A view of f-indexes of inclusion under different axiomatic definitions of fuzzy inclusion

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Abstract. In this paper we analyze the novel constructive definition of f-index of inclusion with respect to four of the most common axiomatic definitions of inclusion measure, namely Sinha-Dougherty, Kitainik, Young and Fan-Xie-Pei. There exist an important difference between the f-index and these axiomatic definitions of inclusion measure: the f-index represents the inclusion in terms of a *mapping* in unit interval, whereas the inclusion measure represents such an inclusion as a *value* in the unit interval.

1 Introduction

Extending crisp operations and relations to the fuzzy case has taken the attention of researchers since Zadeh introduced the notion of fuzzy sets [21]. However, there is not consensus about how to extend some of them and, due to intrinsic features of fuzzy sets, it looks that all those different ways are acceptable; the choice depends on the task or the context where fuzzy sets are defined. One example of this fact is the fuzzy extension of the relation of inclusion, for which there are two different kind of approaches, the *constructive* ones (which provide a formula to represent the inclusion relation) and the *axiomatic* ones (which present some basic properties that must be satisfied by any inclusion measure). In the former case, we can distinguish those based on fuzzy implications [1, 12], probability [17] and overlapping [7, 8, 14]. In the latter class, we can distinguish also between other two subclasses, those allowing a non-null degree of inclusion of some fuzzy sets into the empty set [13, 18] and those that not, which are also related to entropy measures and overlaping [11, 20]. In the literature, one can find many theoretical and practical studies on such families of axiomatic definitions [4–6, 9, 10, 12, 19].

Most of the generalizations in the literature about fuzzy inclusion have a common feature; namely, they are relations that assign a value in the unit interval to each pair of fuzzy sets, A and B, that determines the degree of inclusion of A in B. One exception is [16], where the notion of inclusion is represented by assigning to each pair of fuzzy sets a mapping between the unit interval. Despite this differential feature, in this paper we analize the f-index of inclusion under the view of the four most common axiomatic definitions of measure of inclusion namely, Sinha-Dougherty [18], Kitainik [13], Young [20] and Fan-Xie-Pei [11]. Actually, we show that the f-index of inclusion satisfies, somehow, the axioms of

Sinha-Dougherty with the exception of the relationship with the complementary, which needs to be rewritten in terms of Galois connections.

The structure of this paper is the following. In Section 2 we present the preliminaries which includes the four respective axiomatic definitions of inclusion measures and the notion of the f-index of inclusion. Subsequently, in Section 3, we check each axiom in the context of f-indexes of inclusion, by showing which axioms hold and under which circunstancies. Finally, in Section 4 we present are conclusions and prospects for future work.

2 Preliminaries

A fuzzy set A is a pair $(\mathcal{U}, \mu_A)$ where $\mathcal{U}$ is a non empty set (called the universe of A) and μ_A is a mapping from $\mathcal{U}$ to $[0,1]$ (called membership function of A). In general, the universe is a fixed set for all the fuzzy sets considered and therefore, each fuzzy sets is determined by its membership function. Hence, for the sake of clarity, we identify fuzzy sets with membership functions (i.e., $A(u) = \mu_A(u)$).

On the set of fuzzy sets defined on the universe $\mathcal{U}$, denoted $\mathcal{F}(\mathcal{U})$, we can extend the usual crisp operations of union, intersection and complement as follows. Given two fuzzy sets A and B, we define

- (union) $A \cup B(u) = \max\{A(u), B(u)\}$
- (intersection) $A \cap B(u) = \min\{A(u), B(u)\}$
- (complement) $A^c(u) = n(A(u))$

where $n\colon [0,1] \to [0,1]$ is a negation operator; i.e., n is a decreasing mapping such that $n(0) = 1$ and $n(1) = 0$. In a considerable number of papers, the negation considered is involutive (*strong negation* in fuzzy settings) which adds to n the condition $n(n(x)) = x$ for all $x \in [0,1]$. This condition is crucial to have in fuzzy sets the equality $(A^c)^c = A$. In this paper we assume that *the complement of a fuzzy set is always defined in terms of an involutive negation.*

An implication $I\colon [0,1] \times [0,1]$ is any mapping decreasing in its first component, increasing in the second component and such that $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(0,1) = 0$.

