is not difficult to check that P_A is f_2-weak-contradictory w.r.t. I_A. Note that f_2 is *not* a negation operator since $f_2(1) = \frac{1}{2} \neq 0$ and recall that, as we saw above, there is no negation n satisfying that P_A is n-weak-contradictory w.r.t. I_A. □

  In general, assuming the condition $f(1) \in [0, 1)$ for our class of operators, will allow us for proving that if A is f_1-weak-contradictory w.r.t. B, then there exists f_2 such that B is f_2-weak-contradictory w.r.t. A.

- **A final (technical) requirement.** The last point concerns one practical technicality. In the items above we have justified considering antitonic mappings $f\colon [0, 1] \to [0, 1]$ such that $f(0) = 1$ and $f(1) \in [0, 1)$ (that is, $f(1) \neq 1$). This family of mappings, by considering the point-wise ordering, forms a lattice which is not complete: it is easy to note that the supremum of the whole family is the constant function 1, which does not belong to the family. Including the constant function 1 has two interesting consequences: on the one hand, it provides the structure of complete lattice; on the other hand, it allows us to consider non-contradiction as a special case of weak-contradiction. Hence every pair of fuzzy sets can be considered contradictory up to some degree, including the zero degree.

  As it will be used later, we state here the following result on the set Ω which consists of the antitonic mappings $f\colon [0, 1] \to [0, 1]$ such that $f(0) = 1$.

  Lemma 1: The set Ω is a complete lattice with the usual pointwise ordering between mappings.

B. Properties of f-weak-contradiction

In the rest of the section we provide some theoretical results on the notion of f-weak-contradiction. Such results will be used later to define measures of contradiction and to prove

[1] Note that this inequality coincides with that used in Definition 1.

some of their properties. The first result shows that f-weak-contradiction is preserved when considering subsets of f-weak-contradictory fuzzy sets.

Proposition 1: Let A, B, C and D be four fuzzy sets such that $A \leq C$ and $B \leq D$. If C is f-weak-contradictory w.r.t. D then A is f-weak-contradictory w.r.t. B for any mapping f.

Proof: It is a consequence of the following chain of inequalities, for all $x \in \mathcal{U}$:

$$A(x) \leq C(x) \leq f(D(x)) \leq f(B(x))$$

■

As we explained in the section above, the mappings f in the definition of f-weak-contradiction can be considered as variables. The following result provides a relationship between f-weak-contradictory fuzzy sets w.r.t. different mappings.

Proposition 2: Let A and B be two fuzzy sets and let f_1 and f_2 be two mappings in Ω such that $f_1 \leq f_2$. If A is f_1-weak-contradictory w.r.t. B then A is f_2-weak-contradictory w.r.t. B as well.

Proof: Consider the following chain of inequalities:

$$A(x) \leq f_1(B(x)) \leq f_2(B(x))$$

for all $x \in \mathcal{U}$. ■

From the previous result, we can consider the mapping f in Definition 1 as a degree of contradiction. Note that if A is f-weak-contradictory w.r.t. B, then A is g-weak-contradictory w.r.t. B for all $f \leq g$, as a consequence of the proposition above. In other words, "*the lesser the mapping f, the greatest the number of weak-contradictions*." Furthermore, the mappings

$$f^{\top}(x) = \sup(\Omega)(x) = 1 \qquad \text{and}$$

$$f^{\perp}(x) = \inf(\Omega)(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

determine the weakest and strongest degree of f-weak-contradiction, respectively. In what follows, we introduce a brief study of these two extremal cases of f-weak-contradiction. Let us begin by considering the greatest degree of contradiction; i.e. $f^{\perp}$-weak-contradiction.

Corollary 1: Let A and B be two fuzzy sets such that A is $f^{\perp}$-weak-contradictory w.r.t. B. Then A is f-weak-contradictory w.r.t. B for all $f \in \Omega$.

The following result determines the structure of $f^{\perp}$-weak-contradictory fuzzy sets.

Proposition 3: Let A and B be two fuzzy sets. A is $f_{\perp}$-weak-contradictory w.r.t. B if and only if $B(x) > 0$ implies $A(x) = 0$ for all $x \in \mathcal{U}$.

Proof: Assume that A is $f^{\perp}$-contradictory w.r.t. B. Thus, if $B(x) > 0$, then $A(x) \leq f^{\perp}(B(x)) = 0$; hence $B(x) > 0$ implies $A(x) = 0$.

To prove the converse, firstly, note that if $B(x) = 0$ then the inequality $A(x) \leq f^{\perp}(B(x))$ holds, since $A(x) \leq f^{\perp}(0) = 1$. Secondly, if $B(x) > 0$, then $A(x) = 0$ and the inequality $A(x) \leq f^{\perp}(B(x))$ holds straightforwardly. ■

We focus now on the least degree of contradiction. To begin with, let us note that $f^{\top}$-weak-contradiction does not impose any restriction.

Lemma 2: Let A and B be two fuzzy sets, then A is $f^{\top}$-weak-contradictory w.r.t. B.

Proof: The inequality $A(x) \leq f^{\top}(B(x)) = 1$ trivially holds for all A and B. ■

The really interesting case arises when $f^{\top}$-weak-contradiction is the only weak-contradiction which holds,as this turns out to be an alternative representation of non-contradiction. The following result establishes the structure of a pair of non-contradictory fuzzy sets.

Proposition 4: Let A and B be two fuzzy sets. $f^{\top}$-weak-contradiction is the only weak-contradiction of A w.r.t. B if and only if there exists a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathcal{U}$ such that $B(x_i) = 1$ for all x_i and $\lim A(x_i) = 1$.

Proof: Let us assume firstly that there exists a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathcal{U}$ such that $B(x_i) = 1$ for all x_i and $\lim A(x_i) = 1$. In addition, assume that A is f-weak-contradictory w.r.t. B for some f, that is $A(x_i) \leq f(B(x_i)) = f(1)$ for all x_i. As $\lim A(x_i) = 1$, we have that $f(1) = 1$ and, by antitonicity of f, it should coincide with $f^{\top}$.

For the converse, we prove the reciprocal statement and, thus, assume

(Hyp) for all sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathcal{U}$ such that $B(x_i) = 1$ for all x_i we have $\lim A(x_i) \neq 1$.

Consider the mapping

$$f(x) = \begin{cases} \sup\{A(x) \mid x \in \mathcal{U} \text{ and } B(x) = 1\} & \text{if } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

It is not difficult to check that $f \in \Omega$, that $f \neq f^{\top}$ (since $f(1) \neq 1$ by definition of f and the additional assumption (Hyp)) and that A is f-weak-contradictory w.r.t. B (by definition of f).

Note that the case where there is no sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathcal{U}$ such that $B(x_i) = 1$ is covered in the proof above; in such a case, the mapping f collapses to:

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

■

If the universe where the fuzzy sets has been considered is finite, then the result above can be rewritten is a simpler way.

Corollary 2: Let A and B be two fuzzy sets defined on a finite universe $\mathcal{U}$. $f^{\top}$-weak-contradiction is the only weak-contradiction of A w.r.t. B if and only if there exists $x \in \mathcal{U}$ such that $A(x) = B(x) = 1$.

The two final results of this section are related to the symmetry described in Section II-A. As we discussed above, the contradiction of A w.r.t. B is not necessarily the same that the contradiction of B w.r.t. A but this does not mean that they are unrelated. Assume that $f, g \in \Omega$, respectively, are the mappings defining these weak-contradictions.

The following proposition will relate them in terms of the theory of Galois connections. Recall that a pair of mappings, $f, g\colon [0,1] \to [0,1]$ in our case, forms a *Galois connection* [12] if and only if the equivalence below holds for all $x, y \in [0,1]$:

$$y \leq f(x) \quad \Longleftrightarrow \quad x \leq g(y). \tag{1}$$

Proposition 5: Let (f,g) be a Galois connection in $[0,1]$, and let A and B be two fuzzy sets. Then, A is f-weak-contradictory w.r.t. B if and only if B is g-weak-contradictory w.r.t. A.

