On the construction of adjunctions between a fuzzy preposet and an unstructured set

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Abstract

In this work, we focus on adjunctions, also called isotone Galois connections, in the framework of fuzzy preordered sets (hereafter, fuzzy preposets). Specifically, we present necessary and sufficient conditions so that, given a mapping $f\colon \mathbb{A} \to B$ from a fuzzy preposet $\mathbb{A}$ into an unstructured set B, it is possible to construct a suitable fuzzy preorder relation on B for which there exists a mapping $g\colon B \to \mathbb{A}$ such that the pair (f, g) constitutes an adjunction.

Keywords: Galois connection, Adjunction, Preorder, Fuzzy sets

1. Introduction

The notion of adjunction (or its sibling Galois connection) can be encountered in several research areas, both from a practical and a theoretical

point of view. In the literature, one can find numerous papers on theoretical developments on adjunctions [1, 2, 6, 7, 12, 17] and also on applications thereof [9, 10, 19, 20, 21, 22, 24].

Bělohlávek [1] introduced a fuzzy generalization of the notion of Galois connection and, since then, several papers have appeared on further approaches to either fuzzy adjunctions or fuzzy Galois connections; see [3, 10, 11, 12, 17, 18, 26] for some recent contributions. In some cases, a specific approach is introduced with a particular purpose in mind: for instance, Shi et al. [23] focused on the notion of fuzzy adjunction in view of its use in fuzzy mathematical morphology.

In [25, 26], fuzzy Galois connections on fuzzy posets were introduced as a generalization of Bělohlávek's fuzzy Galois connection, and our approach in this paper is precisely based on this generalization. Specifically, we are interested in constructing a right adjoint (or residual mapping) associated to a given mapping $f\colon \langle A, \rho_A \rangle \to B$ from a fuzzy preposet $\langle A, \rho_A \rangle$ into an unstructured set B. Of course, a convenient fuzzy preorder relation has to be defined on B.

In previous works [14, 15], some of the present authors have studied this problem in the crisp case for a mapping $f\colon \langle A, \leq_A \rangle \to B$ from a partially (pre)ordered set A, and also in the fuzzy case, where the approach was extended to a fuzzy poset $\langle A, \rho_A \rangle$. However, it has been argued that the antisymmetry property of fuzzy order relations is rather restrictive [4, 5], and should be weakened to a version involving a given fuzzy equivalence relation. From that point of view, fuzzy preorder relations are the most natural candidates, as they come along with their own fuzzy equivalence relation (the symmetric kernel relation).

The aim of this paper is to consider a mapping $f\colon \mathbb{A} \to B$ from a fuzzy preposet $\mathbb{A} = \langle A, \rho_A \rangle$ into an unstructured set B, and then characterize those situations in which B can be endowed with a fuzzy preorder relation and an isotone mapping $g\colon B \to A$ can be built such that the pair (f, g) is an adjunction. This problem is more than a mere exercise in generalization since antisymmetry, in practice, is usually a too strong requirement.

Although all the results will be stated in terms of the existence and construction of right adjoints (or residual mappings), all of them can be straightforwardly modified for the existence and construction of left adjoints (or residuated mappings). On the other hand, it is worth to remark that the construction developed in this paper can be extended to the different types of adjunctions (or Galois connections) between fuzzy preposets (see [13]).

The structure of the paper is as follows. In Section 2, the preliminary notions used in the rest of the paper are introduced. Then, the characterization of the existence of right adjoint, together with its construction is given in Section 3. Finally, in Section 4, we state the conclusions and prospects for future work.

2. Preliminary definitions

The most common underlying structure for considering fuzzy generalizations of Galois connections is that of a complete residuated lattice $\mathbb{L} = (L, \leq, \top, \bot, \otimes, \rightarrow)$. We will denote the supremum and infimum operation in the lattice with the symbols $\vee$ and $\wedge$, respectively.

An $\mathbb{L}$-fuzzy set on U is a mapping $X \colon U \rightarrow L$ where $X(u)$ denotes the degree to which u belongs to X; the *core* of X is the (crisp) set of elements $a \in A$ such that $X(a) = \top$.

Let X and Y be $\mathbb{L}$-fuzzy sets, X is said to *be included in Y*, denoted as $X \subseteq Y$, if $X(u) \leq Y(u)$ for all $u \in U$. The union (resp. intersection) of X and Y is defined as the $\mathbb{L}$-fuzzy set $(X \cup Y)(u) = X(u) \vee Y(u)$ (resp. $(X \cap Y)(u) = X(u) \wedge Y(u)$) for each $u \in U$.