Any transformation in the universe $T\colon \mathcal{U} \to \mathcal{U}$ can be extended to $\mathcal{F}(\mathcal{U})$ by defining for each $A \in \mathcal{F}(\mathcal{U})$ the fuzzy set $T(A)(x) = A(T(x))$.

In the rest of this section we deal with different approaches to the notion of inclusion between fuzzy sets which can be found in the literature. We recall below some of them which will be alter considered in the framework of our f-indexes of inclusion in Section 3.

2.1 Sinha-Dougherty axioms

One of the most common measures of inclusion was originally proposed by Sinha-Dougherty in [18].

Definition 1. *A mapping* $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ *is called an* SD-inclusion relation *if it satisfies the following axioms for all fuzzy sets* A, B *and* C*:*

(SD1) $\mathcal{I}(A,B)=1$ *if and only if* $A(u)\le B(u)$ *for all* $u\in\mathcal{U}$.
(SD2) $\mathcal{I}(A,B)=0$ *if and only if there exists* $u\in\mathcal{U}$ *such tat* $A(u)=1$ *and* $B(u)=0$.
(SD3) If $B(u)\le C(u)$ *for all* $u\in\mathcal{U}$ *then* $\mathcal{I}(A,B)\le\mathcal{I}(A,C)$.
(SD4) If $B(u)\le C(u)$ *for all* $u\in\mathcal{U}$ *then* $\mathcal{I}(C,A)\le\mathcal{I}(B,A)$.
(SD5) If $T\colon\mathcal{U}\to\mathcal{U}$ *is a bijective transformation on the universe, then* $\mathcal{I}(A,B)=\mathcal{I}(T(A),T(B))$.
(SD6) $\mathcal{I}(A,B)=\mathcal{I}(B^c,A^c)$.
(SD7) $\mathcal{I}(A\cup B,C)=\min\{\mathcal{I}(A,C),\mathcal{I}(B,C)\}$.
(SD8) $\mathcal{I}(A,B\cap C)=\min\{\mathcal{I}(A,B),\mathcal{I}(A,C)\}$.

Sinha and dougherty included also that $\mathcal{I}(A,B\cup C)\ge\max\{\mathcal{I}(A,B),\mathcal{I}(A,C)\}$ for all fuzzy sets A,B and C. However, we do not consider it here since it is a direct consequence of Axiom (SD3) [2,6].

2.2 Kitainik axioms

In 1987, Kitainik [13] proposed an axiomatic definition of fuzzy subsethood which captures the essential of inclusion measures based on implications. One of the main differences with respect to the axiomatic definition of fuzzy inclusion is the independence with respect to Zadeh's definition, i.e. $(SD1)$.

Definition 2. *A mapping* $\mathcal{I}\colon\mathcal{F}(\mathcal{U})\times\mathcal{F}(\mathcal{U})\to[0,1]$ *is called* K-inclusion relation *if it satisfies the following axioms for all fuzzy sets* A,B *and* C*:*

(K1) $\mathcal{I}(A,B)=\mathcal{I}(B^c,A^c)$.
(K2) $\mathcal{I}(A,B\cap C)=\min\{\mathcal{I}(A,B),\mathcal{I}(A,C)\}$.
(K3) If $T\colon\mathcal{U}\to\mathcal{U}$ *is a bijective transformation on the universe, then* $\mathcal{I}(A,B)=\mathcal{I}(T(A),T(B))$.
(K4) If A *and* B *are crisp then* $\mathcal{I}(A,B)=1$ *if and only if* $A\subseteq B$.
(K5) If A *and* B *are crisp then* $\mathcal{I}(A,B)=0$ *if and only if* $A\not\subseteq B$.

It is not difficult to check that every Kitainik inclusion measure is also a Sinha-Dougherty measure.