Proof: Assume that (f,g) is a Galois connection, then f and g are in Ω, namely, are antitonic and $f(0)=g(0)=1$. On the one hand, antitonicity is straightforward from (1), see [12]. On the other hand, the equality $f(0)=1$ comes from the fact that $g(1)\geq 0$, since by the definition of Galois connection we have:

$$0 \leq g(1) \iff 1 \leq f(0)$$

So, taking into account that $f(x)\in[0,1]$, we obtain $f(0)=1$, (similarly $g(0)=1$).

The equivalence between both weak-contradictions is just a rephrasing of (1), since A is f-weak-contradictory w.r.t. B if and only if $A(x)\leq f(B(x))$ if and only if $B(x)\leq g(A(x))$ if and only if B is g-weak-contradictory w.r.t. A. ■

The previous result establishes an interesting relationship between the f-weak-contradiction of A w.r.t. B and the one of B w.r.t. A. However, not every f-weak contradiction is defined from mappings belonging to one Galois connection, as it is well-known [12] that f is part of a Galois connection in $[0,1]$ if and only if the equality $f(\sup(X))=\inf f(X)$ holds for all $X\subseteq[0,1]$.

Even in the case that A is f-weak-contradictory w.r.t. B with f not being part of a Galois connection, we can always associate an $\bar{f}$ (not necessarily equal to $f^{\top}$) such that B is $\bar{f}$-weak-contradictory w.r.t. A.

Proposition 6: Let A and B be two fuzzy sets such that A is f-weak-contradictory w.r.t. B. Then B is $\overline{f}$-weak-contradictory w.r.t. A, where $\overline{f}$ is defined by

$$\overline{f}(x)=\sup\{y\in[0,1]\mid f(y)\geq x\}.$$

Proof: Certainly, $\overline{f}$ is well-defined and belongs to Ω. So, let us show that $B(u)\leq\overline{f}(A(u))$ for all $u\in\mathcal{U}$.

As A is f-weak-contradictory w.r.t. B we have the inequality $A(u)\leq f(B(u))$; now, using antitonicity of $\bar{f}$, we obtain $\bar{f}\big(f(B(u))\big)\leq\bar{f}\big(A(u)\big)$. Therefore, if we prove that $B(u)\leq\overline{f}\big(f(B(u))\big)$ the proof is finished; but this is just a consequence of the definition of $\overline{f}$, since

$$\overline{f}(f(B(u)))=\sup\{y\in[0,1]\mid f(y)\geq f(B(u))\}\geq B(u).$$

■

C. On the set of weak-contradictions of A w.r.t. B.

We present here, perhaps, the most important properties of f-weak contradiction, since form the basis of the measures of contradiction introduced later in this paper. The results are similar to those presented in [19] on the framework of fuzzy logic programming, but considering mappings in Ω instead of negations. Specifically, in this section we study the structure of the set:

$$\mathcal{F}(A,B)=\big\{f\in\Omega\mid A \text{ is } f\text{-weak-contradictory w.r.t. } B\big\}$$

and we show that $\mathcal{F}(A,B)$ can be characterized by the least operator f such that A is f-weak-contradictory w.r.t. B. Two previous remarks:

- $\mathcal{F}(A,B)$ is not empty by Lemma 2.
- We can guarantee the existence of the infimum of $\mathcal{F}(A,B)$ thanks to the complete lattice structure of Ω (Lemma 1).

We start by proving that $\mathcal{F}(A,B)$ is closed under infima, hence, in particular, $\inf(\mathcal{F}(A,B))\in\mathcal{F}(A,B)$.

Proposition 7: Let A and B be two fuzzy sets and let $\{f_i\}\subseteq\Omega$. If A is f_i-weak-contradictory w.r.t. B for any f_i, then A is $\inf\{f_i\}$-weak-contradictory w.r.t. B.

Proof: It is straightforward to check that $f=\inf\{f_i\}$ is given by $f(x)=\inf\{f_i(x)\}$ for all $x\in[0,1]$. Moreover, as A is f_i-weak-contradictory w.r.t. B for all f_i then $A(u)\leq f_i(B(u))$ for all $u\in\mathcal{U}$. This implies that $A(u)\leq\inf\{f_i(B(u))\}=f(B(u))$ for all $u\in\mathcal{U}$; therefore, A is $\inf\{f_i\}$-weak-contradictory w.r.t. B. ■

As a straightforward consequence of the above result, for all fuzzy sets A and B, there exists the least mapping $f\in\Omega$ verifying that A is f-weak-contradictory w.r.t. B; hereafter, such mapping will be denoted by $f_{A,B}$. Moreover, thanks to Proposition 2, we can characterize the set $\mathcal{F}(A,B)$ as follows:

Corollary 3: Let A and B be two fuzzy sets, then:

$$\mathcal{F}(A,B)=\{f\in\Omega \text{ such that } f_{A,B}\leq f\}$$

where $f_{A,B}=\min\{\mathcal{F}(A,B)\}$.

Therefore, determining the set $\mathcal{F}(A,B)$ is equivalent to determining the least mapping $f\in\Omega$ verifying that A is f-weak-contradictory w.r.t. B. In the remainder of this section, we present the analytic expression of the minimum of $\mathcal{F}(A,B)$, which will be used later to prove the symmetry of the measures of contradiction given in Section III.

Theorem 1: Let A and B be two fuzzy sets. Then the minimum of $\mathcal{F}(A,B)$ is given by the mapping $f_{A,B}$ defined by the formula

$$f_{A,B}(x)=\sup_{u\in\mathcal{U}}\{A(u)\mid x\leq B(u)\}. \tag{2}$$

Proof: We split the proof into three different phases in which we will show that

(A) $f_{A,B}$ is in Ω;
(B) A is $f_{A,B}$-weak-contradictory w.r.t. B;
(C) there is not a mapping $f\in\Omega$ satisfying that $f<f_{A,B}$ and such that A is f-weak-contradictory w.r.t. B.

Note that these three statements above imply that $f_{A,B}$ is the least element in $\mathcal{F}(A,B)$.

Proof of (A): The operator $f_{A,B}$ is obviously well-defined and satisfies the boundary condition $f_{A,B}(0)=1$. Thus, we only have to show that $f_{A,B}$ is antitonic.

Consider $x,y\in[0,1]$ such that $x\leq y$. Then,

$$\{u\in\mathcal{U}\mid y\leq B(u)\}\subseteq\{u\in\mathcal{U}\mid x\leq B(u)\}$$

Therefore, by definition of supremum:

$$\begin{aligned} f_{A,B}(y) &= \sup_{u\in\mathcal{U}}\{A(u)\mid y\leq B(u)\}\\ &\leq \sup_{u\in\mathcal{U}}\{A(u)\mid x\leq B(u)\}\\ &= f_{A,B}(x). \end{aligned}$$

In other words, $f_{A,B}$ is antitonic.

Once we have proved that $f_{A,B}$ is in Ω, we can consider the $f_{A,B}$-weak-contradictory relation.

Proof of (B): Let $u \in \mathcal{U}$, by definition of $f_{A,B}$ and properties of supremum:

$$f_{A,B}(B(u)) = \sup_{v\in\mathcal{U}}\{A(v) \mid B(u) \leq B(v)\} \geq A(u)$$

That is, A is $f_{A,B}$-weak-contradictory w.r.t. B.

Proof of (C): Let $f \in \Omega$ such that $f < f_{A,B}$ and let us show that A is not f-weak-contradictory w.r.t. B.

As $f < f_{A,B}$, there exists $\alpha \in [0,1]$ such that $f(\alpha) < f_{A,B}(\alpha)$. In fact, we will prove that the inequality holds for some element from the image of B, that is, there exists $u \in \mathcal{U}$ such that $f(B(u)) < f_{A,B}(B(u))$.