A binary $\mathbb{L}$-fuzzy relation R on U is an $\mathbb{L}$-fuzzy subset of $U \times U$, i.e. $R \colon U \times U \rightarrow L$, and it is said to be:

(i) *Reflexive* if $R(a, a) = \top$, for all $a \in U$.
(ii) *$\otimes$-Transitive* if $R(a, b) \otimes R(b, c) \leq R(a, c)$, for all $a, b, c \in U$.
(iii) *Symmetric* if $R(a, b) = R(b, a)$, for all $a, b \in U$.
(iv) *Antisymmetric* if $R(a, b) = R(b, a) = \top$ implies $a = b$, for all $a, b \in U$.

The corresponding generalizations of preorder, order, and equivalence relation are the usual ones, namely:[3]

(i) An $\mathbb{L}$-*fuzzy preorder relation* is a fuzzy relation that is reflexive and $\otimes$-transitive.
(ii) An $\mathbb{L}$-*fuzzy order relation* is a fuzzy relation that is reflexive, antisymmetric and $\otimes$-transitive.
(iii) An $\mathbb{L}$-*fuzzy equivalence relation* is a fuzzy relation that is reflexive, symmetric and $\otimes$-transitive.

Definition 1.

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[3]From now on, when no confusion arises, we will omit the prefixes $\mathbb{L}$ and $\otimes$.

(i) A *fuzzy preposet* (fuzzy preposet) is a pair $\mathbb{U} = \langle U, \rho_U \rangle$ in which U is a set and ρ_U is a fuzzy preorder relation on U.

(ii) A *fuzzy partially ordered set* (fuzzy poset) is a pair $\mathbb{U} = \langle U, \rho_U \rangle$ in which U is a set and ρ_U is a fuzzy order relation on U.

Definition 2. Let $\mathbb{U} = \langle U, \rho_U \rangle$ be a fuzzy poset.

(i) The crisp set of *upper bounds* of a fuzzy set X on $\mathbb{U}$ is defined as

$$\mathrm{Up}(X) = \{a \in U \mid X(u) \leq \rho_A(u, a), \text{ for all } u \in U\}.$$

(ii) The *upset* and *downset* of an element $a \in U$ are defined as fuzzy sets $a^{\uparrow}, a^{\downarrow}: U \to L$, where $a^{\downarrow}(u) = \rho_U(u, a)$ and $a^{\uparrow}(u) = \rho_U(a, u)$ for all $u \in U$.

(iii) An element $a \in U$ is called a *maximum* of a fuzzy set X if $X(a) = \top$ and $X \subseteq a^{\downarrow}$. The definition of a minimum is similar.

Note that maximum elements and minimum elements of X are necessarily unique, whenever they exist, because of the antisymmetry property; they are denoted, respectively, as $\max X$ and $\min X$.

Definition 3. Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy preposets.

(i) A mapping $f \colon A \to B$ is said to be *isotone* if $\rho_A(a_1, a_2) \leq \rho_B(f(a_1), f(a_2))$ for all $a_1, a_2 \in A$.

(ii) A mapping $f \colon A \to A$ is said to be *inflationary* if $\rho_A(a, f(a)) = \top$ for all $a \in A$. Similarly, a mapping f is said to be *deflationary* if $\rho_A(f(a), a) = \top$ for all $a \in A$.

Definition 4 ([26]). Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy posets, and consider two mappings $f \colon A \to B$ and $g \colon B \to A$. The pair (f, g) forms an *adjunction* between $\mathbb{A}$ and $\mathbb{B}$, denoted $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ if, for all $a \in A$ and $b \in B$, the equality $\rho_A(a, g(b)) = \rho_B(f(a), b)$ holds.

The mapping f is called the *left adjoint* and the mapping g is called the *right adjoint*.

Note that the definition of fuzzy adjunction does not make explicit use of the particular properties of the fuzzy relations ρ_A and ρ_B and, hence, perfectly makes sense in the case that $\mathbb{A}$ and $\mathbb{B}$ are fuzzy preposets.

Notation 1. *From now on, we will use the following notation: for a mapping $f\colon A \to B$ and a fuzzy subset Y of B, the fuzzy set $f^{-1}(Y)$ is defined as $f^{-1}(Y)(a) = Y(f(a))$, for all $a \in A$.*

Finally, we recall the following theorem which states different equivalent ways to define a fuzzy adjunction.