Moreover, Fodor and Yager showed, by using a representation result already published by Kitianik, that for every K-measure of inclusion $\mathcal{I}$ there exists an implication I such that ,for all fuzzy sets A and B, it holds

$$\mathcal{I}(A,B)=\inf\{I(A(u),B(u))\mid u\in\mathcal{U}\}.$$

2.3 Young axioms

The axioms proposed by Young [20] for a measure of inclusion are based on measures of entropy [15]. Specifically, the following relationship between both measures is proposed: if $\mathcal{I}$ is a measure of inclusion, then $\mathcal{E}(A)=\mathcal{I}(A\cup A^c,A\cap A^c)$ defines a measure of entropy. Based on such an idea, the following axiomatic definition was given:

Definition 3. *A mapping* $\mathcal{I}\colon \mathcal{F}(\mathcal{U})\times\mathcal{F}(\mathcal{U}) \to [0,1]$ *is called* Y-inclusion relation *if it satisfies the following axioms for all fuzzy sets* A, B *and* C*:*

(Y1) $\mathcal{I}(A,B) = 1$ *if and only if* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$*.*

(Y2) If $A(u) \geq 0.5$ *for all* $u \in \mathcal{U}$*, then* $\mathcal{I}(A, A^c) = 0$ *if and only if* $A = \mathcal{U}$*; i.e.,* $A(u) = 1$ *for all* $u \in \mathcal{U}$*.*

(Y3) If $A(u) \leq B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(C,A) \leq \mathcal{I}(B,A)$ *for all fuzzy set* $A \in \mathcal{F}(\mathcal{U})$*.*

(Y4) If $B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$ *for all fuzzy set* $A \in \mathcal{F}(\mathcal{U})$*.*

In the original definition [20] axioms (Y3) and (Y4) are written jointly as one axiom.

2.4 Fan-Xie-Pei axioms

The definition of Young was analyzed and modified slightly by Fan, Xie and Pie [11]. Firstly they criticize the axiom (Y4) and propose to change it by

(FX4) If $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$ for all fuzzy set $A \in \mathcal{F}(\mathcal{U})$.

Subsequently, they propose two other different definitions of measure of inclusion, called weak and strong respectively, by modifying the axioms in Young's definition.

Definition 4. *A mapping* $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ *is said to be a* strong FX-inclusion relation *if it satisfies the following axioms for all fuzzy sets* A, B *and* C*:*

(sFX1) $\mathcal{I}(A,B) = 1$ *if and only if* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$*.*

(sFX2) If $A \neq \varnothing$ *and* $A \cap B = \varnothing$ *then,* $\mathcal{I}(A,B) = 0$*.*

(sFX3) If $A(u) \leq B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(C,A) \leq \mathcal{I}(B,A)$ *and* $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$ *for all fuzzy set* $A \in \mathcal{F}(\mathcal{U})$*.*

Definition 5. *A mapping* $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ *is said to be* weak FX-inclusion relation *if it satisfies the following axioms for all fuzzy sets* A, B *and* C*:*

(wFX1) $\mathcal{I}(\varnothing,\varnothing) = \mathcal{I}(\varnothing,\mathcal{U}) = \mathcal{I}(\mathcal{U},\mathcal{U}) = 1$*; where* $\mathcal{U}(u) = 1$ *for all* $u \in \mathcal{U}$*.*

(wFX2) $\mathcal{I}(A,\varnothing) = 0$

(wFX3) If $A(u) \leq B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(C,A) \leq \mathcal{I}(B,A)$ *and* $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$ *for all fuzzy set* $A \in \mathcal{F}(\mathcal{U})$*.*

In the original paper of Fan, Xie and Pie [11] the reader can find relationships between these measures and fuzzy implications.

2.5 f-indexes of inclusion

Although most of the approaches for extending the relation of inclusion between fuzzy sets consider mappings of the type $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$, as done by Sinha and Dougherty, there are also other approaches that differ from such idea. One example is [16] which assigns to each pair of fuzzy sets a mapping $f\colon [0,1] \to [0,1]$ to represent their degree of inclusion. This approach is based on the definition of f-inclusion given at following.