Let us assume by *reductio ad absurdum* that for all $v \in \mathcal{U}$ such that $\alpha \leq B(v)$ we have $f(B(v)) = f_{A,B}(B(v))$. In such a case, by antitonicity of f, for all $v \in \mathcal{U}$ such that $\alpha \leq B(v)$ we have that $f(\alpha) \geq f(B(v)) = f_{A,B}(B(v))$. Therefore we have the following inequality:

$$\begin{aligned}
f(\alpha) &\geq \sup_{v\in\mathcal{U}}\{f_{A,B}(B(v)) \mid \alpha \leq B(v)\}\\
&\quad\text{(by definition of } f_{A,B})\\
&= \sup_{v\in\mathcal{U}}\Big\{\sup_{u\in\mathcal{U}}\{A(u) \mid B(v) \leq B(u)\} \mid \alpha \leq B(v)\Big\}\\
&\quad\text{(by properties of suprema)}\\
&= \sup_{u\in\mathcal{U}}\{A(u) \mid \alpha \leq B(u)\}\\
&\quad\text{(by definition again)}\\
&= f_{A,B}(\alpha)
\end{aligned}$$

and we obtain $f(\alpha) \geq f_{A,B}(\alpha)$, contradicting our assumption. Hence, we have proved that there exists $u \in \mathcal{U}$ such that:

$$f(B(u)) < f_{A,B}(B(u)).$$

Now, as $f_{A,B}(B(u)) = \sup_{v\in\mathcal{U}}\{A(v) \mid B(u) \leq B(v)\}$, the previous inequality implies that $f(B(u))$ cannot be an upper bound of $\{A(v) \mid B(u) \leq B(v)\}_{v\in\mathcal{U}}$. Thus, there exists $w \in \mathcal{U}$ such that both $B(u) \leq B(w)$ and $f(B(u)) < A(w)$. Now, by antitonicity of f, we have $f(B(w)) \leq f(B(u))$. So, by using also the other inequality we obtain that $f(B(w)) \leq f(B(u)) < A(w)$. In other words, A is not f-weak-contradictory w.r.t. B. ∎

Now that we have the analytic expression of $f_{A,B}$, it is worth to compare $f_{A,B}$ and $f_{B,A}$. Recalling Proposition 5, one might ask whether these two mappings form a Galois connection. We show below that, in general, the answer is negative; and then study the conditions under which the answer is affirmative.

Example 4: Consider $\mathcal{U} = [0,1]$, and the two following fuzzy sets

$$A(u) = \begin{cases} 1 & \text{if } 0 \leq u < 0.5\\ 0 & \text{otherwise}\end{cases} \qquad B(u) = \begin{cases} u & \text{if } 0 \leq u \leq 0.5\\ 0 & \text{otherwise}\end{cases}$$

It is easy to check that

$$f_{A,B}(x) = \begin{cases} 1 & \text{if } 0 \leq u < 0.5\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad f_{B,A}(x) = 0.5$$

The values $x = 0.9$ and $y = 0.5$, for instance, form a counterexample for the equivalence

$$x \leq f_{A,B}(y) \iff y \leq f_{B,A}(x)$$

since the second inequality holds whereas the first one fails. □

The previous example provides a clue to obtain a sufficient condition to obtain a Galois connection: the key fact is that the supremum in the definition of $f_{A,B}$ and $f_{B,A}$ should always be a maximum.

Proposition 8: If the images of A and B are finite, then the pair $\big(f_{A,B}, f_{B,A}\big)$ forms a Galois connection.

III. Measures of contradiction

In this section, we introduce a family of measures to determine how contradictory two fuzzy sets A and B are. The idea underlying all these measures is "*the more f's in Ω such that A is f-weak-contradictory w.r.t. B, the more contradiction between A and B.*" With this idea, measuring contradiction between two fuzzy sets A and B is equivalent to measuring the subset $\mathcal{F}(A,B) \subseteq \Omega$. Moreover, in Section II-B we have seen that $\mathcal{F}(A,B)$ can be characterized by the least mapping f such that A is f-weak-contradictory w.r.t. B (denoted by $f_{A,B}$). And without any doubt, the best way to measure mappings defined from $[0,1]$ to $[0,1]$ is by definite integrals. Actually, the following measure of contradiction between two fuzzy sets was defined in [2] by using this idea.

Definition 2: Let A and B be two fuzzy sets, let $f_{A,B}$ be the least element of $\mathcal{F}(A,B)$. *The measure of contradiction $\mathcal{C}$ between A and B* is defined by:

$$\mathcal{C}(A,B) = 1 - \int_0^1 f_{A,B}(x)dx$$

The measure of contradiction $\mathcal{C}$ satisfies the following properties [2]:

- symmetry; i.e $\mathcal{C}(A,B) = \mathcal{C}(B,A)$.
- antitonicity; i.e. $A_1 \leq A_2$ implies $\mathcal{C}(A_1,B) \geq \mathcal{C}(A_2,B)$.
- $\mathcal{C}(A,B) = 1$ if and only if $B(x) > 0$ implies $A(x) = 0$ for all $x \in \mathcal{U}$.
- $\mathcal{C}(A,B) = 0$ if and only if there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim A(x_i) = \lim B(x_i) = 1$.

It is worth to note that, although $\mathcal{F}(A,B) \neq \mathcal{F}(B,A)$ (see Section II-B), the values $\mathcal{C}(A,B)$ and $\mathcal{C}(B,A)$ coincide. Moreover, those properties imply that $\mathcal{C}$ satisfies the axiomatic definition for *measures of incompatibility* given in [4] in the context of Atanassov Intuitionistic fuzzy sets.

We will introduce a family of measures of contradiction based on the idea underlying the definition above and, naturally, extending $\mathcal{C}$. The key feature of the extension, not developed in [2], is that $\mathcal{C}$ assumes that every pair of values $(\alpha,\beta) \in [0,1]^2$ somehow *"contains the same amount contradiction."* The following example elaborates on this idea.

Example 5: Let us consider the following four fuzzy sets defined on a singleton universe $\{t\}$:

$$A_1(t) = 0.3 \quad A_2(t) = 0.3 \quad B_1(t) = 1 \quad B_2(t) = 0.09$$

Then

$$\mathcal{C}(A_1,A_2) = 1 - 0.3 \cdot 0.3 = 0.91$$

$$\mathcal{C}(B_1, B_2) = 1 - 1 \cdot 0.09 = 0.91$$

since

$$f_{A_1,A_2}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.3 & \text{if } 0 < x \leq 0.3 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{B_1,B_2}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 0.09 \\ 0 & \text{otherwise} \end{cases}$$

The reason for the equality $\mathcal{C}(A_1, A_2) = \mathcal{C}(B_1, B_2)$ is that every element in the sets $\{(x, y) \in [0, 1]^2 \mid x \leq 1 \text{ and } y \leq 0.09\}$ and $\{(x, y) \in [0, 1]^2 \mid x \leq 0.3 \text{ and } y \leq 0.3\}$ has *the same density of contradiction*. Hence, the area of both sets (and therefore also the measure of $\mathcal{C}$) coincides. □

The behavior presented in the example above can be acceptable in some frameworks but not in others. Consider for instance a Fuzzy Answer Set framework [20], [16], [17] where a negation operator is fixed a priori to determine the inconsistency of a Normal Logic Program. In such a framework, the pair $(1, 0.09)$ could be assumed to be completely inconsistent (since it is close to the crisp case $(0, 1)$), whereas the pair $(0.3, 0.3)$ can be considered just partially inconsistent (since there is a considerable membership degree in both fuzzy sets). To deal with this goal, we introduce the notion of *density of contradiction* of a point in the unit square $[0, 1]^2$:

Definition 3: A *density of contradiction* is any mapping $m\colon [0, 1]^2 \to \mathbb{R}^+$ such that:

- $m(x, y) = m(y, x)$ for all $x, y \in [0, 1]$
- $\displaystyle\int_{[0,1]^2} m(x, y)dxdy = 1$

The set of densities of contradiction is denoted by $\mathcal{D}$.