Theorem 1 ([26]). *Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy posets, and consider two mappings $f\colon A \to B$ and $g\colon B \to A$. The following conditions are equivalent:*

(1) $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$.
(2) f and g are isotone, gf is inflationary, and fg is deflationary.
(3) $f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$.
(4) $g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$.
(5) f is isotone and $g(b) = \max f^{-1}(b^{\downarrow})$ for all $b \in B$.
(6) g is isotone and $f(a) = \min g^{-1}(a^{\uparrow})$ for all $a \in A$.

Before stating the following result, let us recall the notion of quotient set associated to a set equipped with an equivalence relation. Let $\sim$ be an equivalence relation on a set A, the *quotient set* of A w.r.t. $\sim$ is the set of all the equivalence classes of the relation $\sim$; this quotient set is usually denoted as $A/_{\sim}$ or $A_{\sim}$, and the equivalence class of an element $a \in A$ is denoted as $[a]_{\sim}$.

Theorem 2 ([15]). *Let $\langle A, \rho_A \rangle$ be a fuzzy poset and consider a mapping $f\colon A \to B$. Let $A_{\equiv_f}$ be the quotient set w.r.t. the kernel relation $\equiv_f$ defined by $a \equiv_f b$ if and only if $f(a) = f(b)$. Then there exists a fuzzy order relation ρ_B on B and a mapping $g\colon B \to A$ such that $(f, g)\colon \langle A, \rho_A \rangle \leftrightharpoons \langle B, \rho_B \rangle$ if and only if the following conditions hold:*

(1) $\max [a]_{\equiv_f}$ exists for all $a \in A$.
(2) $\rho_A(a_1, a_2) \leq \rho_A(\max [a_1]_{\equiv_f}, \max [a_2]_{\equiv_f})$, for all $a_1, a_2 \in A$.

3. The construction of the right adjoint

In this section we generalize Theorem 2 to the framework of fuzzy preposets. The construction will follow that given in [16] as much as possible. First, we recall the notion of transitive closure of a fuzzy relation.

Definition 5 ([8]). Given a fuzzy relation $R \colon U \times U \to L$, the *transitive closure of R* is a fuzzy relation R^{tr} such that the following conditions hold:

(1) R^{tr} is transitive.
(2) $R \subseteq R^{tr}$.
(3) If $R \subseteq R'$ and R' is transitive, then $R^{tr} \subseteq R'$.

Note that a transitive closure always exists and is necessarily unique. Actually, the transitive closure of a fuzzy relation R is the smallest transitive fuzzy relation containing R, and it can be characterized by the following proposition.

Proposition 1 ([8]). *Given a fuzzy relation $R \colon U \times U \to L$, the powers $R^n \colon U \times U \to L$, for $n \in \mathbb{N}$, are recursively defined by the base case $R^1 = R$ and*

$$R^n(a, b) = \bigvee_{x \in U} R^{n-1}(a, x) \otimes R(x, b) \,.$$

The $\otimes$-transitive closure of R is then given by

$$R^{tr}(a, b) = \bigvee_{n=1}^{\infty} R^n(a, b) \,.$$

The symmetric kernel relation $\approx_A$ is a fuzzy equivalence relation that allows to get rid of the lack of antisymmetry, by linking together elements that are 'almost coincident'; formally, the relation $\approx_A$ corresponding to a fuzzy preposet $\langle A, \rho_A \rangle$ is defined as follows:

$$(a_1 \approx_A a_2) = \rho_A(a_1, a_2) \otimes \rho_A(a_2, a_1) \qquad \text{for all } a_1, a_2 \in A \,.$$

The kernel equivalence relation $\equiv_f$ on A associated to a mapping $f \colon A \to B$ is defined as follows, for all $a_1, a_2 \in A$:

$$(a_1 \equiv_f a_2) = \begin{cases} \bot & \text{, if } f(a_1) \neq f(a_2) \,, \\ \top & \text{, if } f(a_1) = f(a_2) \,. \end{cases}$$

Definition 6. Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preposet, and consider a mapping $f \colon A \to B$. The *fuzzy p-kernel*[4] relation $\cong_{Af}$ is the $\otimes$-transitive closure of the union of the fuzzy equivalence relations $\approx_A$ and $\equiv_f$, i.e. $\cong_{Af} = (\approx_A \cup \equiv_f)^{tr}$.

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[4]The prefix p stands for preposet and it is used to distinguish it from the analogous notion in fuzzy posets.

