φ-degrees of inclusion and similarity between L-fuzzy sets

Nicolás Madrid, Manuel Ojeda-Aciego

Universidad de Málaga, Dept. Matemática Aplicada.
Blv. Louis Pasteur 35, 29071 Málaga, Spain.

Abstract

Inclusion is one of the most basic relations between sets. In this paper, we show how to represent the degree of inclusion between two L-fuzzy sets via a function. Specifically, such a function determines the minimal modifications needed in a L-fuzzy set to be included (in Zadeh's sense) into another. To reach such a goal, firstly we present the notion of f-inclusion, which defines a family of crisp binary relations between L-fuzzy sets that are used as indexes of inclusion and, subsequently, we define the φ-degree of inclusion as the most suitable f-inclusion under certain criterium. In addition, we also present three φ-degrees of similarity definable from the φ-degree of inclusion. We show that the φ-degree of inclusion and the φ-degrees of similarities satisfy versions of many common axioms usually required for measures of inclusion and similarity in the literature.

Keywords: Fuzzy sets, Measure of inclusion, Measure of Similarity

1. Introduction

Although the notion of *inclusion* is one of the most basic relations between sets, currently there is not a consensus about how to extend such a notion in fuzzy set theory. Possibly, the best known definition for inclusion is the original one provided by Zadeh in [30], which identifies inclusion between fuzzy sets with the point-wise ordering between membership functions. However, some approaches have criticized such a definition *"for being rigid and for the lack of softness according to the spirit of fuzzy*

Email addresses: `nicolas.madrid@uma.es` (Nicolás Madrid), `aciego@uma.es` (Manuel Ojeda-Aciego)

logic" (quoted from [6]). Basically, one can find three main kinds of approaches in the literature: those based on cardinality [8, 15, 17]; those based on logic implications [1, 3, 12]; and those based on axiomatic definitions [2, 9, 11, 16, 29]. Defining measures of inclusion is not only of a theoretical interest since, for instance, in a framework of Social Science, fuzzy inclusion can be linked with mainstream statistical techniques [27], in a framework of data analysis, with classifiers [18] and the search of redundancy [19], and in a framework of image processing, with fuzzy mathematical morphology [10] and image quality measures [13].

We consider the φ-index[1] of inclusion [22], which is closely related to the axiomatic approaches of Kitainik [16] and Sinha-Dougherty [26] (see [21] for a comparison), and assumes the following motto from [17]: *" A 'good' measure of inclusion should measure violations of Zadeh's inclusion."*. The main difference of the φ-index of inclusion with respect to the other existing measures of inclusion is that, instead of assigning a value in $[0,1]$ (or in a residuated lattice L) to the inclusion between two L-fuzzy sets, the φ-index of inclusion is a mapping from L to L. It is worth remarking that this set of possible φ-indices generalize the usual fuzzy inclusion given by a residuated implication. In this paper, we extend our previous results [21, 22] by introducing the φ-degree of inclusion between L-fuzzy sets. Moreover, we include also three φ-degrees of similarity which can be directly defined from the φ-index of inclusion, and show that all of them satisfy versions of properties usually required by many axiomatic approaches of similarity measures (see [7] for an overview). Similarity relations and measures has been used in several practical areas such as Decision Making, Risk Analysis, or Pattern Recognition [5, 14].

The structure of the paper is given as follows. In Section 2 we introduce the set of φ-indexes of inclusion, which is based on the crisp binary relation of f-inclusion. In this approach the *inclusion* of a fuzzy set A into another B is not modelled by fixing a specific f-inclusion a priori, but by assigning the most suitable f-inclusion that represents the *inclusion* of A in B. The suitability of such a choice is further explained in Section 3, together with properties, motivated by axiomatic definitions of fuzzy in-

[1] We use the prefix φ- to recall that these indexes are *functional* parameters.

clusion measures, which show that it is an adequate representation of the inclusion between L-fuzzy sets. Later, in Section 4 we introduce three φ-degrees of similarity defined from the φ-degrees of inclusion, we provide some properties of them and some relationship between the three φ-degrees of similarity. Finally, in Section 5 we present the conclusions and future work.

2. The set of φ-indexes of inclusion

We will consider hereafter a referential universe $\mathscr{U}$, together with a complete lattice $(L, \leq, \vee, \wedge)$ with 0 and 1 being its greatest and least elements, respectively. Let us recall that an L-fuzzy set A on $\mathscr{U}$ can be identified with its membership function $A\colon \mathscr{U} \to L$.

2.1. The notion of f-inclusion

As stated in the introduction, our approach is based on a graded version of the notion of inclusion between L-fuzzy sets, in which the grades will no longer be elements of L, but certain mappings from L to L.

Not every function from L to L can be used representing degrees of inclusion; our first approach to this kind of parameterized notion of inclusion was introduced originally in [22] in which the interesting functions (the *φ-indexes of inclusion*) had to be deflationary and increasing mappings. The corresponding extension to L-fuzzy sets is given below:

Definition 1. *The set Ω of φ-indexes of inclusion (denoted by Ω) is the set of* increasing *mappings $f\colon L \to L$ satisfying $f(x) \leq x$ for all $x \in L$.*

When considering lattices more general than the unit interval, we will make use of the following equivalent definition of the set Ω in order to deal with incomparable elements.

Lemma 1. *Let L be complete lattice. Then, $f \in \Omega$ if and only if f satisfies the inequalities $f(x) \leq x$ and $f(x) \vee f(y) \leq f(x \vee y)$ for all $x, y \in L$.*

Proof. Note that it is enough to prove that f is increasing if and only if $f(x) \vee f(y) \leq f(x \vee y)$ for all $x, y \in L$. Let us assume firstly that f is increasing and let $x, y \in L$. Then, since $x \leq x \vee y$ and $y \leq x \vee y$, we have $f(x) \leq f(x \vee y)$ and $f(y) \leq f(x \vee y)$. As a consequence, $f(x) \vee f(y) \leq f(x \vee y)$.

To prove the converse let us assume that $f(x) \vee f(y) \leq f(x \vee y)$ for all $x, y \in L$. Then, if $x \leq y$ we have $f(x) \leq f(x) \vee f(y) \leq f(x \vee y) = f(y)$. In other words, f is increasing. $\square$

The definition of f-inclusion is given as follows.

Definition 2 ([22]). *Let A and B be two fuzzy sets and consider $f \in \Omega$. We say that A* is f-included in B *(denoted by $A \subseteq_f B$) if and only if the inequality $f(A(u)) \leq B(u)$ holds for all $u \in \mathscr{U}$.*

Some remarks about the definition above:

- The relation of f-inclusion is a crisp relation.
- The relation of f-inclusion need not be transitive, hence it is not always an ordering relation.
- Different mappings $f \in \Omega$ define different relations of f-inclusion, which can be interpreted to certain extent as a degree of inclusion.

It is worth noting that each mapping f determines bounds between possible truth-values of a fuzzy set included in another. Specifically, fixed $f \in \Omega$, if $A \subseteq_f B$, then the value $f(A(u))$ determines a lower bound of the possible values of $B(u)$, for all $u \in \mathscr{U}$. Moreover, as f is increasing, the greater the value of $A(u)$, the greater the lower bound imposed to the value of $B(u)$ by the inequality $f(A(u)) \leq B(u)$, and thus, the greater the value of $B(u)$ should be. In the particular case of $A(u) = 0$, the f-inclusion does not impose any restriction on the value of $B(u)$, since $f(0) = 0$. This fact represents that the empty set is fully f-included in every fuzzy set for all $f \in \Omega$.

2.2. *Fundamental properties of φ-indexes of inclusion*

Note that Ω has a complete lattice structure with the natural ordering between functions; i.e., given $f, g \in \Omega$, we say that $f \leq g$ if $f(x) \leq g(x)$ for all $x \in L$. In this case, the supremum and infimum in Ω can be defined pointwise, that is, given $f, g \in \Omega$,

$f \vee g(x) = f(x) \vee g(x)$ and $f \wedge g(x) = f(x) \wedge g(x)$ for all $x \in L$. Note that both, $f \vee g$ and $f \wedge g$ are in Ω. As a result, we have the greatest and least elements in Ω given, respectively, by

$$id(x) = x \quad \text{and} \quad \perp(x) = 0 \quad \text{for all } x \in L.$$

The most important feature of a set Ω of φ-indexes of inclusion is the existence of an order between its elements to represent the relationship "*the greater the index, the stronger the inclusion*".

The following proposition shows how the ordering between mappings f and g is reflected when being considered as φ-indexes of inclusion.

Proposition 1. *Let A and B be two fuzzy sets and let $f, g \in \Omega$ such that $f \leq g$. Then, $A \subseteq_g B$ implies $A \subseteq_f B$.*

Proof. Let us assume that $A \subseteq_g B$ then, since $f \leq g$ we have

$$f(A(u)) \leq g(A(u)) \leq B(u)$$

for all $u \in \mathscr{U}$. So $A \subseteq_f B$ as well. □

From the previous proposition, changing the mapping $f \in \Omega$ by a smaller one g in the f-inclusion produces a weaker restriction in the following sense: every pair of fuzzy sets satisfying the f-inclusion with the original mapping will satisfy the restriction with the new one (which is smaller). In other words, the greater the mapping $f \in \Omega$, the stronger the restriction imposed by the f-inclusion. Moreover, note that as a consequence, the mapping $\perp$ determines the weakest f-inclusion whereas the mapping *id* the strongest one. Let us study how are the restrictions imposed by these two extreme f-inclusions. Let us begin by studying the $\perp$-inclusion which will be later used to model *null-inclusion* in Section 3.

Proposition 2. *The relation $A \subseteq_\perp B$ holds for all pairs of L-fuzzy sets A and B.*

Proof. Let A and B be two L-fuzzy sets then, $\perp(A(u)) = 0 \leq B(u)$ for all $u \in \mathscr{U}$. □

This proposition states that every L-fuzzy set is at least $\perp$-included in every other L-fuzzy set. The following result characterises when $A \subseteq_f B$ holds just for $f = \perp$.