The first condition is considered because of the expected symmetry of the measure of contradiction it generates, see below. On the other hand, the second requirement is natural, as in any density distribution. Technically, this second requirement really does not impose a strong condition, and could be substituted by the finiteness of the integral and normalization.

Definition 4: Given two fuzzy sets A and B and a density of contradiction m, we define the measure of contradiction between A and B w.r.t. m as:

$$\mathcal{C}_m(A, B) = 1 - \int_{x=0}^{1} \int_{y=0}^{f_{A,B}(x)} m(x, y)dydx$$

Some remarks about the "measure" $\mathcal{C}_m$:

- Note that $\mathcal{C}_m$ is well-defined for all pair of fuzzy sets A and B since, firstly, the mapping m is integrable by the second item in Definition 3 and, secondly, because the mapping $f_{A,B}$ is antitonic and bounded and, hence, Riemann integrable.
- As with $\mathcal{C}$, we compute the complement w.r.t. 1 of the definite integral, since the lesser the mapping $f_{A,B}$, the more contradictory A w.r.t. B is (i.e the greater $\mathcal{F}(A, B)$).
- Again as with $\mathcal{C}$, the measure of contradiction $\mathcal{C}_m(A, B)$ is not symmetric a priori. However, we show below that, although the contradiction of A w.r.t. B is not necessarily the same as the contradiction of B w.r.t. A (in terms of the sets of weak-contradictions $\mathcal{F}(A, B)$ and $\mathcal{F}(B, A)$), the measure of contradiction coincides in both cases. In other words, we show that $\mathcal{C}_m$ is symmetric; i.e. $\mathcal{C}_m(A, B) = \mathcal{C}_m(B, A)$ for all fuzzy sets A and B and all density of contradiction m.

One of the first requirements for a measure of contradiction to make sense is that $\mathcal{C}_m$ is antitonic for all $m \in \mathcal{D}$. This is established in the following result.

Proposition 9: Let A, B, C and D be four fuzzy sets such that $A \leq C$ and $B \leq D$. Then $\mathcal{C}_m(A, B) \geq \mathcal{C}_m(C, D)$ for all $m \in \mathcal{D}$.

Proof: By using Proposition 1 we have that:

$$\mathcal{F}(A, B) \supseteq \mathcal{F}(C, D)$$

and by Corollary 3, that is equivalent to $f_{A,B} \leq f_{C,D}$. So,

$$\begin{aligned} \mathcal{C}_m(A, B) &= 1 - \int_{x=0}^{1} \int_{y=0}^{f_{A,B}(x)} m(x, y)dydx = \\ &\geq 1 - \int_{x=0}^{1} \int_{y=0}^{f_{C,D}(x)} m(x, y)dydx = \mathcal{C}_m(C, D) \end{aligned}$$

■

As a consequence of the previous result, we obtain that the value of $\mathcal{C}_m(A, B)$ is always in $[0, 1]$.

Corollary 4: Let m be a density of contradiction and let A and B be two fuzzy sets. Then $\mathcal{C}_m(A, B) \in [0, 1]$.

Proof: Let us see that $\mathcal{C}_m(A, B) \geq 0$. Let T be the fuzzy set defined by $T(x) = 1$ for all $x \in \mathcal{U}$. Then, by Proposition 9 we have:

$$\mathcal{C}_m(A, B) \geq \mathcal{C}_m(T, T)$$

By Proposition 4, it easy to see that $\mathcal{F}(T, T) = \{f^\top\}$, hence:

$$\begin{aligned} \mathcal{C}_m(A, B) &\geq 1 - \int_{x=0}^{1} \int_{y=0}^{f^\top(x)} m(x, y)dydx = \\ &= 1 - \int_{x=0}^{1} \int_{y=0}^{1} m(x, y)dydx = 1 - 1 = 0 \end{aligned}$$

The inequality $\mathcal{C}_m(A, B) \leq 1$ is obtained similarly by considering the fuzzy set defined by $O(x) = 0$ for all $x \in \mathcal{U}$. ■

Let us prove now the last statement given in the remarks; i.e. the symmetry of the measure $\mathcal{C}_m$.

Theorem 2: Let A and B be two fuzzy sets and let m be a density of contradiction. Then $\mathcal{C}_m(B, A) = \mathcal{C}_m(A, B)$.

Proof: Note that $\mathcal{C}_m(A, B)$ and $\mathcal{C}_m(B, A)$ determine the volume of the sets:

$$S_{A,B} = \{(x, y, z) \in [0, 1]^3 \mid y \leq f_{A,B}(x),\ z \leq m(x, y)\}$$

$$S_{B,A} = \{(x, y, z) \in [0, 1]^3 \mid y \leq f_{B,A}(x),\ z \leq m(x, y)\}$$

respectively. Let us prove that both volumes coincide.

The idea is to show that $S_{B,A}$ is so closely related to the mirror image of $S_{A,B}$ w.r.t. the plane $\pi \equiv x = y$ that their volumes coincide (since mirror images do no modify volumes). Specifically, if $\tau_\pi(S_{A,B})$ denotes the mirror image of $S_{A,B}$ w.r.t. the plane π, we will prove that

- $\text{int}(S_{B,A}) \subseteq \tau_\pi(S_{A,B})$
- $\text{int}(\tau_\pi(S_{A,B})) \subseteq S_{B,A}$

where int denotes the interior operator of a subset in $\mathbb{R}^3$. This result would imply that the difference between $\tau_\pi(S_{A,B})$ and $S_{B,A}$ is a null set, and both volumes would coincide.

Now, for proving $\text{int}(S_{B,A}) \subseteq \tau_\pi(S_{A,B})$ we will show

$$\begin{aligned}\text{int}(S_{B,A}) &\subseteq \{(x,y,z) \in [0,1]^3 \mid y < f_{B,A}(x),\ z < m(x,y)\}\\ &\subseteq \{(x,y,z) \in [0,1]^3 \mid x \leq f_{A,B}(y),\ z \leq m(y,x)\}\\ &= \tau_\pi(S_{A,B})\end{aligned}$$

Firstly, by symmetry of $m \in \mathcal{D}$ we have that $z < m(x,y)$ is more restrictive than $z \leq m(y,x)$. To finish this part, it is enough to show that $y < f_{B,A}(x)$ implies $x \leq f_{A,B}(y)$ for all $x, y \in [0,1]$:

From $y < f_{B,A}(x) = \sup_{u\in\mathcal{U}}\{B(u) \mid x \leq A(u)\}$ we have that there exists $v \in \mathcal{U}$ such that $y < B(v)$ and $x \leq A(v)$; therefore, $x \leq \sup_{u\in\mathcal{U}}\{A(u) \mid y \leq B(u)\} = f_{A,B}(y)$.

The proof of $\text{int}(\tau_\pi(S_{A,B})) \subseteq S_{B,A}$ is similar. ■

Let us consider some examples to become familiar with the family of measures $\{\mathcal{C}_m\}_{m\in\mathcal{M}}$. These examples are aimed at introducing the study of the extreme cases (i.e. the cases $\mathcal{C}_m(A,B) = 0$ and $\mathcal{C}_m(A,B) = 1$), where the density of contradiction considered plays a crucial role.