Note that $\cong_{Af}$ is also a fuzzy equivalence relation and the fuzzy equivalence classes $[a]_{\cong_{Af}} : A \to L$ are the fuzzy sets defined by

$$[a]_{\cong_{Af}}(x) = (x \cong_{Af} a). \tag{1}$$

In order to facilitate the understanding of the different notions introduced in the construction of the right adjoint, we will illustrate it, step by step, by means of a toy example.

Example 1. Consider the unit interval together with the residuated lattice structure provided by the product t-norm and its residual (Goguen) implication $\mathbb{L} = ([0,1], \leq, 0, 1, \cdot, \to)$. In addition, consider the sets $A = \{a, b, c, d, e, \top\}$ and $B = \{p, q, r, s, t\}$, and the mapping $f \colon A \to B$ defined as $f(a) = f(c) = p$, $f(b) = q$, $f(d) = f(\top) = r$ and $f(e) = s$.

Consider the fuzzy preorder relation ρ_A on A shown in the following table:

ρ_A	a	b	c	d	e	$\top$
a	1	0.5	1	1	1	1
b	0.2	1	1	1	1	1
c	0.2	0.2	1	0.3	1	1
d	0.08	0.2	0.4	1	0.4	1
e	0.2	0.2	1	0.3	1	1
$\top$	0.08	0.2	0.4	0.2	0.4	1

The fuzzy p-kernel relation is the transitive closure of the union of the following two fuzzy relations

$\equiv_f$	a	b	c	d	e	$\top$
a	1	0	1	0	0	0
b	0	1	0	0	0	0
c	1	0	1	0	0	0
d	0	0	0	1	0	1
e	0	0	0	0	1	0
$\top$	0	0	0	1	0	1

$\approx_A$	a	b	c	d	e	$\top$
a	1	0.1	0.2	0.08	0.2	0.08
b	0.1	1	0.2	0.2	0.2	0.2
c	0.2	0.2	1	0.12	1	0.4
d	0.08	0.2	0.12	1	0.12	0.2
e	0.2	0.2	1	0.12	1	0.4
$\top$	0.08	0.2	0.4	0.2	0.4	1

therefore,

$\cong_{Af}$	a	b	c	d	e	$\top$
a	1	0.2	1	0.4	1	0.4
b	0.2	1	0.2	0.2	0.2	0.2
c	1	0.2	1	0.4	1	0.4
d	0.4	0.2	0.4	1	0.4	1
e	1	0.2	1	0.4	1	0.4
$\top$	0.4	0.2	0.4	1	0.4	1

The fuzzy equivalence clases are

$$[a]_{\cong_{Af}} = [c]_{\cong_{Af}} = [e]_{\cong_{Af}} = \{a/1, b/0.2, c/1, d/0.4, e/1, \top/0.4\}$$
$$[b]_{\cong_{Af}} = \{a/0.2, b/1, c/0.2, d/0.2, e/0.2, \top/0.2\}$$
$$[d]_{\cong_{Af}} = [\top]_{\cong_{Af}} = \{a/0.4, b/0.2, c/0.4, d/1, e/0.4, \top/1\}$$

$\square$

Lemma 1. *Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preposet, and consider a mapping $f \colon A \to B$. It then holds that $(a_1 \cong_{Af} a_2) = \top$ if and only if $[a_1]_{\cong_{Af}} = [a_2]_{\cong_{Af}}$.*

PROOF. Consider $a_1, a_2 \in A$ such that $(a_1 \cong_{Af} a_2) = \top$, and let us prove that $[a_1]_{\cong_{Af}}(u) = [a_2]_{\cong_{Af}}(u)$ for all $u \in A$. Given $u \in A$, by using the neutral element of the product, and symmetry and transitivity of $\cong_{Af}$, we have that

$$(a_1 \cong_{Af} u) = \top \otimes (a_1 \cong_{Af} u) = (a_2 \cong_{Af} a_1) \otimes (a_1 \cong_{Af} u) \leq (a_2 \cong_{Af} u).$$

Similarly, $(a_2 \cong_{Af} u) \leq (a_1 \cong_{Af} u)$ and, therefore, $[a_1]_{\cong_{Af}}(u) = [a_2]_{\cong_{Af}}(u)$ for all $u \in A$. $\square$

Notation 2 (p-maximum). *The notions of maximum or minimum element of a fuzzy subset X of a fuzzy preposet are the same as in Definition 2. The absence of antisymmetry makes it possible that there exist several maximum (resp. minimum) elements for X, which will be called p-maximum (resp. p-minimum) elements to make clear that we are working in the framework of a fuzzy preposet. We will write $\text{p-max}\, X$ (resp. $\text{p-min}\, X$) to denote the set of p-maxima (resp. p-minima) of X.*

The following theorem states the different equivalent characterizations of the notion of adjunction between fuzzy preposets. As expected, the general structure of the formulations is preserved, but those concerning the actual definition of the adjoints have to be modified by using the notions of p-maximum and p-minimum element.