Definition 6. *Let A and B be two fuzzy sets and let $f\colon [0,1] \to [0,1]$ be a mapping such that $f(x) \le x$ for all $x \in [0,1]$. We say that A is f-included in B if the inequality $f(A(u)) \le B(u)$ holds for all $u \in \mathcal{U}$.*

Note that the f-inclusion is a crisp relationship in the sense that A is either f-included in B or not. Thus, it somehow reminds the original definition of fuzzy inclusion given by Zadeh. The fuzziness in the definition above is that each f should be considered a degree of inclusion. Let us try to clarify this point. For the sake of presentation, let us define Ω as the set of functions $f\colon [0,1] \to [0,1]$ such that $f(x) \le x$ for all $x \in [0,1]$; i.e., Ω is our set of inclusion degrees. Note that Ω has the structure of a complete lattice, where id (i.e. $id(x) = x$ for all $x \in [0,1]$) and 0 (i.e. $0(x) = 0$ for all $x \in [0,1]$) are the top and least element, respectively.

Given two functions $f, g \in \Omega$ such that $f \le g$ then, $A \subseteq_f B$ implies $A \subseteq_g B$ (see Proposition 7 in Section 3). As a result, the greater the mapping f, the stronger the restriction imposed by the f-inclusion. So, each mapping f in Ω represents a degree of inclusion between fuzzy sets according to the strength of the restriction imposed by the respective f-inclusion relation. In such a way, the mapping 0 represents the null degree of inclusion (actually all pairs of fuzzy sets A and B satisfy $A \subseteq_0 B$) whereas id represents the highest degree (actually if a pair of fuzzy sets A and B satisfy $A \subseteq_{id} B$ then $A \subseteq_f B$ for all $f \in \Omega$). The reader is also referred to [16] for deeper motivational aspects of this set of f-indexes of inclusions.

In order to assign a convenient f-index of inclusion to a pair of fuzzy sets, it can be proved that given two fuzzy sets A and B, the following set

$$\{f \in \Omega \mid A \subseteq_f B\}$$

is closed under suprema. Therefore, its greatest element (denoted hereafter by f_{AB}) seems to be the most appropriated f-index of inclusion for the relation $A \subseteq B$. Moreover, such a mapping is determined by the following theorem.

Theorem 1 ([16]). *Let A and B be two fuzzy sets. Then, the greatest element of $\{f \in \Omega \mid A \subseteq_f B\}$ is*

$$f_{AB}(x) = \min\{x, \inf_{u \in \mathcal{U}} \{B(u) \mid x \le A(u)\}\} \tag{1}$$

Now, in order to provide evidences about why f_{AB} is an appropriated f-index for the relation $A \subseteq B$, in the next section we show that almost all the axioms given by Sinha and Dougherty hold for such an index under a convenient and natural rewriting.

3 Checking axioms for the f-index of inclusion

In the previous section, we have recalled the definition of the f-index of inclusion together with several axiomatic definitions of inclusion measure. To begin with, it is worth to note that the usual notion of an inclusion measure is that of a mapping $\mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$, whereas the f-index is a mapping $\mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to \Omega$, where Ω is the set of functions in the unit interval which are smaller than the identity. Fortunately, thanks to the lattice structure of Ω, the translation of all the axioms is straightforward.

It is also remarkable that many of the axioms in different axiomatic systems are identical or very related. For that reason and for the sake of the presentation, these axioms are grouped together in this section under a common feature, and the relationship with the f-index of inclusion is analyzed jointly.

3.1 Relationship with Zadeh's definition

The original definition of fuzzy inclusion introduced by Zadeh [21] states that, for any two fuzzy sets $A, B \in \mathcal{F}(\mathcal{U})$,

$$A \subseteq B \text{ if and only if } A(u) \leq B(u) \text{ for all } u \in \mathcal{U}.$$

Note that axioms (SD1), (Y1) and (sFX1) are almost identical to that definition. Actually, those axioms can be rewritten as *"the degree of inclusion of A in B is 1 if and only if A is contained in B in Zadeh's sense"*. The following result, already proved in [16, Corollary 2], shows that the f-index of inclusion satisfies exactly this condition.