Example 6: The measure $\mathcal{C}(A,B)$ defined in [2] (and given in Definition 2) can be considered as a special case of the family of measures $\mathcal{C}(A,B)_m$. Specifically when $m(x,y) = 1$ for all $x \in [0,1]$, since:

$$\begin{aligned}\mathcal{C}_m(A,B) = 1 - \int_{x=0}^{1}\int_{y=0}^{f_{A,B}(x)} 1\ dydx =\\ = 1 - \int_0^1 f_{A,B}(x)\ dx = \mathcal{C}(A,B)\end{aligned}$$

□

Example 7: The following example is related to approaches where a negation operator to determine the contradiction is fixed a priori [5], [8], [15], [18]. We can link our measure of contradiction to those approaches by assuming the following equivalence: two fuzzy sets are contradictory if $f_{A,B} \leq n$, where n is the negation operator chosen to represent the contradiction (i.e. the n-weak-contradiction). We can measure the contradiction under such idea by using the family of measures $\{\mathcal{C}_m\}_{m\in\mathcal{D}}$, just considering the following density of contradiction:

$$m(x,y) = \begin{cases} \alpha & \text{if } y \geq n(x)\\ 0 & \text{otherwise}\end{cases}$$

where the value $\alpha \in \mathbb{R}$ is defined as:

$$\alpha = \frac{1}{1 - \int_0^1 n(x)\ dx}$$

Under the consideration above, it is not difficult to verify that the extreme case $\mathcal{C}_m(A,B) = 1$ is equivalent to $f_{A,B} \leq n$, that is, the A and B are fully contradictory if and only if $f_{A,B} \leq n$. □

This example shows that our approach also deals with frameworks in which specific cases of contradictions are fixed a priori. In the example below, we show that we can also deal with a family of points considered non-contradictory, this is, we can cover hypothetic frameworks where not only pairs of values *"close"* to lines $x = 0$ and $y = 0$ are considered fully contradictory, but neighborhoods to the point $(1,1)$ are considered not contradictory.[2]

Example 8: Consider two convex sets V and W in $[0,1]^2$, such that $\{(x,y) \in [0,1]^2 \mid x = 0 \text{ or } y = 0\} \subseteq V$ and $(1,1) \in W$. Then, if we consider the density of contradiction:

$$m(x,y) = \begin{cases} \alpha & \text{if } (x,y) \notin V \cup W\\ 0 & \text{otherwise}\end{cases}$$

where the value $\alpha \in \mathbb{R}$ is defined as:

$$\alpha = \frac{1}{1 - \int_{V^c\cap W^c\cap[0,1]^2} 1\ dxdy}$$

we obtain the following two equivalences for extreme cases when $\mathcal{U}$ is discrete. On the one hand, $\mathcal{C}_m(A,B) = 1$ if and only if for all $u \in \mathcal{U}$ we have that $(A(u), B(u)) \in V$; i.e. for all element in the universe the values given by A and B are fully contradictory. On the other hand, $\mathcal{C}_m(A,B) = 0$ if and only if there is at least one $v \in \mathcal{U}$ such that $(A(v), B(v)) \in W$; i.e.there is an element in the universe such that the values given by A and B are not contradictory. □

Note that in the previous examples, the densities of contradiction considered have always been piecewise constant. However it is also possible to consider *any* density function $d(x,y)$. If the domain of such a density function is contained in $[0,1]^2$, then it can be used directly, otherwise it is necessary to restrict the domain to $[0,1]^2$ and to proceed with a normalization; i.e. by considering a density of contradiction such as

$$m(x,y) = \frac{d(x,y)}{1 - \int_{[0,1]^2} d(x,y)\ dxdy}$$

Example 9: In the case of the normal density distribution function $\mathcal{N}(\mu,\Sigma)$ given by

$$\mathcal{N}(\mu,\Sigma)(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^2|\Sigma|}}\,\mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)}$$

for all $\mathbf{x} \in \mathbb{R}^{\mathbf{2}}$, the density of contradiction from it is:

$$m(x,y) = \frac{\mathcal{N}(\mu,\Sigma)(x,y)}{1 - \int_{[0,1]^2} \mathcal{N}(\mu,\Sigma)(x,y)\ dxdy}$$

In the case of considering the normal distribution, the characterization of the extreme cases coincide with those given in [2] for the measure $\mathcal{C}$. □

As we have seen, the characterization of the extreme cases in the measures $\mathcal{C}_m$ depends strongly on the density of contradiction considered; specifically, on the support of $m(x,y)$. Let us begin by characterizing the case $\mathcal{C}_m(A,B) = 1$.

Proposition 10: Let A and B be two fuzzy sets and let m be a density of contradiction. Then $\mathcal{C}_m(A,B) = 1$ if and only if for each $u \in \mathcal{U}$, the set

$$\Xi_u = \{(x,y) \in \mathit{supp}(m) \mid x \leq A(u) \text{ and } y \leq B(u)\} \quad (3)$$

is a null set.

[2]The idea underlying these neighborhood considerations is that the rule $p \to \neg q$ holds if $p = 0$ or $q = 0$ whereas does not if $p = 1$ and $q = 1$.

Proof: We will prove that $\mathcal{C}_m(A,B) \neq 1$ if and only if there is $u \in \mathcal{U}$ such that Ξ_u is not a null set. Note that $\mathcal{C}_m(A,B) \neq 1$ is equivalent to say that

$$\int_{x=0}^{1}\int_{y=0}^{f_{A,B}(x)} m(x,y)dydx \neq 0$$

which, assuming $D = \{(x,y) \in supp(m) \mid y \leq f_{A,B}(x)\}$, is equivalent to

$$\int_D m(x,y)dydx \neq 0$$

which is equivalent to

$$Area(D) \neq 0.$$

So, let as show that the area of D is not zero if and only if there exists $u \in \mathcal{U}$ such that the set Ξ_u defined in (3) is not null.

If the area of D is not zero, then, by the antitonicity of $f_{A,B}$ and how D is defined, there is a non-empty ball contained in the interior of D.So, there is $(\alpha,\beta) \in D$ and $\varepsilon > 0$ such that

$$[\alpha-\varepsilon,\alpha]\times[\beta-\varepsilon,\beta] \subseteq \mathrm{int}(D).$$

It is not difficult to check that there exists $u \in \mathcal{U}$ such that the ball above is included in Ξ_u: in effect, as $(\alpha,\beta) \in \mathrm{int}(D)$, we have that $\alpha < f_{A,B}(\beta) = \sup_{u\in\mathcal{U}}\{A(u) \mid \beta \leq B(u)\}$, so there is $u \in \mathcal{U}$ such that $\alpha < A(u)$ y $\beta \leq B(u)$. In other words, there is $u \in \mathcal{U}$ such that

$$\begin{aligned}
&[\alpha-\varepsilon,\alpha]\times[\beta-\varepsilon,\beta] \subseteq \\
&\qquad \subseteq \{(x,y) \in \mathrm{supp}(m) \mid x \leq \alpha \text{ and } y \leq \beta\} \\
&\qquad \subseteq \{(x,y) \in \mathrm{supp}(m) \mid x \leq A(u) \text{ and } y \leq B(u)\} \\
&\qquad = \Xi_u
\end{aligned}$$

As a result, we have Ξ_u is non-null for certain value of u.

To prove the converse, let us assume that there is $u \in \mathcal{U}$ such that Ξ_u is non-null. Now, using that $x \leq A(u)$ and $y \leq B(u)$ implies $y \leq B(u) \leq f_{B,A}(A(u)) \leq f_{B,A}(x)$, we have that

$$\begin{aligned}
\Xi_u &\stackrel{(3)}{=} \{(x,y) \in \mathrm{supp}(m) \mid x \leq A(u) \text{ and } y \leq B(u)\} \\
&\subseteq \{(x,y) \in \mathrm{supp}(m) \mid y \leq f_{B,A}(x)\}
\end{aligned}$$

Finally, by symmetry of $\mathcal{C}_m$ (Theorem 2) we have that

$$Area\big(\{(x,y) \in \mathrm{supp}(m) \mid y \leq f_{B,A}(x)\}\big) = Area(D)$$

and D is non-null. ■

In what follows, in order to characterize the case $\mathcal{C}_m(A,B) = 0$, we will write

$$E_m = \overline{\mathrm{int}(\mathrm{supp}(m))}$$

that is, the closure of the interior of the support of the density of contradiction m; basically, this is the application of a morphological filter to the support of m, which amounts to deleting isolated points and lines.

Note that E_m is not empty because of the equality $\int_{x=0}^{1}\int_{y=0}^{1} m(x,y)dydx = 1$ (required by definition of density of contradiction). In the statement of the following result, we will consider that E_m inherits the pointwise ordering within the real unit square and, therefore, makes sense to consider its maximal elements (which exist since E_m is closed by definition).