Theorem 3 ([13]). *Let $\mathbb{A} = \langle A, \rho_A \rangle, \mathbb{B} = \langle B, \rho_B \rangle$ be two fuzzy preposets, and consider two mappings $f \colon \mathbb{A} \to \mathbb{B}$ and $g \colon \mathbb{B} \to \mathbb{A}$. The following statements are equivalent:*

(1) $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$.

(2) f and g are isotone, gf is inflationary, and fg is deflationary.

(3) $f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$.

(4) $g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$.

(5) f is isotone and $g(b) \in$ p-max $f^{-1}(b^{\downarrow})$ for all $b \in B$.

(6) g is isotone and $f(a) \in$ p-min $g^{-1}(a^{\uparrow})$ for all $a \in A$.

The notion of Hoare ordering between crisp subsets is generalized below (including a weak and a strong version). We then prove that the three notions coincide in a particular case that will be used in the statements of the main results of the paper.

Definition 7. Consider a fuzzy preposet $\mathbb{A} = \langle A, \rho_A \rangle$. We define the following fuzzy relations on the powerset of A:

(i) $(C \sqsubseteq_W D) = \bigvee\limits_{c \in C} \bigvee\limits_{d \in D} \rho_A(c, d)$,

(ii) $(C \sqsubseteq_H D) = \bigwedge\limits_{c \in C} \bigvee\limits_{d \in D} \rho_A(c, d)$,

(iii) $(C \sqsubseteq_S D) = \bigwedge\limits_{c \in C} \bigwedge\limits_{d \in D} \rho_A(c, d)$,

for all crisp subsets C, D of A.

Lemma 2. *Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preposet, and consider fuzzy subsets X, Y of A such that* p-min $X \neq \varnothing \neq$ p-min Y, *then*

$$\big(\text{p-min}\, X \sqsubseteq_W \text{p-min}\, Y\big) = \big(\text{p-min}\, X \sqsubseteq_H \text{p-min}\, Y\big) = \big(\text{p-min}\, X \sqsubseteq_S \text{p-min}\, Y\big)$$

and their value coincides with $\rho_A(x, y)$ for any $x \in$ p-min X and $y \in$ p-min Y.

PROOF. Firstly, note that if $u_1, u_2 \in$ p-min X, then $\rho_A(u_1, u_2) = \top$, by definition of p-min X.

Secondly, $\rho_A(x_1, y_1) = \rho_A(x_2, y_2)$ for all $x_1, x_2 \in$ p-min X, $y_1, y_2 \in$ p-min Y. Indeed, $\rho_A(x_1, y_1) \geq \rho_A(x_1, x_2) \otimes \rho_A(x_2, y_1) = \top \otimes \rho_A(x_2, y_1) \geq \rho_A(x_2, y_2) \otimes \rho_A(y_2, y_1) = \rho_A(x_2, y_2)$. Analogously, $\rho_A(x_2, y_2) \geq \rho_A(x_1, y_1)$. $\square$

As a consequence of the previous result, we will use the following notation

Notation 3. *Given a subset $S \subset A$ and an element $a \in A$, we will write*

$$\varphi_S(a) \stackrel{\text{def}}{=} \text{p-min}(\text{Up}([a]_{\cong_{Af}}) \cap S). \qquad (2)$$

Remark 1. Note that, due to Lemma 2, it holds that $\big(\varphi_S(a_1) \sqsubseteq_H \varphi_S(a_2)\big) = \rho_A(x, y)$ for any $x \in \varphi_S(a_1)$ and $y \in \varphi_S(a_2)$, and this justifies that, in order to simplify the notation, we write $\rho_A(\varphi_S(a_1), \varphi_S(a_2))$ instead of $\big(\varphi_S(a_1) \sqsubseteq_H \varphi_S(a_2)\big)$.

The next technical lemma will be used hereinafter, namely, twice in the proof of Theorem 4.