Proposition 1. *Let A and B be two fuzzy sets. Then,* $f_{AB} = id$ *if and only if* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$.

It is convenient to mention that axioms (K4) and (wFX1) are weaker assumptions than Zadeh's inclusion, and therefore, they are also satisfied by f-indexes.

3.2 The case of null inclusion

Different axiomatic systems treat differently the particular case of the null inclusion, some of them introduce a characterization (if and only if) of the situations in which the degree of inclusion is zero, whereas others simply state a condition (if ... then) that it should satisfy. The axioms related to null inclusion are (SD2), (K5), (Y2), (sFX2) and (wFX2).

It is worth to mention here that, these axioms are slightly controversial. In fact, Sinha and Dougherty stated in [18]:

> *"[...] Axiom 2 may seem unnatural to many readers. In particular, this axiom causes 'problems' if we wish to model the entropy of a set via Kosko's method [...]"* and also
> *" [...] one may want to add another requirement that* $\mathcal{I}(A, \varnothing) = 0$*. This is not consistent with Axiom 2."*

The importance of not assuming $\mathcal{I}(A, \varnothing) = 0$ (note that this equality does not hold in our approach either) is that

> *"$\mathcal{I}(A, \varnothing)$ denotes the degree to which A can be classified as the empty set".*

In other words, (SD2) is contradictory with axioms (Y2), (sFX2) and (wFX2). Sinha and Dougherty also propose in [18] the following weak version of Axiom (SD2): for every pair of fuzzy sets A and B

$(SD2^*)$ $\big(\exists u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0\big)$ implies $\mathcal{I}(A, B) = 0$.

This weaker version of the axiom $(SD2)$ holds in the context of f-indexes.

Proposition 2 ([16]). *Let A and B be two fuzzy sets. If there exists $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$, then $f_{AB} = 0$.*

Note that the previous result implies, in general, that the f-index of inclusion does not satisfy either axiom (Y2) or (sFX2) or (wFX2), as the following example shows.

Example 1. Consider $\mathcal{U} = \{a, b\}$ and let A be the fuzzy set defined by $A(a) = 0.6$ and $A(b) = 0.8$. Then, by equation (1), we have that:

$$f_{A\varnothing}(x) = \begin{cases} 0 & \text{if } x \leq 0.8 \\ x & \text{otherwise} \end{cases}$$

which is obviously different from the function 0. This fact contradicts (sFX2) and (wFX2), since for any measure of inclusion $\mathcal{I}$ holding such axioms we have $\mathcal{I}(A, \varnothing) = 0$. Moreover, by equation (1) again, we have

$$f_{A,A^c}(x) = \begin{cases} 0.2 & \text{if } 0.2 \leq x \leq 0.8 \\ x & \text{otherwise} \end{cases}$$

Note that $f_{A,A^c} \neq 0$ and that for any measure of inclusion $\mathcal{I}$ holding (Y2), we have $\mathcal{I}(A, A^c) = 0$.

We study now, more in depth, the relationship of f-indexes with the axioms (SD2) and (K4). For this, let us recall the following characterization for the index $f = 0$.

Proposition 3 ([16]). *Let A and B be two fuzzy sets. $f_{AB} = 0$ if and only if there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$ such that $A(u_n) = 1$ and $\lim B(u_n) = 0$.*

As a result, we have that axiom (K4) holds for the f-index of inclusion.

Corollary 1. *Let A and B be two crisp sets then, $f_{AB} = 0$ if and only if $A \subseteq B$.*

Finally, note that Proposition 3 is very close but not equal to axiom $(SD2)$. The difference is that the for the f-index to be 0 there should not exist an element fully in A that fully not in B but, instead, it is sufficient that for each $\varepsilon < 0$ there exists an element fully in A which is in B in degree smaller than ε. Obviously, if the underlying universe $\mathcal{U}$ is finite, we get exactly axiom (SD2).