Proposition 3. *Let A and B be two fuzzy sets. $A \subseteq_f B$ implies $f = \perp$ if and only if there is a nonempty set of elements in the universe $\{u_i\}_{i\in\mathbb{I}} \subseteq \mathscr{U}$ such that $A(u_i) = 1$ for all $i \in \mathbb{I}$ and $\bigwedge_{i\in\mathbb{I}} B(u_i) = 0$.*

Proof. Let us assume firstly that the only f-inclusion of A in B is $A \subseteq_\perp B$ and, by reductio ad absurdum, also that for all set $\{u_i\}_{i\in\mathbb{I}} \subseteq \mathscr{U}$ such that $A(u_i) = 1$ for all $i \in \mathbb{I}$ we have $\bigwedge_{i\in\mathbb{I}} B(u_i) \neq 0$. Then, let us consider the value

$$\alpha = \bigwedge_{u\in\mathscr{U}} \{B(u) \mid A(u) = 1\}.$$

Note that in the case there is no $u \in \mathscr{U}$ such that $A(u) = 1$ then, $\alpha = 1$. Therefore, α is different from 0 by the assumption. Consider the mapping $f\colon L \to L$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ \alpha & \text{if } x = 1 \end{cases}$$

Note that $f \in \Omega$, and let us show that $A \subseteq_f B$. Given $u \in \mathscr{U}$, if $A(u) \neq 1$ then, the inequality $f(A(u)) \leq B(u)$ holds straightforwardly. If $A(u) = 1$ then,

$$f(A(u)) = f(1) = \alpha = \bigwedge_{u'\in\mathscr{U}} \{B(u') \mid A(u') = 1\} \leq B(u).$$

Then, we have found $f \in \Omega$ such that $f \neq \perp$ (since $\alpha \neq 0$) and $A \subseteq_f B$, which contradicts that the only f-inclusion of A in B is $A \subseteq_\perp B$.

Let us assume now that there is a set $\{u_i\}_{i\in\mathbb{I}} \subseteq \mathscr{U}$ such that $A(u_i) = 1$ for all $i \in \mathbb{I}$ and $\bigwedge_{i\in\mathbb{I}} B(u_i) = 0$. Consider $f \in \Omega$ such that $A \subseteq_f B$ and let us show that $f = \perp$. Note that by monotonicity of f, it is enough to prove that $f(1) = 0$. From $A \subseteq_f B$ we have

$$f(A(u)) = f(1) \leq B(u) \qquad \text{for all } u \in \{u_i\}_{i\in\mathbb{I}}$$

which implies, taking into account that $\bigwedge_{i\in\mathbb{I}} B(u_i) = 0$, that

$$f(1) \leq \bigwedge_{i\in\mathbb{I}} B(u_i) = 0.$$

□

Note that, given two L-fuzzy sets A and B, if there exist an element $u \in \mathscr{U}$ such that $A(u) = 1$ and $B(u) = 0$, by the previous proposition, the only f-inclusion of A in B is

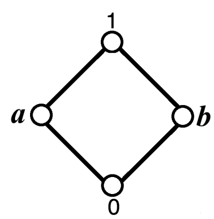
$\subseteq_\perp$ (i.e. null-inclusion). The converse is not true in general as the following examples show:

Example 1. *Let $\mathscr{U} = [1,\infty)$ and $L = [0,1]$. Let us consider the following two fuzzy sets given by $A(u) = 1$ and $B(u) = \frac{1}{u}$ for all $u \in [1,\infty)$. Note that there is no $u \in [1,\infty)$ such that $A(u) = 1$ and $B(u) = 0$, however the only f-inclusion of A in B is the $\perp$-inclusion since*

$$f(A(u)) = f(1) \leq B(u) = \frac{1}{u} \quad \text{for all } u \in [1,\infty)$$

holds just when $f(1) = 0$.

Example 2. *Let $\mathscr{U} = \{u_1, u_2\}$ and let L be the complete lattice given by the following Hasse diagram:*



Let us consider the two L-fuzzy sets A and B given by $A(u_1) = A(u_2) = 1$, $B(u_1) = a$ and $B(u_2) = b$. Note that there is no $u \in \mathscr{U}$ such that $A(u) = 1$ and $B(u) = 0$, however if A is f-included in B then:

$$f(1) = f(A(u_1)) \leq B(u_1) = a$$
$$f(1) = f(A(u_2)) \leq B(u_2) = b$$

which implies $f(1) \leq a \wedge b = 0$ or equivalently, that $f = \perp$.

Note that, whenever the only f-inclusion of A in B is the $\perp$-inclusion, Proposition 3 only ensures that we can find an element u such that the value of $A(u)$ is 1 and the value of $B(u)$ is as close to 0 as desired. A simplified formulation in terms of the existence of an $u \in \mathscr{U}$ such that $A(u) = 1$ and $B(u) = 0$ can be provided in some cases. The first case is when the lattice is totally ordered and the universe is finite.

Corollary 1. *Let L be a linearly ordered set let A and B be two L-fuzzy sets defined on a finite universe $\mathscr{U}$. Then there exists $u \in \mathscr{U}$ such that $A(u) = 1$ and $B(u) = 0$ if and only if $A \subseteq_f B$ implies $f = \perp$.*

Proof. It is a direct consequence of Proposition 3. $\square$

The second case is valid for infinite universes but requires the existence of a certain element in the lattice between 0 and the rest of elements.

Corollary 2. *Let L be a lattice with an element $a \in L$ such that $0 < a \leq l$ for all $l \in L \setminus \{0\}$. Let A and B be two L-fuzzy sets. Then there exists $u \in \mathscr{U}$ such that $A(u) = 1$ and $B(u) = 0$ if and only if $A \subseteq_f B$ implies $f = \perp$.*

Proof. The direct implication is straightforward.

Conversely, let A and B be two L-fuzzy sets and let us assume that $A \subseteq_f B$ implies $f = \perp$. Then, by Proposition 3, there is a set of elements in the universe $\{u_i\}_{i\in\mathbb{I}} \subseteq \mathscr{U}$ such that $A(u_i) = 1$ for all $i \in \mathbb{I}$ and $\bigwedge_{i\in\mathbb{I}} B(u_i) = 0$. Let us assume also, by reductio ad absurdum, that for all $u \in \{u_i\}_{i\in\mathbb{I}}$ we have that $B(u) \neq 0$. Then, by hypothesis $0 < a \leq B(u)$ for all $u \in \{u_i\}_{i\in\mathbb{I}}$, hence $0 < a \leq \bigwedge_{i\in\mathbb{I}} B(u_i)$, which is a contradiction. $\square$

The third case is a corollary from the previous result, and focuses on the case in which the lattice of truth-values is, in fact, a finite chain.

Corollary 3. *Let L be a finite chain and let A and B be two L-fuzzy sets. Then, there exist $u \in \mathscr{U}$ such that $A(u) = 1$ and $B(u) = 0$ if and only if $A \subseteq_f B$ implies $f = \perp$.*

Proof. The result is a direct consequence of Corollary 2. $\square$

Let us continue by studying the strongest case, i.e., the *id*-inclusion, which is characterised by the following result.

Proposition 4. *Let A and B be two L-fuzzy sets. The following statements are equivalent:*

1. $A \subseteq_{id} B$.
2. $A(u) \leq B(u)$ *for all* $u \in \mathscr{U}$.
3. $A \subseteq_f B$ *for all* $f \in \Omega$.

Proof. (1) $\Longleftrightarrow$ (2) is straightforward. (1) $\Longleftrightarrow$ (3) holds from Proposition 1, taking into account that *id* is the greatest element in Ω. $\square$

This result states that the highest possible φ-inclusion coincides with Zadeh's inclusion and it implies the satisfability of f-inclusion for all the rest of φ-indexes f. Moreover, note that the consideration of Zadeh's inclusion as the highest φ-index of inclusion is closely related to the main axiomatic approaches of measures of inclusion between fuzzy sets [2, 22, 29].

2.3. f-inclusion, reflexivity, antisymmetry and transitivity.

We have already motivated the use of mappings in Ω to represent the degree of inclusion between L-fuzzy sets in the section above. Here we show that, despite f-inclusion is not a partial order relation in general, it is a useful tool to deal with subsethood within fuzzy set theory. Specifically, we provide links between the f-inclusion and the notions of *reflexivity, antisymmetry* and *transitivity* on crisp binary relations. The first result shows that the f-inclusion is always reflexive.

Proposition 5. *Let A be a L-fuzzy set, then $A \subseteq_f A$ for all $f \in \Omega$.*

Proof. The result comes from the fact that $f(x) \leq x$ for all $f \in \Omega$ and $x \in L$. Therefore, $f(A(u)) \leq A(u)$ for all $u \in \mathscr{U}$. □

The f-inclusion is not transitive in general, but relates to it in terms of composition of mappings, as the following result states.

Proposition 6. *Let A, B and C be three L-fuzzy sets and let $f, g \in \Omega$. Then, $A \subseteq_f B$ and $B \subseteq_g C$ implies $A \subseteq_{g \circ f} C$.*

Proof. From $A \subseteq_f B$ and $B \subseteq_g C$ we have the inequalities $f(A(u)) \leq B(u)$ and $g(B(u)) \leq C(u)$ for all $u \in \mathscr{U}$, respectively. From the former inequality and by using that g is increasing, we have $g(f(A(u))) \leq g(B(u))$ for all $u \in \mathscr{U}$. So by adding the latter inequality we have $g(f(A(u))) \leq g(B(u)) \leq C(u)$ for all $u \in \mathscr{U}$; or equivalently, $A \subseteq_{g \circ f} C$. □

Concerning antisymmetry, it is not hard to prove that from $A \subseteq_f B$ and $B \subseteq_f A$ we cannot guarantee that $A = B$ (except for the extremal case $f = id$). However, both f-inclusions can be used to bound the difference between A and B, as we will see later in Section 4.

2.4. The φ-index of inclusion as an extension of the crisp case

The use of an φ-indexes as degrees of inclusion between L-fuzzy sets would make no sense if it did not extend the notion of inclusion in the crisp case. Such an extension can be analyzed from two different points of view: on the one hand, by restricting the approach to the crisp case and, on the other hand, by considering crisp sets within the L-fuzzy framework.

To begin with, L-fuzzy set theory becomes standard set theory if the lattice considered is $L = \{0,1\}$. In such a case, the set of φ-indexes Ω would consist of just two mappings $\perp$ and id. Note that the $\perp$-inclusion does not impose any restriction (see Proposition 2) and the id-inclusion holds if and only if $A(u) \leq B(u)$ for all $u \in \mathscr{U}$; i.e., if $A \subseteq B$ in the crisp sense. As a consequence, given two (crisp) sets A and B, then either A is not included in B, which is represented by the fact that A is f-included in B just for $\perp$, or A is included in B, which is represented by the fact that A is f-included in B for all indexes (namely, for $\perp$ and id).