Proposition 11: Let A and B be two fuzzy sets and let m be a density of contradiction. Then $\mathcal{C}_m(A,B) = 0$ if and only if for all maximal (α,β) in E_m there is a sequence $\{u_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim A(u_i) \geq \alpha$ and $\lim B(u_i) \geq \beta$.

Proof: Firstly, by symmetry of $\mathcal{C}_m$ (Theorem 2), we have $\mathcal{C}_m(A,B) = \mathcal{C}_m(B,A)$. In addition, $\mathcal{C}_m(B,A) = 0$ if and only if $\int_{x=0}^{1}\int_{y=0}^{f_{B,A}(x)} m(x,y)dydx = 1$.

Recalling that we have $\int_{x=0}^{1}\int_{y=0}^{1} m(x,y)dydx = 1$ by definition of m, we can conclude that $\mathcal{C}_m(B,A) = 0$ if and only if $\int_{x=0}^{1}\int_{y=f_{B,A}(x)}^{1} m(x,y)dydx = 0$.

For any maximal element $(\alpha,\beta) \in E_m$, we have that $\beta \leq f_{B,A}(\alpha)$. In effect, assuming it does not hold, there would exist[3] a non-null subset $E_{\alpha,\beta}$ containing (α,β) such that

$$E_{\alpha,\beta} \subseteq \{(x,y) \in [0,1]^2 \mid y > f_{B,A}(x)\} \cap \mathrm{supp}(m),$$

and, as a result, we would have

$$\int_{x=0}^{1}\int_{y=f_{B,A}(x)}^{1} m(x,y)dydx \geq \int_{E_{\alpha,\beta}} m(x,y)dxdy > 0$$

contradicting $\int_{x=0}^{1}\int_{y=f_{B,A}(x)}^{1} m(x,y)dydx = 0$.

Now, as $\beta \leq f_{B,A}(\alpha) = \sup_{u\in\mathcal{U}}\{B(u) \mid \alpha \leq A(u)\}$ we can assume that there is a sequence $\{u_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim A(u_i) \geq \alpha$ and $\lim B(u_i) \geq \beta$.

For the converse, firstly, let us remark that, by definition of E_m, we have $\iint_{E_m} m = \iint_{[0,1]^2} m = 1$. If we denote the set of maximal elements in E_m by M, then obviously we have

$$E_m \subseteq \bigcup_{(\alpha,\beta)\in M} \{(x,y) \in [0,1]^2 \mid x \leq \alpha,\ y \leq \beta\}$$

Secondly, given a maximal $(\alpha,\beta) \in E_m$, and $\delta < \alpha$, by the hypothesis, we have that $\beta \leq f_{B,A}(\delta)$. For the proof of this fact, consider a sequence $\{u_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim A(u_i) \geq \alpha$ and $\lim B(u_i) \geq \beta$, then there exists $u \in \mathcal{U}$ such that $A(u) \geq \delta$ and $B(u) \geq \beta$. Now, the inequality $\beta \leq f_{B,A}(\delta) = \sup_{u\in\mathcal{U}}\{B(u) \mid \delta \leq A(u)\}$ is obvious.

We only need to note the following chain of inclusions

$$\begin{aligned}
\mathrm{int}(E_m) &\subseteq \bigcup_{(\alpha,\beta)\in M} \{(x,y) \in [0,1]^2 \mid x < \alpha,\ y \leq \beta\} \\
&\subseteq \{(x,y) \in [0,1] \mid \ y \leq f_{B,A}(x)\}
\end{aligned}$$

where the second one follows from $\beta \leq f_{B,A}(\delta)$, by substituting x for δ. Therefore

$$\int_{x=0}^{1}\int_{y=0}^{f_{B,A}(x)} m(x,y)dydx \geq \int_{E_m} m(x,y)dydx = 1,$$

which concludes the proof. ■

The statement of the two proposition above can be combined and rephrased in terms of the set E_m as in the next corollaries. In our framework, the existence of values of $(A(u),B(u))$ close enough to $(1,1)$ implies zero degree of contradiction between A and B. Dually, if all the values are close enough to $(0,0)$, the measure of contradiction should be equal to 1.

[3]This follows from topological considerations due to the particular construction of E_m as the closure of the interior.

Corollary 5: Let A and B be two fuzzy sets and let m be a density of contradiction such that $E_m \neq [0,1]^2$. Let $\{V_i\}_{i\in\mathbb{I}}$ be the family of maximal rectangles in $\overline{E_m}^c$ such that $(0,0) \in V_i$ for all $i \in I$, and $\{W_i\}_{i\in\mathbb{I}}$ the family of maximal rectangles in $\overline{E_m}^c$ such that $(1,1) \in W_i$ for all $i \in I$. Then:

- $\mathcal{C}_m(A,B) = 1$ if and only if for all $u \in \mathcal{U}$ we have $(A(u), B(u)) \in V_i$ for some $i \in \mathbb{I}$.
- $\mathcal{C}_m(A,B) = 0$ if and only if for all $i \in I$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$ such that

$$(\lim A(u_n), \lim B(u_n)) \in W_i.$$

Proof: By Proposition 10 we know that $\mathcal{C}_m(A,B) = 1$ if and only if for all $u \in \mathcal{U}$ the set

$$\{(x,y) \in \operatorname{supp}(m) \mid x \leq A(u) \text{ and } y \leq B(u)\}$$

is a null set. This is equivalent to the fact that for all $u \in \mathcal{U}$:

$$\{(x,y) \in [0,1]^2 \mid x \leq A(u) \text{ and } y \leq B(u)\} \subseteq \overline{E_m}^c$$

Finally, as the set above is a rectangle containing $(0,0)$, and the inclusion is equivalent to $(A(u), B(u)) \in V_i$ for some $i \in \mathbb{I}$.

To prove the other statement just note that, on the one hand, for all maximal $(\alpha,\beta) \in E_m$, the set

$$\{(x,y) \in [0,1] \mid x \geq \alpha,\ y \geq \beta\}$$

is a rectangle containing $(1,1)$ and is included in $\overline{E_m}^c$. Moreover, on the other hand, for each W_i it is easy to check that there is a maximal element $(\alpha,\beta) \in E_m$ which also belongs to W_i, since otherwise W_i would not be maximal. Finally, one has just to note that the statements below are equivalent:

- for all maximal $(\alpha,\beta) \in E_m$ there exists a sequence $\{u_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$ satisfying that $\lim A(u_i) \geq \alpha$ and $\lim B(u_i) \geq \beta$
- for all W_i there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$ such that $(\lim A(u_n), \lim B(u_n)) \in W_i$.

■

The corollary above required the fact $E_m \neq [0,1]^2$. If this would not be the case, that is $E_m = [0,1]^2$, the resulting statements are greatly simplified.

Corollary 6: Let A and B be two fuzzy sets and let m be a density of contradiction such that $E_m = [0,1]^2$ for all $(x,y) \in [0,1]^2$. Then:

- $\mathcal{C}_m(A,B) = 1$ if and only if either $A(u) = 0$ or $B(u) = 0$ for all $u \in \mathcal{U}$.
- $\mathcal{C}_m(A,B) = 0$ if and only if there exists a sequence $\{u_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim A(u_i) = \lim B(u_i) = 1$.

Proof: For the first statement, Proposition 10 states that $\mathcal{C}_m(A,B) = 1$ if and only if the sets of the form $\{(\alpha,\beta) \in \operatorname{supp}(m) \mid \alpha \leq A(u),\ \beta \leq B(u)\}$ are null-sets. We have just to note that if $E_m = [0,1]^2$, then $\mathcal{C}_m(A,B) = 1$ if and only if either $A(u) = 0$ or $B(u) = 0$.