Lemma 3. *Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy preposets, and consider two mappings $f \colon A \to B$ and $g \colon B \to A$ such that $(f, g) \colon \mathbb{A} \rightleftharpoons \mathbb{B}$. For all $a \in A$, it holds that $[a]_{\cong_{Af}} \subseteq gf(a)^{\downarrow}$.*

PROOF. Recall that the fuzzy relation $\cong_{Af}$ has been defined as the transitive closure of the union of the fuzzy relations $\approx_A$ and $\equiv_f$. Hereafter, we will write R to refer to this union. Using Proposition 1 and properties of the supremum, it suffices to prove that any power R^n satisfies the following inequality:
$$aR^n u \leq \rho_A(u, gf(a)) \qquad \text{for all } u \in A. \qquad (3)$$

(i) For $n = 1$ and $u \in A$, we have that

$$\begin{aligned} aRu &= (a \approx_A u) \vee (a \equiv_f u) \\ &= \big(\rho_A(a, u) \otimes \rho_A(u, a)\big) \vee (a \equiv_f u) \\ &\leq \rho_A(u, a) \vee (a \equiv_f u). \end{aligned}$$

Considering the two possible values of $a \equiv_f u$:

- If $(a \equiv_f u) = \bot$, then due to the monotonicity of f and the adjunction property, we have that

$$\rho_A(u, a) \leq \rho_B(f(u), f(a)) = \rho_A(u, gf(a)).$$

- If $(a \equiv_f u) = \top$, then inequality (3) degenerates to a tautology. Specifically, using $f(a) = f(u)$ and the adjunction property, we have

$$\rho_A(u, gf(a)) = \rho_B(f(u), f(a)) = \rho_B(f(a), f(a)) = \top.$$

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(ii) Assume inequality (3) holds for $n-1$, and let us prove it holds for n.

$$
\begin{aligned}
aR^n u &= \bigvee_{x \in A} aR^{n-1}x \otimes xRu \\
&\leq \bigvee_{x \in A} \rho_A(x, gf(a)) \otimes \big((x \approx_A u) \vee (x \equiv_f u)\big) \\
&= \bigvee_{x \in A} \rho_A(x, gf(a)) \otimes \big((\rho_A(x,u) \otimes \rho_A(u,x)) \vee (x \equiv_f u)\big) \\
&\leq \bigvee_{x \in A} \rho_A(x, gf(a)) \otimes \big(\rho_A(u,x) \vee (x \equiv_f u)\big).
\end{aligned}
$$

Once again, we reason on each $x \in A$ separately, considering the possible values of $x \equiv_f u$, and using the monotonicity of f and the hypothesis $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ when necessary:

- If $(x \equiv_f u) = \bot$, then the result follows due to commutativity of $\otimes$ and the transitivity of ρ_A.
- If $(x \equiv_f u) = \top$, then $f(x) = f(u)$ implies

$$
\rho_A(x, gf(a)) = \rho_B(f(x), f(a)) = \rho_B(f(u), f(a)) = \rho_A(u, gf(a)).
$$

$\square$

We can now state some necessary conditions for the existence of fuzzy adjunctions between fuzzy preposets. The result obtained resembles that in the crisp case [16].

Theorem 4. *Let* $\mathbb{A} = \langle A, \rho_A \rangle$ *and* $\mathbb{B} = \langle B, \rho_B \rangle$ *be fuzzy preposets, and consider two mappings* $f \colon A \to B$ *and* $g \colon B \to A$ *such that* $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$. *The following statements hold:*

(1) $gf(A) \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_{Af}}$.

(2) $\varphi_{gf(A)}(a) \neq \varnothing$, *for all* $a \in A$ *(the element* $gf(a)$ *is in this set).*

(3) $\rho_A(a_1, a_2) \leq \rho_A\big(\varphi_{gf(A)}(a_1), \varphi_{gf(A)}(a_2)\big)$ *for all* $a_1, a_2 \in A$.

PROOF.

Item 1. Consider $a \in A$ and let us show that $gf(a) \in \text{p-max}[gf(a)]_{\cong_{Af}}$.