Let us analyze now the extension of the crisp inclusion from the second point of view, i.e., by considering crisp sets within the fuzzy framework. Recall that a fuzzy set A is crisp if $A(u) \in \{0,1\}$ for all $u \in \mathscr{U}$. For crisp sets, we have the following result.

Proposition 7. *Let A and B be two crisp sets. If $A \subseteq_f B$ for $f \neq \perp$, then $A \subseteq_g B$ for all $g \in \Omega$.*

Proof. Let us assume that $A \subseteq_f B$ for $f \neq \perp$. Then, by monotonicity of f we have that $f(1) = \alpha \neq 0$. Then, for all $u \in \mathscr{U}$ such that $A(u) = 1$ we have

$$f(A(u)) = f(1) = \alpha \leq B(u).$$

Since $B(u) \in \{0,1\}$, then for all $u \in \mathscr{U}$ such that $A(u) = 1$ we have that $B(u) = 1$ as well. By Proposition 4 we have finally that $A \subseteq_g B$ for all $g \in \Omega$. □

As a consequence of the previous result, we have two extreme situations for any pair A and B of crisp sets: either A is only f-included in B for $\perp$, or A is f-included in B for all $f \in \Omega$; which can be identified with the no inclusion and the inclusion of A in B in the crisp sense, respectively.

2.5. The φ-indexes of inclusion as weak forms of Zadeh's fuzzy inclusion

The definition of inclusion between fuzzy sets provided by Zadeh [30] states that a fuzzy sets A is contained in another B if $A(u) \leq B(u)$ for all $u \in \mathscr{U}$. Note the similarity of such a definition with our Definition 2, since the only difference is the use of a function $f \in \Omega$ to modulate the membership values of A. The original definition of Zadeh has been criticized in some approaches for being rigid and lacking the spirit of fuzzy logic [6, 26]:*"This rigid definition unfortunately does not do justice to the spirit of fuzzy set theory: we may want to talk about a fuzzy set being "more or less" a subset of another one [...]"*. Our φ-indexes of inclusion approach adapt Zadeh's original idea by incorporating a function that determines *how much we have to modify A so that it is included in B in the sense of Zadeh*.

2.6. Relationship with measures of inclusion based on residuated implications

Among the constructive measures of inclusion existing in the literature, those based on fuzzy implications [1, 12] have a significative importance. Let us recall that an L-fuzzy implication is defined as a mapping $\mathscr{I}\colon L \times L \to L$ which is decreasing in the first component, increasing in the second component, and such that $\mathscr{I}(0,0) = \mathscr{I}(0,1) = \mathscr{I}(1,1) = 1$ and $\mathscr{I}(1,0) = 0$. Given two L-fuzzy sets A and B and an L-fuzzy implication $\mathscr{I}$, the degree of inclusion of A in B w.r.t. $\mathscr{I}$ is defined as:

$$Inc_{\mathscr{I}}(A,B) = \bigwedge_{u \in \mathscr{U}} \mathscr{I}(A(u),B(u)) \tag{1}$$

If the implication $\mathscr{I}$ is residuated (which is a common assumption) then, there exists a conjunction $\mathscr{C}\colon L \times L \to L$ increasing in both arguments with $\mathscr{C}(1,0) = \mathscr{C}(0,1) = \mathscr{C}(0,0) = 0$, $\mathscr{C}(1,1) = 1$, $\mathscr{C}(1,x) = x$ for all $x \in L$ and such that

$$\mathscr{C}(c,a) \leq b \quad \Longleftrightarrow \quad c \leq \mathscr{I}(a,b)$$

for all $a,b,c \in L$. Moreover, $\mathscr{C}$ is right continuous in the first argument; i.e., for $X \subseteq L$ we have that $\mathscr{C}(\bigvee X, y) = \bigvee\{\mathscr{C}(x,y) \mid x \in X\}$.

Let us assume now that $\alpha \leq Inc_{\mathscr{I}}(A,B)$ for some $\alpha \in L$. Then we have the following chain of statements

$$\alpha \leq Inc_{\mathscr{I}}(A,B) = \bigwedge_{u \in \mathscr{U}} \mathscr{I}(A(u),B(u))$$

$$\alpha \leq \mathscr{I}(A(u),B(u)) \qquad \text{for all } u \in \mathscr{U}$$

$$\mathscr{C}(\alpha,A(u)) \leq B(u) \qquad \text{for all } u \in \mathscr{U}$$

Note that this last inequality is an instance of Definition 2 for the function $f_\alpha \colon L \to L$ defined by $f_\alpha(x) = \mathscr{C}(\alpha,x)$: it satisfies $f_\alpha(x) \leq x$ for all $x \in L$ (the inequality comes from the monotonicity of $\mathscr{C}$ and that $\mathscr{I}(1,x) = x$ for all $x \in L$) and $f_\alpha(x) \vee f_\alpha(y) \leq f_\alpha(x \vee y)$ (since $\mathscr{C}$ is right continuous on the second argument).

In summary, we can represent the restriction imposed by any degree α of a measure of inclusion $Inc_{\mathscr{I}}$ based on residuated implications (Equation (1)) with the notion of f_α-inclusion. Note that f_α-inclusion is defined by choosing just a mapping in Ω, whereas the restriction $Inc_{\mathscr{I}}(A,B) \geq \alpha$ requires the choice of an implication operator and a threshold value $\alpha \in L$.

3. Representing the inclusion of fuzzy sets with a φ-index

3.1. Defining the φ-degree of inclusion of two L-fuzzy sets

Working with a fixed φ-index f and determining whether two L-fuzzy sets either do or do not satisfy the relation $\subseteq_f$ would have the same shortcoming of Zadeh's definition of inclusion. As we emphasized in the previous section, mappings in Ω can (and must) be considered as degrees of inclusion in the sense *"the greater the mapping, the stronger the inclusion"*. In this way, given two L-fuzzy sets A and B, the greatest $f \in \Omega$ verifying that $A \subseteq_f B$ would be a suitable one to represent the degree of inclusion of A in B. Therefore, the first task is to prove the existence of such a greatest element. Formally, given A and B, let us consider the following subset of Ω:

$$\Lambda(A,B) = \{f \in \Omega \mid A \subseteq_f B\}$$

It is not difficult to check that Ω inherits the structure of complete lattice and, therefore, we can guarantee the existence of the supremum of $\Lambda(A,B)$. The following result shows that such a supremum is in fact a maximum.

Lemma 2. *Let A and B be two L-fuzzy sets and consider $\{f_i\}_{i\in\mathbb{I}} \subseteq \Omega$. If A is f_i-included in B for all $i \in \mathbb{I}$, then A is $\bigvee_{i\in\mathbb{I}} f_i$-included in B.*

Proof. It is well known that $\bigvee_{i\in\mathbb{I}} f_i$ is given by $f(x) = \bigvee_{i\in\mathbb{I}} f_i(x)$ for all $x \in L$. Moreover, since Ω is a complete lattice, $\bigvee_{i\in\mathbb{I}} f_i \in \Omega$ as well. Now, since A is f_i-included in B for all $i \in \mathbb{I}$, we have that $f_i(A(u)) \leq B(u)$ for all $u \in \mathscr{U}$. Therefore, $\bigvee_{i\in\mathbb{I}} f_i(A(u)) \leq B(u)$ for all $u \in \mathscr{U}$, or equivalently, A is $\bigvee_{i\in\mathbb{I}} f_i$-included in B. □

As a direct consequence of the previous proposition, there exists the greatest element of $\Lambda(A,B)$ and, hence, the following definition of φ-degree of inclusion between L-fuzzy sets A and B makes sense.

Definition 3. *Let A and B be two L-fuzzy sets, the* φ-degree of inclusion *of A in B, denoted by $Inc(A,B)$, is defined as the maximum of $\Lambda(A,B)$.*

Note firstly that the φ-degree of inclusion of A in B just depends on L without any other a priori assumption or any kind of parameter [12]. Secondly, note that thanks to Proposition 1, the set $\Lambda(A,B)$ of mappings f in Ω such A is f-included in B is characterized by $Inc(A,B)$, since:

$$\Lambda(A,B) = \{f \in \Omega \mid A \subseteq_f B\} = \{f \in \Omega \mid f \leq Inc(A,B)\}$$

3.2. Analytic expression of the φ-degree of inclusion

In this section we will obtain an analytic expression of the φ-degree of inclusion which will be used in subsequent sections. Specifically, we show that given two L-fuzzy sets A and B, the φ-degree of inclusion of A in B can be obtained in terms of the the following auxiliary function:

$$f_{A,B}(x) = \bigwedge_{u\in\mathscr{U}} \{B(u) \mid x \leq A(u)\} \tag{2}$$

In fact, we will show below that $Inc(A,B) = f_{A,B} \wedge id$.

The proof of such a result has been divided into three lemmas. The first lemma shows that $f_{A,B} \wedge id$ is in Ω, so it makes sense to talk about $(f_{A,B} \wedge id)$-inclusion.

Lemma 3. *Let $f_{A,B}$ be the mapping defined in equation (2), then $f_{A,B} \wedge id$ is in Ω.*

Proof. Obviously, we have that $(f_{A,B} \wedge id)(x) \leq x$.

Now, consider $x, y \in L$ and let us show that

$$(f_{A,B} \wedge id)(x) \vee (f_{A,B} \wedge id)(y) \leq (f_{A,B} \wedge id)(x \vee y)$$

For this, let us see firstly that $f_{A,B}$ satisfies the inequality $f_{A,B}(x) \leq f_{A,B}(x \vee y)$ and $f_{A,B}(y) \leq f_{A,B}(x \vee y)$. Since

$$\{B(u) \mid x \vee y \leq A(u)\} \subseteq \{B(u) \mid x \leq A(u)\}$$

by definition of infimum we have:

$$f_{A,B}(x \vee y) = \bigwedge_{u \in \mathscr{U}} \{B(u) \mid x \vee y \leq A(u)\} \geq \bigwedge_{u \in \mathscr{U}} \{B(u) \mid x \leq A(u)\} = f_{A,B}(x)$$

Similarly, we can obtain the other inequality. Now, again by properties of supremum and infimum we have that

$$f_{A,B}(x \vee y) \wedge (x \vee y) \geq f_{A,B}(x) \wedge x$$
$$f_{A,B}(x \vee y) \wedge (x \vee y) \geq f_{A,B}(y) \wedge y$$

as a result

$$f_{A,B}(x \vee y) \wedge (x \vee y) \geq (f_{A,B}(x) \wedge x) \vee (f_{A,B}(y) \wedge y)$$

as we wanted to prove. In other words, $f_{A,B} \wedge id$ is in Ω. □

Let us show now that for every pair of L-fuzzy sets A and B, A is $(f_{A,B} \wedge id)$-included in B.