For the second statement, just note that the only maximal element in E_m is the pair $(1,1)$. So applying Proposition 11 the result is straightforward. ■

As seen in Example 7, there are frameworks in which a negation operator is fixed before studying weak-contradiction. We can also deal with this idea by considering a density of contradiction m which assigns 0 to those pairs satisfying the weak-contradiction requirement. In such a case we have the following result.

Corollary 7: Let A and B be two fuzzy sets and let m_f be a density of contradiction such that $m_f(x,y) = 0$ if and only if $x > f(y)$ with $f \in \Omega$. Then:

- $\mathcal{C}_m(A,B) = 1$ if and only if A is f-weak contradictory w.r.t. B.
- $\mathcal{C}_m(A,B) = 0$ if and only if there exists a sequence $\{u_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim A(u_i) = \lim B(u_i) = 1$.

Proof: The first statement follows directly from Proposition 10, and the second one from Corollaries 5 and 6. ■

Another interesting framework (which, to the best of our knowledge, has not appears in the literature) would be to fix a priori *two* operators in Ω to represent contradiction: one to determine when two fuzzy sets are contradictory and the other to represent when not. In that way, given $f_1, f_2 \in \Omega$ we can consider that two fuzzy sets A and B are fully contradictory if they are f_1-weak-contradictory, and zero-contradictory if $f_2 \leq f_{A,B}$, that is, if A is f-weak-contradictory w.r.t. B then $f \geq f_2$. Note that is essential to require the inequality $f_1 < f_2$.

Corollary 8: Let A and B be two fuzzy sets, let $f_1, f_2 \in \Omega$ with $f_1 < f_2$ and let m be a density of contradiction such that $m(x,y) = 0$ if and only if $x < f_1(y)$ or $x < f_2(y)$. Then:

- $\mathcal{C}_m(A,B) = 1$ if and only if A is f_1-weak contradictory w.r.t. B.
- $\mathcal{C}_m(A,B) = 0$ if and only if $f_2 \leq f_{A,B}$

Proof: The first statement follows directly from Proposition 10, and the second one from Corollaries 5 and 6. ■

It is worth to note that Proposition 11, and also the second items in Corollaries 5, 6, and 7 are given in terms of sequences in the universe $\mathcal{U}$. Hence, if the universe considered would be finite, the sequences can be substituted by the existence of two elements in the universe satisfying the property.

Corollary 9: Let A and B be two fuzzy sets and let m be a density of contradiction. Then $\mathcal{C}_m(A,B) = 0$ if and only if for all maximal $(\alpha,\beta) \in E_m$ there is $u \in \mathcal{U}$ such that $A(u) \geq \alpha$ and $B(u) \geq \beta$.

A similar consequence can be obtained if the membership functions of A and B are *continuous*.

Corollary 10: Let m be a density of contradiction, let $\mathcal{U}$ be a bounded set of $\mathbb{R}$ and let A and B be two fuzzy sets defined on $\mathcal{U}$ with continuous membership functions. Then $\mathcal{C}_m(A,B) = 0$ if and only if for all maximal $(\alpha,\beta) \in E_m$ there is $u \in \mathcal{U}$ such that $A(u) \geq \alpha$ and $B(u) \geq \beta$.

Proof: Assume that $\mathcal{C}(A,B) = 0$. By Proposition 11, we know that there exists a sequence $\{u_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim A(u_i) = \lim B(u_i) = 1$. Let us show that there exists an element $u \in \{u_i\}_{i\in\mathbb{N}}$ such that $A(u) = B(u) = 1$. As $\mathcal{U}$ is bounded, then $\{u_i\}_{i\in\mathbb{N}}$ is bounded as well. Thus we can ensure there exists a convergent subsequence of $\{u_i\}_{i\in\mathbb{N}}$; let us denote such subsequence by $\{v_i\}_{i\in\mathbb{N}}$. Let us show that $u = \lim v_i$ is the searched element. Firstly, let us show that $u \in \mathcal{U}$. As $[0,1]$ is a closed set of $\mathbb{R}$ and the membership function of A is continuous, the set $\mathcal{U} = A^{-1}([0,1])$ is closed. Therefore, as $\{v_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$ then $u = \lim v_i \in \mathcal{U}$. On the other hand, by continuity we have $A(u) = A(\lim v_i) = \lim A(v_i) = 1$. Similarly we can prove that $B(u) = 1$.

The converse implication is a direct consequence of Proposition 11. ■

To finish with, if we assume the requirement assumed either in Corollary 9 or 10 (i.e. either finite universes or continuous memberships), the second item in Corollaries 5, 6, and 7 can be rewritten, respectively, as:

- $\mathcal{C}_m(A, B) = 0$ if and only if for all W_i there exists $u_i \in \mathcal{U}$ such that $(A(u_i), B(u_i)) \in W_i$.
- $\mathcal{C}_m(A, B) = 0$ if and only if there exists $u \in \mathcal{U}$ such that $A(u) = B(u) = 1$.
- $\mathcal{C}_m(A, B) = 0$ if and only if there exists $u \in \mathcal{U}$ such that $A(u) = B(u) = 1$.

IV. On the definition of $\mathcal{C}_m$ in terms of negations

In Section II we introduced and motivated the set of operators Ω to represent the idea of contradiction via the notion of f-weak-contradiction. The question now is, is it really necessary to consider the set of operators Ω to define the measures $\mathcal{C}_m$? What would happen if we define a measure of contradiction by using just negation operators or involutive negation operators? In this section, we answer these questions and motivate the use of operators in Ω with this study.

First of all, take into account that in the definition of $\mathcal{C}_m$ we required the existence of the least operator f such that A is f-weak-contradictory w.r.t. B. Besides, to guarantee the existence of such an operator, the structure of complete lattice is needed. However, the set of involutive negation operators with the usual order does not have such a complete lattice structure. Contrariwise, the set of negation operators does have a complete lattice structure with the usual ordering.

We redefine the measure $\mathcal{C}_m$ in terms of negations as follows: Let A and B be two fuzzy sets, consider then the infimum $n_{A,B}$ of the set of negations n such that A is n-weak-contradictory w.r.t. B. Let m be a density of contradiction. Then we can define the measure of contradiction $\mathcal{C}_m^*$ as follows:

$$\mathcal{C}_m^*(A, B) = 1 - \int_{x=0}^{1} \int_{y=0}^{n_{A,B}(x)} m(x, y)dydx \qquad (4)$$

An interesting feature is that, given two fuzzy sets A and B, the set of negations n such that A is n-weak-contradictory w.r.t. B could be the emptyset. That would happen if and only if there exists $u \in \mathcal{U}$ such that $A(u) > 0$ and $B(u) = 1$. Curiously enough, we have the following result:

Proposition 12: Let A and B be two fuzzy sets and let m be a density of contradiction. Then:

$$\mathcal{C}_m^*(A, B) = \begin{cases} \mathcal{C}_m(A, B) & \text{if } B(u) = 1 \text{ implies } A(u) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof: Assume firstly that $B(u) = 1$ implies $A(u) = 0$. Let us denote by $\mathcal{F}^*(A, B)$ the set of negations such that A is n-weak-contradictory with respect to B. As Ω contains all the negations, we have that $\mathcal{F}^*(A, B) \subseteq \mathcal{F}(A, B)$. This implies that $f_{A,B} \leq n_{A,B}$. To see the other inclusion, we only have to note that, thanks to Theorem 1, the operator $f_{A,B}$ is actually a negation operator, so $f_{A,B} = n_{A,B}$ and then $\mathcal{C}_m^*(A, B) = \mathcal{C}_m(A, B)$.

To prove the other case, note that there is no negation n such that A is n-weak-contradictory with respect to B. So $\mathcal{F}^*(A, B) = \varnothing$; since the infimum of the empty set is the supremum of the set, we have that

$$n_{A,B} = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

Now, the rest is straightforward. ■

The result above presents a clear difference between defining the measure of contradiction by using either negation operators or operators in Ω. However, note that in most cases, both definitions coincide. Note that, in the cases where both measures of contradiction differ, $\mathcal{C}_m^*$ has a non-continuous behavior, as the following example shows.