Corollary 2. *Let A and B be two fuzzy sets on a finite universe $\mathcal{U}$. $f_{AB} = 0$ if and only if there exists $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$.*

3.3 About monotonicity

The axioms related to the monotonicity of measures of inclusion are (SD3), (SD4), (Y3),(Y4), (FX4), (sFX3) and (wFX3).

We show that (SD3) and (SD4) hold for the f-indexes of inclusion as a consequence of the following result which establishes some monotonic properties for the f-index

Proposition 4 ([16]). *Let A, B, C, and D be four fuzzy sets such that $A(u) \leq B(u)$ and $C(u) \leq D(u)$ for all $u \in \mathcal{U}$ then, $B \subseteq_f C$ implies $A \subseteq_f D$.*

As a direct consequence, the axioms (SD3) and (SD4) hold in the context of f-indexes as well.

Corollary 3. *Let A, B and C be three fuzzy sets:*

- *if $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $f_{AB} \leq f_{AC}$;*
- *if $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $f_{CA} \leq f_{BA}$.*

The rest of axioms, namely (Y3),(Y4), (FX4), (sFX3) and (wFX3) also hold for the f-indexes since are weaker forms of (SD3) and (SD4).

3.4 Transformation Invariance

The only two axioms related to transformations on the universe $\mathcal{U}$ are (SD5) and (K3), and are identical. Let us recall that the axiom (SD5) states that for any fuzzy inclusion $\mathcal{I}$, if $T : \mathcal{U} \to \mathcal{U}$ is a transformation (i.e. a one-to-one mapping) on the universe, then

$$\mathcal{I}(A, B) = \mathcal{I}(T(A), T(B))$$

for all fuzzy sets A and B. This axiom comes from the crisp environment, where the inclusion relationship is not modified if it is applied any kind of transformation; as reflexion, translations, etc. Let us see that it is also satisfied in the context of f-indexes.

Proposition 5. *Let A and B be two fuzzy sets and let $T : \mathcal{U} \to \mathcal{U}$ be a transformation on $\mathcal{U}$, then $f_{AB} = f_{T(A)T(B)}$.*

Proof. Since f_{AB} and $f_{T(A)T(B)}$ are, by definition, the suprema of the sets $\{f \in \Omega \mid A \subseteq_f B\}$ and $\{f \in \Omega \mid T(A) \subseteq_f T(B)\}$, respectively, we prove the result by showing that both sets are the same. Consider $f \in \Omega$ such that $A \subseteq_f B$; then, for all $u \in \mathcal{U}$ we have $f(A(u)) \leq B(u)$ which, by the bijectivity of T, is equivalent to say that for all $u \in \mathcal{U}$ we have $f(A(T(u))) \leq B(T(u))$, which is equivalent to $T(A) \subseteq_f T(B)$.

3.5 Relationship with the complement

The complement appears in axioms (Y2), (SD6) and (K1). The case of (Y2) is more related with the null degree of inclusion than with the relation between inclusion and complement, and for such a reason, it was studied in Section 3.2; the other two axioms (SD6) and (K1) are identical, and state that that for any fuzzy inclusion $\mathcal{I}$ and every pair of fuzzy sets A and B, the equality $\mathcal{I}(A, B) = \mathcal{I}(B^c, A^c)$ holds.

In general, neither the equality $f_{AB} = f_{B^cA^c}$ nor the relation $A \subseteq_f B$ implies $B^c \subseteq_f A^c$ holds for $f \in \Omega$. However, it is possible to establish some relationships between both f-indexes via adjoint pairs. Let us recall that two mappings $f, g\colon [0, 1] \to [0, 1]$ form an adjoint pair if

$$f(x) \le y \iff x \le g(y) \qquad \text{for all } x \in [0, 1] \tag{2}$$

The first result connects the f-indexes of inclusion of A in B with those related to the inclusion of B in A via the negation n used to define the complement and adjoint pairs.

Proposition 6. *Let A and B be two fuzzy sets and let (f, g) be an adjoint pair. Then $A \subseteq_f B$ if and only if $B^c \subseteq_{n\circ g\circ n} A^c$.*

Proof. Let us begin by proving that $f \in \Omega$ if and only if $n \circ g \circ n \in \Omega$.