Lemma 4. *Let A and B be two L-fuzzy sets, then A is $(f_{A,B} \wedge id)$-included in B.*

Proof. For all $u \in \mathscr{U}$, we have that

$$f_{A,B}(A(u)) \wedge A(u) \leq f_{A,B}(A(u)) = \bigwedge_{v \in \mathscr{U}} \{B(v) \mid A(u) \leq A(v)\} \leq B(u)$$

Therefore, we have proven that $(f_{A,B} \wedge id)(A(u)) \leq B(u)$ for all $u \in \mathscr{U}$, which means that A is $(f_{A,B} \wedge id)$-included in B. □

Finally, the third lemma shows, basically, the maximality of $f_{A,B} \wedge id$ in $\Lambda(A, B)$.

Lemma 5. *Let A and B be two L-fuzzy sets and consider $f \in \Omega$. Then $f_{A,B} \wedge id < f$ implies that A is not f-included in B.*

Proof. The hypothesis $f_{A,B} \wedge id < f$ means $f_{A,B}(x) \wedge x \leq f(x)$ for all $x \in L$ and there exists at least one element $\alpha \in L$ such that $f_{A,B}(\alpha) \wedge \alpha < f(\alpha)$. Let us prove firstly that there exists $u_0 \in \mathscr{U}$ such that $\alpha \leq A(u_0)$ and $f_{A,B}(A(u_0)) \wedge A(u_0) < f(A(u_0))$.

By *reductio ad absurdum*, assume that $f(A(v)) = f_{A,B}(A(v)) \wedge A(v)$ for all $v \in \mathscr{U}$ satisfying $\alpha \leq A(v)$. Moreover, by monotonicity of f, we have

$$f(\alpha) \leq f(A(v)) = f_{A,B}(A(v)) \wedge A(v) \leq f_{A,B}(A(v))$$

Thus, putting together the previous inequality for all $v \in \mathscr{U}$ satisfying $\alpha \leq A(v)$, definition of $f_{A,B}$ and properties of infima, we obtain the following chain of inequalities

$$\begin{aligned} f(\alpha) &\leq \bigwedge_{v \in \mathscr{U}} \{f_{A,B}(A(v)) \mid \alpha \leq A(v)\} \\ &= \bigwedge_{v \in \mathscr{U}} \Big\{ \bigwedge_{u \in \mathscr{U}} \{B(u) \mid A(v) \leq A(u)\} \mid \alpha \leq A(v) \Big\} \\ &\leq \bigwedge_{u \in \mathscr{U}} \{B(u) \mid \alpha \leq A(u)\} = f_{A,B}(\alpha) \end{aligned}$$

So we have obtained that $f(\alpha) \leq f_{A,B}(\alpha)$. Using now that $f \leq id$ (since $f \in \Omega$), we have that $f(\alpha) \leq f_{A,B}(\alpha) \wedge \alpha$ which contradicts our assumption $f_{A,B}(\alpha) \wedge \alpha < f(\alpha)$. Hence, there exists $u_0 \in \mathscr{U}$ such that $\alpha \leq A(u_0)$ and $f_{A,B}(A(u_0)) \wedge A(u_0) < f(A(u_0))$.

This inequality together with the fact that $f \leq id$ leads to the fact that $f(A(u_0))$ cannot be a lower bound of $f_{A,B}(A(u_0)) = \bigwedge_{v \in \mathscr{U}} \{B(v) \mid A(u_0) \leq A(v)\}$. Hence, there exists $w_0 \in \mathscr{U}$ such that $A(u_0) \leq A(w_0)$ and $f(A(u_0)) \not\leq B(w_0)$. By using the monotonicity of f in the former inequality we have that $f(A(u_0)) \leq f(A(w_0))$.

Let us assume, once again by reductio ad absurdum, that A is f-included in B. Then, in particular, we would have $f(A(w_0)) \leq B(w_0)$. By transitivity, we have that $f(A(u_0)) \leq f(A(w_0)) \leq B(w_0)$, which contradicts that $f(A(u_0)) \not\leq B(w_0)$. Therefore, A is not f-included in B. □

By joining the three previous lemmas we can obtain the following result:

Theorem 1. *Let A and B be two fuzzy sets, then $Inc(A,B) = f_{A,B} \wedge id$.*

Proof. We have to show that $f_{A,B} \wedge id$ is the maximum element of $\Lambda(A,B)$. By Lemmas 3 and 4 we have that $f_{A,B} \wedge id \in \Lambda(A,B)$. By the complete lattice structure of Ω and Lemma 2, the maximum of $\Lambda(A,B)$ exists and is $Inc(A,B)$. Now, by Lemma 5, we have that there are no upper bounds of $Inc(A,B)$ strictly greater than $f_{A,B} \wedge id$. But as $f_{A,B} \wedge id \in \Lambda(A,B)$ then, necessarily $f_{A,B} \wedge id$ is the maximum element of $\Lambda(A,B)$; i.e., $f_{A,B} \wedge id = Inc(A,B)$. ☐

Example 3. *On the universe $\mathscr{U} = \{u_1,u_2,u_3,u_4,u_5,u_6,u_7\}$, let us consider the fuzzy sets A and B on the unit interval given by the following table:*

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
A	0.2	0.3	0.5	0.7	0.9	0.7	0.2
B	0.4	0.2	0.9	0.5	0.4	0.6	0.4

The φ-degree of inclusion $Inc(A,B)$ can be determined by Theorem 1. To begin with, let us compute $f_{A,B}$ by distinguishing cases:

$$f_{A,B}(x) = \bigwedge_{u\in\mathscr{U}} \{B(u) \mid x \leq A(u)\} = \begin{cases} 0.2 & \text{if} \quad x \leq 0.3 \\ 0.4 & \text{if} \quad 0.3 < x \leq 0.9 \\ 1 & \text{if} \quad 0.9 < x \leq 1 \end{cases}$$

Now, by computing the infimum $f_{A,B} \wedge id$ we obtain

$$Inc(A,B)(x) = f_{A,B}(x) \wedge x = \begin{cases} 0.2 & \text{if} \quad 0.2 < x \leq 0.3 \\ 0.4 & \text{if} \quad 0.4 < x \leq 0.9 \\ x & \text{otherwise}. \end{cases}$$

In the rest of the section we show different properties of $Inc(A,B)$, following the initial study done by the standard axiomatic approaches of inclusion measures between fuzzy sets [16, 26, 29].

3.3. Null-inclusion and full-inclusion

Recall that the cases of null-inclusion and full-inclusion were already discussed for the general case of φ-indexes. In terms of φ-degree inclusion we can rephrase them as follows:

Proposition 8. *Let A and B two L-fuzzy sets, then:*

- $Inc(A,B) = \perp$ *if and only if there is a set of elements in the universe* $\{u_i\}_{i\in\mathbb{I}} \subseteq \mathscr{U}$ *such that* $A(u_i) = 1$ *for all* $i \in \mathbb{I}$ *and* $\bigwedge_{i\in\mathbb{I}} B(u_i) = 0$.
- $Inc(A,B) = id$ *if and only if* $A(u) \le B(u)$ *for all* $u \in \mathscr{U}$.

3.4. Transitivity of φ-degree of inclusion

Once again, the previous result about φ-indexes (Proposition 6) can be rephrased in terms of f-inclusion as follows:

Proposition 9. *Let A,B and C be fuzzy sets, then* $Inc(B,C) \circ Inc(A,B) \le Inc(A,C)$.

It is worth noting that the standard fuzzy version of transitivity, namely $Inc(A,B) \wedge Inc(B,C) \le Inc(A,C)$ does not hold in general.

Example 4. *Consider the fuzzy sets* A, B *and* C *on the singleton* $\mathscr{U} = \{u\}$ *given by* $A(u) = 0.7$, $B(u) = 0.6$ *and* $C(u) = 0.4$. *Then, we have:*

$$Inc(A,B)(x) = \begin{cases} 0.6 & \text{if } x \in (0.6, 0.7] \\ x & \text{otherwise.} \end{cases} \qquad Inc(B,C)(x) = \begin{cases} 0.4 & \text{if } x \in (0.4, 0.6] \\ x & \text{otherwise.} \end{cases}$$

$$Inc(A,B) \wedge Inc(B,C)(x) = \begin{cases} 0.4 & \text{if } x \in (0.4, 0.6] \\ 0.6 & \text{if } x \in (0.6, 0.7] \\ x & \text{otherwise.} \end{cases}$$

However,

$$Inc(A,C)(x) = \begin{cases} 0.4 & \text{if } x \in (0.4, 0.7] \\ x & \text{otherwise.} \end{cases}$$

and then, $Inc(A,C) \not\geq Inc(A,B) \wedge Inc(B,C)$.

3.5. Monotonicity of φ-degree of inclusion

Another of the most common properties required to inclusion measures is to be coherent with the standard ordering between fuzzy sets. With respect to the φ-degree of inclusion we have the following result:

Proposition 10. *Let* A, B *and* C *be three L-fuzzy sets:*

- *if $B(u) \leq C(u)$ for all $u \in \mathscr{U}$ then, $Inc(A,B) \leq Inc(A,C)$;*
- *if $B(u) \leq C(u)$ for all $u \in \mathscr{U}$ then, $Inc(C,A) \leq Inc(B,A)$.*

Proof. It is a consequence of the fact that, for any fuzzy sets A,B,C, and D such that $A(u) \leq B(u)$ and $C(u) \leq D(u)$ for all $u \in \mathscr{U}$ then, $B \subseteq_f C$ implies $A \subseteq_f D$, which straightforwardly follows from the following chain of inequalities for a $u \in \mathscr{U}$

$$f(A(u)) \leq f(B(u)) \leq C(u) \leq D(u).$$

□

3.6. Transformation invariance

Another very common property required to inclusion measures between L-fuzzy sets is the invariance under transformations of the universe [16, 26]. We follow here the usual terminology, in which these transformations are just bijective mappings in the universe.

A transformation $T\colon \mathscr{U} \to \mathscr{U}$ in the universe $\mathscr{U}$ can be extended to the set of L-fuzzy sets $L^{\mathscr{U}}$ (i.e., the fuzzy powerset of $\mathscr{U}$) by defining $\hat{T}(A)(u) = A(T(u))$ for all fuzzy set $A \in L^{\mathscr{U}}$. The following result shows that the φ-degree of inclusion is invariant under transformations.