By definition of p-maximum element of a fuzzy set, we have to prove that it is an element of its core, and also an upper bound. To begin with, it is

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straightforward that $[gf(a)]_{\cong_{Af}}(gf(a)) = \top$, therefore we just have to prove the inclusion $[gf(a)]_{\cong_{Af}} \subseteq (gf(a))^{\downarrow}$ between fuzzy sets, that is, we have to prove that $[gf(a)]_{\cong_{Af}}(u) \le \rho_A\big(u, gf(a)\big)$ for all $u \in A$ or equivalently, that

$$gf(a)R^n u \le \rho_A(u, gf(a)) \qquad \text{for all } u \in A \text{ and for any } n \ge 1. \qquad (4)$$

On the one hand, by Lemma 3, $gf(a)R^n u \le \rho_A(u, gfgf(a))$, for all $u \in A$. On the other hand, since the composition fg is deflationary and g is isotone, we have that $\top = \rho_B(fg(f(a)), f(a)) \le \rho_A(gfgf(a), gf(a))$, which implies $\rho_A(gfgf(a), gf(a)) = \top$. Hence, for all $u \in A$, we have

$$\rho_A(u, gfgf(a)) = \rho_A(u, gfgf(a)) \otimes \rho_A(gfgf(a), gf(a)) \le \rho_A(u, gf(a))$$

Summarizing, inequality (4) holds.

Item 2. Note that the set of upper bounds and the image involved in this condition are crisp sets. Specifically, we will prove that $gf(a)$ belongs to p-$\min(\mathrm{Up}([a]_{\cong_{Af}}) \cap g(f(A)))$.

To begin with, we have to check that $gf(a) \in \mathrm{Up}([a]_{\cong_{Af}}) \cap gf(A)$. As it is obvious that $gf(a) \in gf(A)$, we just have to show that $gf(a)$ is an upper bound of the fuzzy set $[a]_{\cong_{Af}}$, which is straightforwardly deduced from Lemma 3.

Finally, for the minimality, we have to check that $\rho_A(gf(a), x) = \top$ for all $x \in \mathrm{Up}([a]_{\cong_{Af}}) \cap g(f(A)))$. Consider $x \in \mathrm{Up}([a]_{\cong_{Af}}) \cap g(f(A))$, then there exists $a_1 \in A$ such that $x = gf(a_1)$ and $(a \cong_{Af} u) \le \rho_A(u, x)$ for all $u \in A$. Particularly, considering $u = a$ and using the monotonicity of g and the adjunction property, we have that

$$\begin{aligned} \top = (a \cong_{Af} a) \le \rho_A(a, x) &= \rho_A(a, gf(a_1)) \\ &= \rho_B(f(a), f(a_1)) \\ &\le \rho_A(gf(a), gf(a_1)) = \rho_A(gf(a), x). \end{aligned}$$

Item 3. Consider $a_1, a_2 \in A$. Since f and g are monotone mappings, we have

$$\rho_A(a_1, a_2) \le \rho_A(gf(a_1), gf(a_2)).$$

From this inequality, we directly obtain the required condition

$$\rho_A(a_1, a_2) \le \rho_A\big(\varphi_{gf(A)}(a_1), \varphi_{gf(A)}(a_2)\big),$$

since we have just proved above that $gf(a) \in \varphi_{gf(A)}(a)$. $\qquad\qquad \square$

Corollary 1. *Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preposet, let B be an unstructured set, and consider a mapping $f\colon A \to B$. If f is the left adjoint of an adjunction, then there exists a subset $S \subseteq A$ such that*

(1) $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_{Af}}$.

(2) $\varphi_S(a) \neq \varnothing$, for all $a \in A$.

(3) $\rho_A(a_1, a_2) \leq \rho_A\big(\varphi_S(a_1), \varphi_S(a_2)\big)$ for all $a_1, a_2 \in A$.

The second part of this section is devoted to establishing the sufficient conditions for the existence of the right adjoint. So, given $f\colon A \to B$ with the conditions above, we will construct a fuzzy preorder relation on B together with a mapping $g\colon B \to A$, which will turn out to be a right adjoint of f.

Definition 8. Consider a fuzzy preposet $\mathbb{A} = \langle A, \rho_A \rangle$ together with a mapping $f\colon A \to B$ and a subset $S \subseteq A$ satisfying the following hypotheses, for all $a \in A$, and $a_1, a_2 \in A$:

$$S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_{Af}}, \tag{5}$$

$$\varphi_S(a) \neq \varnothing, \tag{6}$$

$$\rho_A(a_1, a_2) \leq \rho_A(\varphi_S(a_1), \varphi_S(a_2)) \tag{7}$$

For any $a_0 \in A$, we define the fuzzy relation $\rho_B^{a_0}\colon B \times B \to L$ as follows

$$\rho_B^{a_0}(b_1, b_2) = \rho_A(\varphi_S(a_1), \varphi_S(a_2))$$

where $a_i \in f^{-1}(b_i)$ if $f^{-1}(b_i) \neq \varnothing$ and $a_i = a_0$ otherwise, for any $i \in \{1, 2\}$.