Example 10: Consider the family of fuzzy sets A_ε given by $A_\varepsilon(u) = \varepsilon$ for all $u \in \mathcal{U}$ and the fuzzy set B defined by $B(u) = 1$ for all $u \in \mathcal{U}$. Then, given a density of contradiction m such that $m(x, y) \neq 0$ for all $(x, y) \in [0, 1]^2$, the measures $\mathcal{C}_m^*(A_\varepsilon, B)$ are given by:[4]

$$\mathcal{C}_m^*(A_\varepsilon, B) = \begin{cases} 1 & \text{if } \varepsilon = 0 \\ 0 & \text{otherwise} \end{cases}$$

Note the highly unstable behavior of the measure for values close to 0. This behavior occurs because the set $\mathcal{F}^*(A, B)$ can be empty, whereas this never happens with $\mathcal{F}(A, B)$. Specifically, for the measures $\mathcal{C}_m(A_\varepsilon, B)$, in the case that $m(x, y) = 1$ for all $(x, y) \in [0, 1]^2$, are:

$$\mathcal{C}_m(A_\varepsilon, B) = 1 - \varepsilon$$

□

Another consequence of Proposition 12 is that the new measure of contradiction $\mathcal{C}_m^*$ is not symmetric. This can be easily seen.

Example 11: Consider the sets A_ε and B introduced in the previous example, then for $1 \neq \varepsilon \neq 0$ we have

$$\mathcal{C}_m^*(A_\varepsilon, B) = 0 \neq 1 - \varepsilon = \mathcal{C}_m^*(B, A_\varepsilon)$$

□

Summarizing, should we have used negation operators in the definition of $\mathcal{C}_m$, some undesirable behavior can arise such as the instability of the measure and the loss of symmetry.

V. Conclusion and future work

In this paper we have presented a measure of contradiction between fuzzy sets based on the notion of f-weak-contradiction. The measure has been defined in three steps. First, we have introduced and motivated the notion of f-weak-contradiction. Moreover, some results concerning with such a notion have been presented; among them we stress on those showing that mappings f can be considered as degrees of contradictions. The second step lied in defining formally the measure $\mathcal{C}_m(A, B)$. Such definition is motivated by the idea of considering the least mapping $f \in \Omega$ verifying

[4]The results of the measures follow easily by using Proposition 12 and Corollary 6.

that A is f-weak-contradictory w.r.t. B and we have proved two interesting properties, namely, antitonicity and symmetry. In addition, we have related the measure $\mathcal{C}_m(A,B)$ with different frameworks by fixing a specific kind of density of contradiction. In that way, we have characterized the two extremal cases of the measure $\mathcal{C}_m(A,B)$ according to the density of contradiction considered.

Finally, in the third step, we have seen that, although a similar measure of contradiction can be defined by using the notion of N-contradiction, two unexpected features appear in such a case: non-symmetry and a non-continuous behavior. Therefore, the consideration of the set of mappings Ω used to define the notion of f-weak-contradiction, seems to be crucial in the definition of $\mathcal{C}_m$.

As a future work we would like to use the same idea underlying in $\mathcal{C}_m(A,B)$ to measure another relationships between fuzzy sets like *inclusion* or *similarity*. Moreover, some preliminaries results led us to foresee a strong relationship between the measure $\mathcal{C}_m(A,B)$ and the notion of *overlap index* [1], which measures up to what extent two fuzzy sets share information.

REFERENCES

[1] H. Bustince, J. Fernandez, R. Mesiar, J. Montero, and R. Orduna, Overlap functions, *Nonlinear Analysis, Theory, Methods and Applications* 72:1488-1499, 2010.

[2] H. Bustince, N. Madrid, and M. Ojeda-Aciego. A measure of contradiction based on the notion of N-weak-contradiction. In *FUZZ-IEEE 2013, IEEE International Conference on Fuzzy Systems*, pp. 1–6, 2013.

[3] H. Bustince, M. Pagola, R. Mesiar, E. Hullermeier, F. Herrera. Grouping, Overlap, and Generalized Bientropic Functions for Fuzzy Modeling of Pairwise Comparisons. *IEEE Transactions on Fuzzy Systems* 20(3): 405–415, 2012.

[4] E. Castiñeira, S. Cubillo, and W. Montilla. Measuring incompatibility between Atanassov's Intuitionistic Fuzzy Sets. *Information Sciences* 180:820–833, 2010.

[5] E. Castiñeira, C. Torres-Blanc, and S. Cubillo. Obtaining contradiction measures on intuitionistic fuzzy sets from fuzzy connectives. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 20(1):41–58, 2012.

[6] J. L. Castro, E. Trillas, and J. M. Zurita. Searching Potential Conflicts in Medical Expert Systems. *Fuzzy Systems and Artificial Intelligence*, 3 (1):79–94, 1994.

[7] L. Cholvy and C. Garion A logic to reason on contradictory beliefs with a majority approach. *IJCAI Workshop on Inconsistency in Data and Knowledge* 2001.

[8] S. Cubillo, C. Torres, and E. Castiñeira. Self-contradiction and contradiction between two Atanassov's intuitionistic fuzzy sets. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 16(3):283–300, 2008.

[9] S. Cubillo and E. Castiñeira. Measuring contradiction in fuzzy logic. *International Journal of General Systems*, 34(1):39–59, 2005.

[10] S. Cubillo and E. Castiñeira. Measures of self-contradiction on Atanassov's intuitionistic fuzzy sets: an axiomatic model. *International Journal of Intelligent Systems*, 24:863–888, 2009.

[11] J. Fodor and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, Springer, 1994.

[12] B. Ganter and Rudolf Wille. *Formal Concept Analysis: Mathematical Foundations*. Springer-Verlag, 1998

[13] J. Grant and A. Hunter. Measuring inconsistency in knowledgebases. *J. Intelligent Information Systems*, 27:159–184, 2006.

[14] S. Harabagiu, A. Hickl, and F. Lacatusu. Negation, contrast and contradiction in text processing. In *Proceedings of the 21st national conference on Artificial intelligence - Volume 1*, AAAI'06, pages 755–762. AAAI Press, 2006.

[15] N. Madrid and M. Ojeda-Aciego. Towards a fuzzy answer set semantics for residuated logic programs. In *Web Intelligence/IAT Workshops*, pages 260–264, 2008.

[16] N. Madrid and M. Ojeda-Aciego. On coherence and consistence in fuzzy answer set semantics for residuated logic programs. *Lect. Notes in Computer Science*, 5571:60–67, 2009.

[17] N. Madrid and M. Ojeda-Aciego. On the measure of incoherence in extended residuated logic programs. In *IEEE Intl Conf on Fuzzy Systems (FUZZ-IEEE'09)*, pages 598–603, 2009.

[18] N. Madrid and M. Ojeda-Aciego. Measuring inconsistency in fuzzy answer set semantics. *IEEE Transactions on Fuzzy Systems*, 19(4):605–622, 2011.

[19] N. Madrid and M. Ojeda-Aciego. On least coherence-preserving negations. In *31st Meeting of the North-American Fuzzy Information Processing Society, NAFIPS'12*, pp. 1–6, 2012.

[20] N. Madrid and M. Ojeda-Aciego. On the measure of incoherent information in extended multi-adjoint logic programs. In *IEEE Symposium on Foundations of Computational Intelligence, FOCI 2013*, pages 30–37, 2013.

[21] R.G. Nelson. *Scattering theory of waves and particles.* Springer-Verlag, 1982.

[22] A.Pradera, G. Beliakov, and H. Bustince. Aggregation functions and contradictory information. *Fuzzy Sets and Systems* 191:41–61, 2012

[23] M. Thimm. Inconsistency measures for probabilistic logics. *Artificial Intelligence* 197:1–24, 2013.

[24] E. Trillas, C. Alsina, and J. Jacas. On contradiction in fuzzy logic. *Soft Computing* 3(4):197–199, 1999