Proposition 11. *Let A and B be two L-fuzzy sets and let $T\colon \mathscr{U} \to \mathscr{U}$ be a transformation on $\mathscr{U}$, then $Inc(A,B) = Inc(\hat{T}(A), \hat{T}(B))$.*

Proof. Since $f_{A,B}$ and $f_{\hat{T}(A),\hat{T}(B)}$ are, respectively, the suprema of the sets $\{f \in \Omega \mid A \subseteq_f B\}$ and $\{f \in \Omega \mid \hat{T}(A) \subseteq_f \hat{T}(B)\}$, we can obtain the result by showing that both sets are the same. Consider $f \in \Omega$ such that $A \subseteq_f B$. Then, for all $u \in \mathscr{U}$ we have $f(A(u)) \leq B(u)$ which, by the bijectivity of T, is equivalent to say that for all $u \in \mathscr{U}$ we have $f(A(T(u))) \leq B(T(u))$, which is equivalent to $\hat{T}(A) \subseteq_f \hat{T}(B)$. □

3.7. Relationship with union and intersection between L-fuzzy sets

The first result shows that the equivalence $A \subseteq B \cap C$ if and only if $A \subseteq B$ and $A \subseteq C$ is also preserved by the φ-degree of inclusion. Let us recall that the intersection of two L-fuzzy sets A and B is defined by $A \cap B(u) = A(u) \wedge B(u)$ for all $u \in \mathscr{U}$.

Theorem 2. *Let A, B and C be three L-fuzzy sets then*

$$Inc(A, B \cap C) = Inc(A, B) \wedge Inc(A, C).$$

Proof. By Theorem 1 we have that $Inc(A, B \cap C) = f_{A,B\cap C} \wedge id$, where $f_{A,B\cap C}$ is the mapping given by

$$f_{A,B\cap C}(x) = \bigwedge_{u \in \mathscr{U}} \{B(u) \wedge C(u) \mid x \leq A(u)\}.$$

By properties of infimum we have:

$$f_{A,B\cap C}(x) = \left(\bigwedge_{u \in \mathscr{U}} \{B(u) \mid x \leq A(u)\} \right) \wedge \left(\bigwedge_{u \in \mathscr{U}} \{C(u) \mid x \leq A(u)\} \right) = f_{A,B}(x) \wedge f_{A,C}(x)$$

for all $x \in L$. Finally, we have that

$$Inc(A, B \cap C) = f_{A,B\cap C} \wedge id = (f_{A,B} \wedge f_{A,C}) \wedge id =$$
$$(f_{A,B} \wedge id) \wedge (f_{A,C} \wedge id) = Inc(A, B) \wedge Inc(A, C).$$

□

In general, the equivalence $A \cup B \subseteq C$ if and only if $A \subseteq C$ and $B \subseteq C$ is not preserved in terms of the equality but in terms of the inequality. Let us recall that the union of two L-fuzzy sets A and B is defined by $A \cup B(u) = A(u) \vee B(u)$ for all $u \in \mathscr{U}$.

Theorem 3. *Let A, B and C be three L-fuzzy sets then*

$$Inc(A \cup B, C) \leq Inc(A, C) \wedge Inc(B, C).$$

Proof. To prove the inequality, just take into account that if $A \cup B \subseteq_f C$ then $A \subseteq_f C$ and $B \subseteq_f C$ since, by the monotonicity of f:

$$f(A(u)) \leq f(A(u) \vee B(u)) \leq C(u)$$

for all $u \in \mathscr{U}$. As a result, $Inc(A \cup B, C) \leq Inc(A, C) \wedge Inc(B, C)$. □

The following example shows a case where the strict inequality $Inc(A \cup B, C) < Inc(A, C) \wedge Inc(B, C)$ holds.

Example 5. *Consider the complete lattice L given in Example 2 and the L-fuzzy sets A, B and C on the singleton universe $\mathcal{U} = \{u\}$ given by $A(u) = a$, $B(u) = b$ and $C(u) = 0$. Then, by following the formula given in Theorem 1, we have that:*

$$Inc(A,C)(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = a \\ b & \text{if } x = b \\ 1 & \text{if } x = 1 \end{cases} \qquad Inc(B,C)(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = b \\ a & \text{if } x = a \\ 1 & \text{if } x = 1 \end{cases}$$

Since $A \vee B(u) = 1$, by Proposition 8 we have that $Inc(A \cup B, C)(x) = 0$. However,

$$Inc(A,C) \wedge Inc(B,C)(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The following result shows that if the lattice L is a totally ordered set, then the equality $Inc(A \cup B, C) = Inc(A,C) \wedge Inc(B,C)$ holds. A preliminary version for the unit interval was proved in [22].

Theorem 4. *Let $(L, \leq)$ be a totally ordered set and let A, B and C be three L-fuzzy sets, then*

$$Inc(A \cup B, C) = Inc(A,C) \wedge Inc(B,C).$$

Proof. By Theorem 1, $Inc(A \cup B, C) = f_{A \cup B, C} \wedge id$. Now, for all $x \in L$, we have

$$\begin{aligned} f_{A \cup B, C}(x) &= \bigwedge_{u \in \mathcal{U}} \{C(u) \mid x \leq A(u) \vee B(u)\} \\ &= \left(\bigwedge_{u \in \mathcal{U}} \{C(u) \mid x \leq A(u)\} \right) \wedge \left(\bigwedge_{u \in \mathcal{U}} \{C(u) \mid x \leq B(u)\} \right) \\ &= f_{A,C}(x) \wedge f_{B,C}(x). \end{aligned}$$

where the second equality is given by the linearity of $(L, \leq)$. Finally, we have that

$$\begin{aligned} Inc(A \cup B, C) = f_{A \cup B, C} \wedge id = (f_{A,B} \wedge f_{B,C}) \wedge id = \\ (f_{A,B} \wedge id) \wedge (f_{B,C} \wedge id) = Inc(A,B) \wedge Inc(B,C). \end{aligned}$$

□

3.8. Relationship with the complement of L-fuzzy sets

The extension to the fuzzy case of the relationship between complement and inclusion, namely $A \subseteq B$ if and only if $B^c \subseteq A^c$, has to take into account that the notion of complement is not unique, since it depends on the negation operator considered. Given an involutive negation operator $n\colon L \to L$ (i.e., n is monotonically decreasing and $n^2 = n \circ n = id$), we can define the complement of an L- fuzzy set A with respect to n by the equality $A^c(u) = n(A(u))$ for all $u \in \mathscr{U}$. Hereafter, we asume that there is a *fixed* involutive negation n used to define the complement.

In general, the equality $Inc(A,B) = Inc(B^c,A^c)$ does not hold. However, it is possible to establish some relationships between both φ-degree in terms of adjoint pairs. Let us recall that two mappings $f,g\colon L \to L$ form an adjoint pair if

$$f(x) \leq y \iff x \leq g(y) \qquad \text{for all } x \in L \tag{3}$$

Given an adjoint pair (f,g), our first result links, in some sense, the f inclusion of A in B with the g-inclusion of B^c in A^c in terms of the negation n.

Proposition 12. *Let A and B be two fuzzy sets and let (f,g) be an adjoint pair. Then $A \subseteq_f B$ if and only if $B^c \subseteq_{n \circ g \circ n} A^c$.*

Proof. By properties of the adjunction, we have that equalities $f(x \vee y) = f(x) \vee f(y)$ and $g(x \wedge y) = g(x) \wedge g(y)$ hold. Now, since n is involutive and decreasing we have:

$$n \circ g \circ n(x \vee y) = n \circ g(n(x) \wedge n(y)) = n\big(g(n(x)) \wedge g(n(y))\big) = n \circ g \circ n(x) \vee n \circ g \circ n(y)$$

Let us now prove that $f \in \Omega$ if and only if $n \circ g \circ n \in \Omega$. For this, we have just to prove that $f(x) \leq x$ for all $x \in L$ if and only if $n \circ g \circ n(x) \leq x$ for all $x \in L$. Let us assume that $f(x) \leq x$ for all $x \in L$; then, by the adjoint property we have the following chain of equivalences for all $x \in L$.

$$f(x) \leq x \iff f(n(x)) \leq n(x) \iff n(x) \leq g(n(x)) \iff x \geq n(g(n(x)))$$

Therefore, $f \in \Omega$ if and only if $n \circ g \circ n \in \Omega$.

Let us assume now that $A \subseteq_f B$. Then, for any $u \in \mathscr{U}$ we have:

$$f(A(u)) \leq B(u) \iff A(u) \leq g(B(u)) \iff n(A(u)) \geq n(g(B(u))).$$

Finally, by using that $n \circ n = id$, we have that

$$f(A(u)) \leq B(u) \iff n(A(u)) \geq n(g(n(n(B(u))))),$$

or equivalently, $B^c \subseteq_{n \circ g \circ n} A^c$. $\square$

4. The notion of φ-degree of similarity

There is a very close relationship between equality and inclusion: two sets are equal if one is included in the other and vice versa. In this section, we deal with similarity as a suitable generalization of equality to the L-fuzzy case, and introduce three different φ-degree of similarity defined by using the φ-degree of inclusion given in Section 3 and study their properties. These three degrees are defined by considering three common approaches used in the literature to define measures of similarities from measures of inclusion. Specifically, given two L-fuzzy sets A and B, the three respective φ-degrees of similarity are defined as follows:

- The φ_{eq}-degree of similarity between A and B, considers the pair $Inc(A,B)$ and $Inc(B,A)$;
- The $\varphi_{\cup\cap}$-degree of similarity, is defined as the φ-degree of inclusion of $A \cup B$ in $A \cap B$, i.e., $Inc(A \cup B, A \cap B)$;
- The $\varphi_{\wedge}$-degree of similarity, consider the infimum between $Inc(A,B)$ and $Inc(B,A)$.

In the subsequent sections we present the formal definitions and a detailed study of the mentioned degrees of similarity.

4.1. The φ_{eq}-degree of similarity

This notion considers the φ-degrees of inclusion of A into B and of B into A, in this sense, it somehow generalizes the equality of sets, hence its name φ_{eq}-degree.