On the one hand, since (f, g) forms an adjoint pair, both mappings f and g are monotonic. This is a straightforward consequence of the definition: by Equation (2) and from $f(x) \le f(x)$ we get $x \le g \circ f(x)$ for all $x \in [0, 1]$. Now, monotonicity comes from adjointness, since

$$x_1 \le x_2 \le g \circ f(x_2) \iff f(x_1) \le f(x_2) \qquad \text{for all } x_1, x_2 \in [0, 1]$$

The monotonicity of g is proved similarly.

On the other hand, let us assume $f \in \Omega$, that is, $f(x) \le x$ for all $x \in [0, 1]$. Then by the adjoint property we have the following chain of equivalences for all $x \in [0, 1]$.

$$f(x) \le x \iff f(n(x)) \le n(x) \iff n(x) \le g(n(x)) \iff x \ge n(g(n(x)))$$

So, $f \in \Omega$ if and only if $n \circ g \circ n \in \Omega$.

Let us assume now that $A \subseteq_f B$. Then, for any $u \in \mathcal{U}$ we have:

$$f(A(u)) \le B(u) \iff A(u) \le g(B(u)) \iff n(A(u)) \ge n(g(B(u))).$$

Finally, by using that $n \circ n = id$, we have that

$$f(A(u)) \le B(u) \iff n(A(u)) \ge n(g(n(n(B(u))))),$$

or equivalently, $B^c \subseteq_{n\circ g\circ n} A^c$.

The following theorem shows that the f-index of A included in B and B^c included in A^c are related by adjointness in the case of a finite universe.

Theorem 2. *Let A and B be two fuzzy sets on a finite universe $\mathcal{U}$. Then, $(f_{AB}, n \circ f_{B^cA^c} \circ n)$ forms an adjoint pair.*

Proof. Let us begin by noticing that f_{AB} is always the left adjoint of an isotone Galois connection. This is equivalent to prove that $f_{AB}\left(\sup_{i\in\mathbb{I}} x_i\right) = \sup_{i\in\mathbb{I}} f_{AB}(x_i)$ and that equality comes from the structure of f_{AB}

$$f_{AB}(x) = \min\{x, \inf_{u\in\mathcal{U}}\{B(u) \mid x \leq A(u)\}\}$$

given by Equation (1) and the fact that the universe is finite.

To prove now that $(f_{AB}, n \circ f_{B^cA^c} \circ n)$ forms an adjoint pair we use a result that states that if (f_{AB}, g) is an isotone Galois connection, then $g(y) = \sup\{x \in [0,1] \mid f_{AB}(x) \leq y\}$. So let us check that $g(y) = n \circ f_{B^cA^c} \circ n(y)$ for all $y \in [0,1]$:

$$g(y) = \sup\{x \in [0,1] \mid f_{AB}(x) \leq y\}$$

(By definition of f_{AB})

$$= \sup\{x \in [0,1] \mid \min\{x, \min_{u\in\mathcal{U}}\{B(u) \mid x \leq A(u)\} \leq y\}$$

(By associativity of min)

$$= \sup_{u\in\mathcal{U}}\{x \in [0,1] \mid x \leq y, B(u) \leq y, x \leq A(u)\}$$

(By n involutive)

$$= \sup_{u\in\mathcal{U}}\{n(n(x)) \in [0,1] \mid x \leq n(n(y)), B(u) \leq n(n(y)), n(n(x)) \leq A(u)\}$$

(By n decreasing and involutive)

$$= n\left(\inf_{u\in\mathcal{U}}\{n(x) \in [0,1] \mid n(y) \leq n(x), n(y) \leq n\left(B(u)\right), n(A(u)) \leq n(x)\}\right)$$

(By associativity of min)

$$= n\left(\inf\{n(x) \in [0,1] \mid \min\{n(y), \min_{u\in\mathcal{U}}\{n(A(u)) \mid n(y) \leq n(B(u))\} \leq n(x)\}\right)$$

$$= n\left(\min\{n(y), \min_{u\in\mathcal{U}}\{n(A(u)) \mid n(y) \leq n(B(u))\}\right) = n(f_{B^cA^c}(n(y)))$$

3.6 Relationship with union and intersection

The axioms related with union and intersection are (SD7), (SD8) and (K8). Note that (SD8) coincides with (K8). They state that for any fuzzy inclusion $\mathcal{I}$ and three fuzzy sets A, B and C we have the following equalities:

- $\mathcal{I}(A \cup B, C) = \min\{\mathcal{I}(A, C), \mathcal{I}(B, C)\}$
- $\mathcal{I}(A, B \cap C) = \min\{\mathcal{I}(A, B), \mathcal{I}(A, C)\}$

Once again, let us recall a result concerning ordering between f-indexes.

Proposition 7 ([16]). *Let A and B be two fuzzy sets and let $f, g \in \Omega$ such that $f \geq g$. Then, $A \subseteq_f B$ implies $A \subseteq_g B$.*

As a consequence of Propositions 1 and 7, we obtain:

Corollary 4. *Let A, B and C be three fuzzy sets and let $f, g \in \Omega$.*

- *If $A \subseteq_f C$ and $B \subseteq_g C$, then $A \cup B \subseteq_{f \wedge g} C$.*
- *If $A \subseteq_f B$ and $A \subseteq_g C$, then $A \subseteq_{f \wedge g} B \cap C$.*

But we can go further and prove the following theorem.

Theorem 3. *Let A, B and C be three fuzzy sets then,*

$$f_{A \cup B, C} = \min\{f_{AC}, f_{BC}\} \quad \textit{and} \quad f_{A, B \cap C} = \min\{f_{AB}, f_{AC}\}.$$

Proof. Let us prove the first equality $f_{A \cup B, C} = \min\{f_{AC}, f_{BC}\}$. By Theorem 1 and definition of $A \cup B$, we know that

$$f_{A \cup B, C} = \min\{x, \inf_{u \in \mathcal{U}} \{C(u) \mid x \leq \max\{A(u), B(u)\}\}\}$$

and by properties of infimum and maximum we have:

$$f_{A \cup B, C} = \min\{x, \inf_{u \in \mathcal{U}} \{C(u) \mid x \leq A(u)\}, \inf_{u \in \mathcal{U}} \{C(u) \mid x \leq B(u)\}\}$$

which is equivalent to say $f_{A \cup B, C} = \min\{f_{AC}, f_{BC}\}$.

The second equality, i.e., $f_{A \cup B, C} = \min\{f_{AC}, f_{BC}\}$, is proved similarly. By Theorem 1 and definition of $B \cap C$, we have that

$$f_{A, B \cap C} = \min\{x, \inf\{\min_{u \in \mathcal{U}}\{B(u), C(u)\} \mid x \leq A(u)\}\}$$

and by properties of infimum and maximum we have:

$$f_{A \cup B, C} = \min\{x, \inf_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\}, \inf_{u \in \mathcal{U}} \{C(u) \mid x \leq A(u)\}\}$$

which is equivalent to say $f_{A, B \cap C} = \min\{f_{AB}, f_{AC}\}$.

4 Conclusions and future work

We have studied the relationships between the f-index of inclusion presented in [16] and some axiomatic inclusion measures used commonly in the literature, namely, Sinha-Dougherty [18], Kitainik [13], Young [20] and Fan-Xie-Pei [11]. Despite the f-index cannot be considered, by definition, any of those inclusion measures, we show that it is very close to the Sinha-Dougherty axioms. Actually we show that for a finite universe all the axioms of Sinha-Dougherty (and therefore also those of Kitainik) hold excepti the one related to the complement (SD6). With respect to the complements, there is a natural relationship between the f-index of A in B and the one of B^c in A^c by means of Galois connections.

As future work it would be interesting to continue the motivation of the f-index of inclusion as a convenient representation of the relationship $A \subseteq B$. Moreover, it would be interesting also to establish relationships with the n-weak contradiction [3] and to define an f-index of similarity.

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