Definition 4. *Let A and B be two L-fuzzy sets, then the φ_{eq}-degree of similarity between A and B, denoted $S_{eq}(A,B)$, is defined by*

$$S_{eq}(A,B) := \{Inc(A,B), Inc(B,A)\}$$

Note that the values of the φ_{eq}-degree are subsets of Ω of cardinality 2 or 1 (it might be the case that both degrees coincide). Identifying the singletons $\{f\}$ with $\{f,f\}$ these values can be ordered by suitably extending the componentwise ordering as follows: Given $\{f_1,g_1\},\{f_2,g_2\}$, we say that $\{f_1,g_1\} \leq \{f_2,g_2\}$ if and only if either $f_1 \leq f_2$ and $g_1 \leq g_2$ or $g_1 \leq f_2$ and $f_1 \leq g_2$.

The following results motivate the use of the φ_{eq}-degree of similarity to model the similarity according to common approaches of similarity measures (see [7]).

Proposition 13. *Let A and B be two L-fuzzy sets, then $S_{eq}(A,B) = S_{eq}(B,A)$.*

Proof. Obvious. □

Proposition 14. *Let A,B and C be three L-fuzzy sets such that $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathscr{U}$, then $S_{eq}(A,C) \leq S_{eq}(A,B)$ and $S_{eq}(A,C) \leq S_{eq}(B,C)$.*

Proof. Let us prove firstly the inequality $S_{eq}(A,C) \leq S_{eq}(A,B)$. By Proposition 8 we have $Inc(A,C) = Inc(A,B) = id$ and by Proposition 10 $Inc(C,A) \leq Inc(B,A)$. Therefore, $S_{eq}(A,C) \leq S_{eq}(A,B)$. The proof of the second inequality is similar. □

We characterize below the two extreme φ_{eq}-degrees of similarity, namely, $\{id,id\}$ and $\{\bot,\bot\}$. The first result shows that the greatest φ_{eq}-degree is equivalent to equality.

Proposition 15. *Let A and B be two L-fuzzy sets. Then $A = B$ if and only if $S_{eq}(A,B) = \{id,id\}$.*

Proof. From Proposition 8 we have that $Inc(A,B) = id$ and $Inc(B,A) = id$ is equivalent to say that $A(u) \leq B(u)$ and $B(u) \leq A(u)$ for all $u \in \mathscr{U}$; i.e., $A = B$. □

The lowest φ_{eq}-degree of similarity is characterized in the following corollary.

Corollary 4. *Let A and B be two fuzzy sets. $S_{eq}(A,B) = \{\bot,\bot\}$ if and only if there are two sets in the universe $\{u_i\}_{i\in\mathbb{I}},\{v_j\}_{j\in\mathbb{J}} \subseteq \mathscr{U}$ such that $A(u_i) = 1$ and $B(v_j) = 1$ for all $i \in \mathbb{I}$, $j \in \mathbb{J}$ and $\bigwedge_{j\in\mathbb{J}} A(v_j) = 0$ and $\bigwedge_{i\in\mathbb{I}} B(u_i) = 0$.*

Proof. Direct consequence of Proposition 3. □

From the previous result we can obtain three results that resemble common axioms usually required in the literature for measures of similarity (see [7, 26, 29]). Firstly, in the case of L-fuzzy sets on finite totally ordered lattices, the lowest φ_{eq}-degree of similarity is equivalent to the existence of two elements in the universe such that one fully belongs to $A \cap B^c$ and the other to $A^c \cap B$.

Corollary 5. *Let L be a finite totally ordered lattice and let A and B be two L-fuzzy sets. $S_{eq}(A,B) = \{\perp, \perp\}$ if and only if there exist two elements in the universe $u_1, u_2 \in \mathscr{U}$ such that $A(u_1) = B(u_2) = 0$ and $A(u_2) = B(u_1) = 1$.*

Proof. Direct consequence of Corollary 4. □

There is also an interesting relationship between the φ_{eq}-degree of similarity $\{\perp, \perp\}$ and the complement of normal fuzzy sets. Let us recall that an L-fuzzy set A is called normal if there exists $u \in \mathscr{U}$ such that $A(u) = 1$.

Corollary 6. *Let A be a normal L-fuzzy set such that A^c is also normal, then $S_{eq}(A,A^c) = \{\perp, \perp\}$. If L is finite and totally ordered, then the converse also holds.*

Proof. Direct consequence of Corollary 4. □

The last result relating the φ_{eq}-degree of similarity $\{\perp, \perp\}$ and axiomatic approaches given in terms of crisp sets.

Corollary 7. *Let A be a L-fuzzy set on a finite universe $\mathscr{U}$. If A is crisp and $\varnothing \neq A \neq \mathscr{U}$ then, $S_{eq}(A,A^c) = \{\perp, \perp\}$.*

Proof. Direct consequence of Corollary 4. □

The φ_{eq}-degree of similarity is related to the intersection between fuzzy sets as follows: the similarity between $A \cap C$ and $B \cap C$ is always greater than the similarity between A and B. It is worth mentioning that such a property is related to divergence measures [24].

Proposition 16. *Let A, B and C be three L-fuzzy sets, then the following inequality holds $S_{eq}(A,B) \leq S_{eq}(A \cap C, B \cap C)$.*

Proof. By Theorem 2, Proposition 8 and Proposition 10 we have that:

$$Inc(A\cap C, B\cap C) = Inc(A\cap C, B) \wedge Inc(A\cap C, C) = Inc(A\cap C, B) \geq Inc(A, B)$$

The inequality $Inc(B\cap C, A\cap C) \geq Inc(B, A)$ is obtained similarly. □

A similar result can be obtained for the case of union when the lattice considered is totally ordered.

Proposition 17. *Let L be a totally order lattice and let A, B and C be three L-fuzzy sets then, $S_{eq}(A, B) \leq S_{eq}(A\cup C, B\cup C)$.*

Proof. Similar to the previous proof, but applying Theorem 4 instead of Theorem 2. □

4.2. The $\varphi_{\cup\cap}$-degree of similarity

The next approach to similarity is based on a very usual construction technique of measures of similarity from measures of inclusion. Basically, we can determine the similarity between two fuzzy sets A and B by measuring the inclusion of $A\cup B$ in $A\cap B$.

Definition 5. *Let A and B be two L-fuzzy sets, then the $\varphi_{\cup\cap}$-degree of similarity $S_{\cup\cap}(A, B)$ is defined as the φ-degree of inclusion of $A\cup B$ in $A\cap B$; i.e.,*

$$S_{\cup\cap}(A, B) = Inc(A\cup B, A\cap B).$$

The following results present some interesting properties in order to motivate the use of $S_{\cup\cap}(A, B)$ for dealing with similarity.

Proposition 18. *Let A and B be two L-fuzzy sets, then $S_{\cup\cap}(A, B) = S_{\cup\cap}(B, A)$.*

Corollary 8. *Let A, B and C be L-fuzzy sets such that $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathscr{U}$, then:*

- $S_{\cup\cap}(A, C) \leq S_{\cup\cap}(A, B)$
- $S_{\cup\cap}(A, C) \leq S_{\cup\cap}(B, C)$

Proof. Firstly, note that thanks to the ordering $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathscr{U}$ we have that $A \cup C = B \cup C = C$, $A \cap B = A \cap C = A$, $A \cup B = B$ and $B \cap C = B$. As a result, the two items are consequences of Proposition 10, since

$$S_{\cup\cap}(A,C) = Inc(A \cup C, A \cap C) = Inc(C,A) \leq Inc(B,A) = Inc(A \cup B, A \cap B) = S_{\cup\cap}(A,B)$$

and

$$S_{\cup\cap}(A,C) = Inc(A \cup C, A \cap C) = Inc(C,A) \leq Inc(C,B) = Inc(C \cup B, C \cap B) = S_{\cup\cap}(B,C).$$

□

The highest $\varphi_{\cup\cap}$-degree of similarity is characterised as follows.

Proposition 19. *Let A and B be two L-fuzzy sets, then $A = B$ if and only if $S_{\cup\cap}(A,B) = id$.*

Proof. If $A = B$ then $Inc(A \cup B, A \cap B) = Inc(A,A) = id$.

Conversely, by Proposition 8, $Inc(A \cup B, A \cap B) = id$ implies $A \cup B(u) \leq A \cap B(u)$ for all $u \in \mathscr{U}$. Then, for all $u \in \mathscr{U}$

$$A(u) \leq A(u) \vee B(u) \leq A(u) \wedge B(u) \leq B(u)$$

and

$$B(u) \leq A(u) \vee B(u) \leq A(u) \wedge B(u) \leq A(u).$$

In other words, $A(u) = B(u)$. □

The lowest $\varphi_{\cup\cap}$-degree of similarity is characterized in the following corollary.

Corollary 9. *Let A and B be two L-fuzzy sets. $S_{\cup\cap}(A,B) = \perp$ if and only if there is a subset in the universe $\{u_i\}_{i \in \mathbb{I}} \subseteq \mathscr{U}$ such that $A(u_i) \vee B(u_i) = 1$ for all $i \in \mathbb{I}$ and $(\bigwedge_{i \in \mathbb{I}} A(u_i)) \wedge (\bigwedge_{i \in \mathbb{I}} B(u_i)) = 0$.*

Proof. Direct consequence of Proposition 3. □

As in the previous section for the φ_{eq}-degree of similarity, it is interesting to rewrite some direct consequences of the previous result to be in terms of the axiomatic approaches of measures of similarity. In the case of L-fuzzy sets on finite and totally ordered lattices, the lowest $\varphi_{\cup\cap}$-degree of similarity between two fuzzy sets A and B is equivalent to the existence of at least one element $u \in \mathscr{U}$ such $A \cup B(u) = 1$ but $A \cap B(u) = 0$. The formal result is given below in a more general environment.

Corollary 10. *Let L be a lattice with an element $a \in L$ such that $0 < a \leq l$ for all $l \in L \setminus \{0\}$. Let A and B be two L-fuzzy sets, the $S_{\cup\cap}(A,B) = \perp$ if and only if there exists $u \in \mathscr{U}$ such that either $A(u) = 0$ and $B(u) = 1$, or $A(u) = 1$ and $B(u) = 0$.*

As in the case of the φ_{eq}-degree of similarity, there is a relationship between the φ-index $\perp$ and the complement when we restrict the index to crisp sets.

Corollary 11. *If A is crisp, then $S_{\cup\cap}(A,A^c) = \perp$.*

Finally, there is also a relation between the $\varphi_{\cup\cap}$-degree of similarity of union and intersections of L-fuzzy sets.

Proposition 20. *Let A,B and C be three L-fuzzy sets.*

- *If L is distributive then, $S_{\cup\cap}(A,B) \leq S_{\cup\cap}(A \cap C, B \cap C)$.*
- *If L is totally ordered then, $S_{\cup\cap}(A,B) \leq S_{\cup\cap}(A \cup C, B \cup C)$.*

Proof. By applying Theorem 2, Proposition 8 and Proposition 10 we obtain the first item as follows:

$$\begin{aligned}
S_{\cup\cap}(A \cap C, B \cap C) &= Inc((A \cap C) \cup (B \cap C), (A \cap C) \cap (B \cap C)\\
&= Inc((A \cup B) \cap C, (A \cap B) \cap C)\\
&= Inc((A \cup B) \cap C, A \cap B) \wedge Inc((A \cup B) \cap C), C)\\
&= Inc((A \cup B) \cap C, A \cap B)\\
&\geq Inc(A \cup B, A \cap B) = S_{\cup\cap}(A,B)
\end{aligned}$$

By applying Theorem 4, Proposition 8 and Proposition 10 we obtain the second item as follows:

$$
\begin{aligned}
S_{\cup\cap}(A\cup C, B\cup C) &= Inc((A\cup C)\cup(B\cup C),(A\cup C)\cap(B\cup C)\\
&= Inc((A\cup B)\cup C,(A\cap B)\cup C)\\
&= Inc((A\cup B),(A\cap B)\cup C)\wedge Inc(C,(A\cap B)\cup C)\\
&= Inc((A\cup B),(A\cap B)\cup C)\\
&\geq Inc(A\cup B, A\cap B) = S_{\cup\cap}(A,B)
\end{aligned}
$$

□

4.3. The $\varphi_\wedge$-degree of similarity

This third approach to similarity between sets is based on the simultaneous consideration of the φ-degrees of inclusion of one set into the other.

Formally, it is possible to define the relation of f*-similarity* between two L-fuzzy sets A and B as $A =_f B$ if and only if $A \subseteq_f B$ and $B \subseteq_f A$.

From this relation, and following a similar reasoning that in Section 3, we can define an $\varphi_\wedge$-degree of similarity by considering the supremum of the set

$$\Upsilon(A,B) = \{f \in \Omega \mid A =_f B\}$$

which, indeed, is a maximum as shown below.

Proposition 21. *Let A and B be two L-fuzzy sets and consider $\{f_i\}_{i\in\mathbb{I}} \subseteq \Omega$. If A is f_i-similar to B for all $i \in \mathbb{I}$, then A is $\bigvee_{i\in\mathbb{I}} f_i$-similar to B.*

Proof. By definition of the f-similarity, we have that A is f_i-similar to B and B is f_i-similar to A for all $i \in \mathbb{I}$. Then, by Lemma 2, we have that A is $\bigvee_{i\in\mathbb{I}} f_i$-included in B and B is $\bigvee_{i\in\mathbb{I}} f_i$-included in A. In other words, A is $\bigvee_{i\in\mathbb{I}} f_i$-similar to B. □

Then, we can provide the following definition.

Definition 6. *Let A and B be two L-fuzzy sets and consider the set $\Upsilon(A,B) = \{f \in \Omega \mid A =_f B\}$, then the $\varphi_\wedge$-degree of similarity $S_\wedge(A,B)$ is defined as*

$$S_\wedge(A,B) = \max\Upsilon(A,B) = \max\{f \in \Omega \mid A =_f B\}.$$

The analytic expression of the index $S_\wedge(A,B)$ can be provided in terms of the φ-degrees of inclusion $Inc(A,B)$ and $Inc(B,A)$, as the following result shows.

Theorem 5. *Let A and B be two L-fuzzy sets, then*

$$S_\wedge(A,B) = Inc(A,B) \wedge Inc(B,A).$$

Proof. Let us show that $Inc(A,B) \wedge Inc(B,A)$ is the maximum of $\Upsilon(A,B)$. Firstly, by Proposition 1, we have that $Inc(A,B) \wedge Inc(B,A) \in \Upsilon(A,B)$.

Now, let us assume $f \in \Upsilon(A,B)$ and let us see that $f \leq Inc(A,B) \wedge Inc(B,A)$. By definition of $\Upsilon(A,B)$ we have that $A \subseteq_f B$ and $B \subseteq_f A$ and, hence, $f \leq Inc(A,B)$ and $f \leq Inc(B,A)$. Therefore, $Inc(A,B) \wedge Inc(B,A)$ is the maximum of $\Upsilon(A,B)$. □

The next results present some interesting properties of $S_\wedge(A,B)$.

Proposition 22. *Let A and B be two L-fuzzy sets, then* $S_\wedge(A,B) = S_\wedge(B,A)$.

Proof. It is a consequence of Theorem 5. □

Corollary 12. *Let A,B and C be L-fuzzy sets satisfying $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathscr{U}$, then:*

- $S_\wedge(A,C) \leq S_\wedge(A,B)$
- $S_\wedge(A,C) \leq S_\wedge(B,C)$

Proof. The two items are consequences of Theorem 5 and Proposition 10, since

$$\begin{aligned} S_\wedge(A,C) &= Inc(A,C) \wedge Inc(C,A) = Inc(C,A) \\ &\leq Inc(B,A) = Inc(A,B) \wedge Inc(B,A) = S_\wedge(A,B) \end{aligned}$$

and

$$\begin{aligned} S_\wedge(A,C) &= Inc(A,C) \wedge Inc(C,A) = Inc(C,A) \\ &\leq Inc(C,B) = Inc(C,B) \wedge Inc(B,C) = S_\wedge(B,C) \end{aligned}$$

□

The highest $\varphi_\wedge$-degree of similarity is characterized as follows.

Proposition 23. *Let A and B be two L-fuzzy sets on a finite universe $\mathscr{U}$. Then, $A = B$ if and only if $S_\wedge(A,B) = id$.*

Proof. If $A = B$ then $S_\wedge(A,B) = Inc(A,B) \wedge Inc(B,A) = Inc(A,A) = id$. Conversely, since id is the maximum element in Ω, if $S_\wedge(A,B) = Inc(A,B) \wedge Inc(B,A) = id$ then, $Inc(A,B) = Inc(B,A) = id$. Thus, by Proposition 8, $A(u) \leq B(u)$ and $B(u) \leq A(u)$. for all $u \in \mathscr{U}$. In other words, $A = B$. $\square$

The lowest $\varphi_\wedge$-degree of similarity is characterized in the following corollary.

Corollary 13. *Let A and B be two L-fuzzy sets. $S_\wedge(A,B) = \perp$ if and only if there is a subset in the universe $\{u_i\}_{i\in\mathbb{I}} \subseteq \mathscr{U}$ such that either $A(u_i) = 1$ for all $i \in \mathbb{I}$ and $(\bigwedge_{i\in\mathbb{I}} B(u_i)) = 0$ or $B(u_i) = 1$ for all $i \in \mathbb{I}$ and $(\bigwedge_{i\in\mathbb{I}} A(u_i)) = 0$.*

Proof. Direct consequence of Proposition 3. $\square$

Once again, let us rewrite some direct consequences of the previous result to put it in relation with the axiomatic approaches of measures of similarity. In the case of L-fuzzy sets on finite and totally ordered lattices, the lowest $\varphi_\wedge$-degree of similarity between two fuzzy sets A and B is equivalent to the existence of at least one element $u \in \mathscr{U}$ such $A \cup B(u) = 1$ but $A \cap B(u) = 0$. For the sake of a higher generality, the result is given in the following terms.

Corollary 14. *Let L be a lattice with an element $a \in L$ such that $0 < a \leq l$ for all $l \in L \setminus \{0\}$. Let A and B be two fuzzy sets. $S_\wedge(A,B) = \perp$ if and only if there exist $u \in \mathscr{U}$ such that either $A(u) = 0$ and $B(u) = 1$ or $A(u) = 1$ and $B(u) = 0$.*

There is also a relationship between the φ-index $\perp$ and the complement when we restrict the index to crisp sets.

Corollary 15. *If A is crisp, then $S_\wedge(A, A^c) = \perp$.*

Finally, there is also a relation between the $\varphi_\wedge$-degree of similarity of union and intersections of fuzzy sets.

Proposition 24. *Let A, B and C be three L-fuzzy sets then,*

$$S_\wedge(A,B) \leq S_\wedge(A \cap C, B \cap C).$$

Proof. By applying Theorem 2, Proposition 8 and Proposition 10 we obtain the first item as follows:

$$\begin{aligned} S_\wedge(A\cap C, B\cap C) &= Inc((A\cap C),(B\cap C))\wedge Inc((B\cap C),(A\cap C)) = \\ &= Inc((A\cap C),B)\wedge Inc((A\cap C),C)\wedge Inc((B\cap C),A)\wedge Inc((B\cap C),C) \\ &= Inc((A\cap C),B)\wedge Inc((B\cap C),A) \geq S_\wedge(A,B) \end{aligned}$$

Proposition 25. *Let L be a totally ordered lattice and let A,B and C be three L-fuzzy sets. Then*

$$S_\wedge(A,B) \leq S_\wedge(A\cup C, B\cup C).$$

By applying Theorem 4, Corollary 8 and Proposition 10 we obtain the second item as follows:

$$\begin{aligned} S_\wedge(A\cup C, B\cup C) &= Inc((A\cup C),(B\cup C))\wedge Inc((B\cup C),(A\cup C)) \\ &= Inc(A,(B\cup C))\wedge Inc(C,(B\cup C))\wedge Inc(B,(A\cup C))\wedge Inc(C,(A\cup C)) \\ &= Inc(A,(B\cup C))\wedge Inc(B,(A\cup C)) \\ &\geq S_\wedge(A,B) \end{aligned}$$

□

4.4. Relationship between the different φ-degrees of similarity

Since the three approaches have been defined on the basis of the φ-degree of inclusion, it is likely that they should be related to each other. The first result shows that the ordering of the S_{eq}- similarity is maintained by $S_\wedge$.

Proposition 26. *Let A, B, C and D be L-fuzzy sets, then $S_{eq}(A,B) \leq S_{eq}(C,D)$ implies $S_\wedge(A,B) \leq S_\wedge(C,D)$.*

Proof. If $S_{eq}(A,B) \leq S_{eq}(C,D)$ then, either $Inc(A,B) \leq Inc(C,D)$ and $Inc(B,A) \leq Inc(D,C)$ or $Inc(A,B) \leq Inc(D,C)$ and $Inc(B,A) \leq Inc(C,D)$. If $Inc(A,B) \leq Inc(C,D)$ and $Inc(B,A) \leq Inc(D,C)$ then, we have:

$$S_\wedge(A,B) = Inc(A,B)\wedge Inc(B,A) \leq Inc(C,D)\wedge Inc(D,C) = S_\wedge(C,D)$$

The other case is similar. □

Fixed A and B, both $S_\wedge(A,B)$ and $S_{\cup\cap}(A,B)$ are elements of Ω, so it makes sense to wonder the relationship between both indexes.

Proposition 27. *Let A and B be two L-fuzzy sets, then*

$$S_{\cup\cap}(A,B) \leq S_\wedge(A,B)$$

Proof. By the definition of $S_{\cup\cap}$ and $S_\wedge$ and Theorem 3 we have

$$\begin{aligned} S_{\cup\cap}(A,B) &= Inc(A\cup B, A\cap B) \\ &\leq Inc(A, A\cap B) \wedge Inc(B, A\cap B) \\ &= Inc(A,A) \wedge Inc(A,B) \wedge Inc(B,A) \wedge Inc(B,B) \\ &\leq Inc(A,B) \wedge Inc(B,A) = S_\wedge(A,B) \end{aligned}$$

□

The inequality proved above might be strict, as the following example shows.

Example 6. *Consider again the L-fuzzy sets of Example 5. In this case, we have:*

$$S_\wedge(A,B) = \begin{cases} 1 & \textit{if } x = 1 \\ 0 & \textit{otherwise.} \end{cases} \qquad \textit{and} \qquad S_{\cup\cap}(A,B) = \perp.$$

Nevertheless, the equality holds when the complete lattice L considered is totally ordered.

Proposition 28. *Let L be a totally ordered lattice and let A and B be two L-fuzzy sets, then $S_{\cup\cap}(A,B) = S_\wedge(A,B)$.*

Proof. The proof is similar to the proof of Proposition 27 by applying Theorem 4 instead of Theorem 3. □

5. Conclusions and future work

We have shown how to use functions to represent the degrees of inclusion and similarity for L-fuzzy sets. To the best of our knowledge, this is the first approach which provides indexes of inclusion and similarity between L-fuzzy sets by means of functions. Specifically, we have defined the notion of f-inclusion, a binary crisp relation between L-fuzzy sets, $A \subseteq_f B$, which is somehow states how much we have to modify A so that it is included in B in Zadeh's sense. Then, the φ-degree of inclusion of A into B, denoted $Inc(A,B)$ is defined as the maximum f such that A is f-included in B, which corresponds to the minimum modification necessary. We have also shown that the φ-index of inclusion satisfies the required properties to be considered as such and, moreover, an analytic expression for $Inc(A,B)$ has been presented. Among those properties satisfied by the φ-index of inclusion, we put attention on those related to the Kitainik [16] and Sinha-Dougherty [26] axioms, since all those axioms are satisfied after a conveniently rewriting in terms of functions. The only noteworthy variation is the use of adjoint pairs in the relation of the index of inclusion of complements of fuzzy sets (see Proposition 12).

Finally, we have introduced three φ-degrees of similarity in terms of the φ-degree of inclusion, and have shown that they are related to some axiomatic approaches to similarity by proving some properties of these three degrees of similarity.

The four following lines of future research naturally appear as a logical continuation of this work:

1. Use φ-indexes in order to define *a real valued* measures of inclusion for each pair of L-fuzzy sets (this was done to measure *contradiction* in [4]). This way, the comparison with axiomatic approaches of subsethood could be done fairly, without the need to modify or adapt the axioms to the L-fuzzy case. Moreover, a comparison with other constructive measures of inclusion, as those based on (T,N)-implications [25] or on aggregations [6], will be of our interest as well.
2. Considering specific subsets of Ω enables us to obtain some properties that does not hold in when considering the full set Ω, for instance, in results concerning the complements. In this line, it is necessary to identifiy the required additional properties, besides those in Definition 1, and study consequences for the satisfiability of the properties given in Section 3. This also includes the analytical expression of the new φ-degree of inclusion.
3. As shown in Section 2.6, f-inclusion can be closely related to residuated impli-

cations. Therefore, it is natural to analize whether the index of inclusion could be seen as a certain kind of logic implication. Thus, given the information provided by two L-fuzzy sets A and B, $Inc(A,B)$ could be associated with an implication to represent the information we can infer about B from the information of A.

4. Residuated pairs are fundamental for the development of different generalizations of fuzzy logic programming [20, 23]. Furthermore, in [28] inferred rules are linked to monotonic mappings which can be interpreted as generalizations of fuzzy implications and they might be considered as φ-indexes of inclusion as well. Therefore, we aim at using our index of inclusion to construct knowledge databases based on If-Then fuzzy rules in the framework of generalized fuzzy logic programming.

References

[1] W. Bandler and L. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:13–30, 1980.

[2] B. Bouchon-Meunier, M. Rifqi, and S. Bothorel. Towards general measures of comparison of objects. *Fuzzy Sets and Systems*, 84:143–153, 1996.

[3] P. Burillo, N. Frago, and R. Fuentes. Inclusion grade and fuzzy implication operators. *Fuzzy Sets and Systems*, 114:417–429, 2000.

[4] H. Bustince, N. Madrid, and M. Ojeda-Aciego. The notion of weak-contradiction: definition and measures. *IEEE Transactions on Fuzzy Systems*, 23:1057–1069, 2015.

[5] R. Chutia and M. Gogoi. Fuzzy risk analysis in poultry farming using a new similarity measure on generalized fuzzy numbers. *Computers and Industrial Engineering*, 115:543–558, 2018.

[6] C. Cornelis, C. Van der Donck, and E. Kerre. Sinha-Dougherty approach to the fuzzification of set inclusion revisited. *Fuzzy Sets and Systems*, 134:283–295, 2003.

[7] I. Couso, L. Garrido, and L. Sánchez. Similarity and dissimilarity measures between fuzzy sets: A formal relational study. *Information Sciences*, 229:122–141, 2013.

[8] B. De Baets, H. Meyer, and H. Naessens. On rational cardinality-based inclusion measures. *Fuzzy Sets and Systems*, 128:169–183, 2002.

[9] G. Deng, Y. Jiang, and J. Fu. Monotonic similarity measures between fuzzy sets and their relationship with entropy and inclusion measure. *Fuzzy Sets and Systems*, 287:97–118, 2016.

[10] E. R. Dougherty and A. T. Popov. Fuzzy mathematical morphology based on fuzzy inclusion. In E. E. Kerre and M. Nachtegael, editors, *Fuzzy Techniques in Image Processing*, pages 76–100. Physica-Verlag HD, Heidelberg, 2000.

[11] J. Fan, W. Xie, and J. Pei. Subsethood measure: New definitions. *Fuzzy Sets and Systems*, 106:201–209, 1999.

[12] J. Fodor and R. Yager. Fuzzy Set-Theoretic Operators and Quantifiers. In D. Dubois and H. Prade, editors, *Fundamentals of Fuzzy Sets*, volume 7 of *The Handbooks of Fuzzy Sets Series*, pages 125–193. Springer US, 2000.

[13] A. Ghareeb and S. Rida. Image quality measures based on intuitionistic fuzzy similarity and inclusion measures. *Journal of Intelligent and Fuzzy Systems*, 34:4057–4065, 2018.

[14] G. Hesamian and J. Chachi. On similarity measures for fuzzy sets with applications to pattern recognition, decision making, clustering, and approximate reasoning. *Journal of Uncertain Systems*, 11:35–48, 2017.

[15] B. Huang, H. Li, G. Feng, and Y. Zhuang. Inclusion measure-based multi-granulation intuitionistic fuzzy decision-theoretic rough sets and their application to issa. *Knowledge-Based Systems*, 138:220–231, 2017.

[16] L. M. Kitainik. Fuzzy inclusions and fuzzy dichotomous decision procedures. In J. Kacprzyk and S. A. Orlovski, editors, *Optimization Models Using Fuzzy Sets and Possibility Theory*, pages 154–170. Springer Netherlands, Dordrecht, 1987.

[17] B. Kosko. Fuzzy entropy and conditioning. *Information Sciences*, 40:165–174, 1986.

[18] L. Kuncheva. Using measures of similarity and inclusion for multiple classifier fusion by decision templates. *Fuzzy Sets and Systems*, 122(3):401–407, 2001.

[19] Z. Ma, W. Zhang, and W. Ma. Assessment of data redundancy in fuzzy relational databases based on semantic inclusion degree. *Information Processing Letters*, 72:25–29, 1999.

[20] N. Madrid and M. Ojeda-Aciego. Measuring inconsistency in fuzzy answer set semantics. *IEEE Transactions on Fuzzy Systems*, 19:605–622, 2011.

[21] N. Madrid and M. Ojeda-Aciego. A view of f-indexes of inclusion under different axiomatic definitions of fuzzy inclusion. *Lecture Notes in Artificial Intelligence Science*, 10564:307–318, 2017.

[22] N. Madrid, M. Ojeda-Aciego, and I. Perfilieva. f-inclusion indexes between fuzzy sets. In *Proc. of IFSA-EUSFLAT*, 2015.

[23] J. Medina, M. Ojeda-Aciego, and P. Vojtáš. Similarity-based unification: a multi-adjoint approach. *Fuzzy Sets and Systems*, 146:43–62, 2004.

[24] S. Montes, I. Couso, P. Gil, and C. Bertoluzza. Divergence measure between fuzzy sets. *International Journal of Approximate Reasoning*, 30:91–105, 2002.

[25] J. Pinheiro, B. Bedregal, R. H. Santiago, and H. Santos. A study of (t,n)-implications and its use to construct a new class of fuzzy subsethood measure. *International Journal of Approximate Reasoning*, 97(1-16), 2018.

[26] D. Sinha and E. R. Dougherty. Fuzzification of set inclusion: Theory and applications. *Fuzzy Sets and Systems*, 55:15–42, 1993.

[27] M. Smithson. Fuzzy set inclusion: Linking fuzzy set methods with mainstream techniques. *Sociological Methods & Research*, 33:431–461, 2005.

[28] U. Straccia and N. Madrid. A top-k query answering procedure for fuzzy logic programming. *Fuzzy Sets and Systems*, 205:1–29, 2012.

[29] V. R. Young. Fuzzy subsethood. *Fuzzy Sets and Systems*, 77:371–384, 1996.

[30] L. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.