Note that this definition might depend on the possible choices of a_i. The following lemma, based on Remark 1, shows that the value of $\rho_B^{a_0}$ actually is independent of this choice.

Lemma 4. *The fuzzy relation $\rho_B^{a_0}$ is well defined, and it is a fuzzy preorder relation on B.*

PROOF. The definition does not depend on the choice of the preimages a_i since, if other preimages $\bar{a}_i$ would have been chosen, then $(a_i \equiv_f \bar{a}_i) = \top$ and, hence, by Lemma 1, the fuzzy sets corresponding to the equivalence

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classes $[a_i]_{\cong_{Af}}$ and $[\bar{a}_i]_{\cong_{Af}}$ would coincide and $\varphi_S(a_i) = \varphi_S(\bar{a}_i)$. Moreover, due to Remark 1, we have that

$$\rho_A(\varphi_S(a_1), \varphi_S(a_2)) = \rho_A(x, y) \quad \text{for any } x \in \varphi(a_1) \text{ and } y \in \varphi(a_2) \,.$$

We will now prove that $\rho_B^{a_0}$ is a fuzzy preorder relation on B.

Reflexivity If $b \in f(A)$ and $f(a) = b$, then $\rho_B^{a_0}(b, b) = \rho_A(\varphi_S(a), \varphi_S(a)) = \top$; otherwise, if $b \notin f(A)$, then $\rho_B^{a_0}(b, b) = \rho_A(\varphi_S(a_0), \varphi_S(a_0)) = \top$.

Transitivity Follows directly from the definition of $\rho_B^{a_0}$ and the transitivity of ρ_A. $\qquad\square$

Example 2. For the fuzzy preposet $\langle A, \rho_A \rangle$, together with the mapping $f \colon A \to B$ defined in Example 1, it is not difficult to check that the subset $S = \{c, \top\}$ satisfies the three conditions (5), (6) and (7), because

$$\text{p-max}\,[a]_{\cong_{Af}} \cup \text{p-max}\,[b]_{\cong_{Af}} \cup \text{p-max}\,[d]_{\cong_{Af}} = \{c, e\} \cup \{b\} \cup \{\top\}\,,$$

and $\varphi_S(a) = \varphi_S(b) = \varphi_S(c) = \varphi_S(e) = \{c\}$ and $\varphi_S(d) = \varphi_S(\top) = \{\top\}$. Thus, the fuzzy preorder relation ρ_B^c is the following

ρ_B^c	p	q	r	s	t
p	1	1	1	1	1
q	1	1	1	1	1
r	0.4	0.4	1	0.4	0.4
s	1	1	1	1	1
t	1	1	1	1	1

$\qquad\square$

We can now focus on the definition of suitable mappings $g \colon B \to A$ such that (f, g) forms an adjoint pair.

Lemma 5. *Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preposet, consider a mapping $f \colon A \to B$, and let S be a subset of A satisfying hypotheses (5)–(7). Given $a_0 \in A$, there exists a mapping $g \colon B \to A$ such that $(f, g) \colon \langle A, \rho_A \rangle \leftrightarrows \langle B, \rho_B^{a_0} \rangle$ where $\rho_B^{a_0}$ is the fuzzy preorder relation introduced in Definition 8.*

PROOF. There exist a number of suitable definitions of $g \colon B \to A$, and all of them can be specified as follows:

(C1) If $b \in f(A)$, then $g(b)$ could be any element in $\varphi_S(x_b)$ for some $x_b \in f^{-1}(b)$.

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(C2) If $b \notin f(A)$, then $g(b)$ could be any element in $\varphi_S(a_0)$.

The existence of g is clear by the axiom of choice, since for all $b \in f(A)$, the sets $f^{-1}(b)$ are nonempty (so x_b can be chosen for all $b \in f(A)$) and, moreover, by hypothesis (6), $\varphi_S(x_b)$ and $\varphi_S(a_0)$ are nonempty as well.

Now, we have to prove that g is a right adjoint to f, that is, for all $a \in A$ and $b \in B$ the following equality holds

$$\rho_B^{a_0}\big(f(a), b\big) = \rho_A\big(a, g(b)\big).$$

From the definition of $\rho_B^{a_0}$ (see Definition 8), it follows that

$$\rho_B^{a_0}(f(a), b) = \rho_A(\varphi_S(a), \varphi_S(w)).$$

where w satisfies either $w \in f^{-1}(b)$ if $b \in f(A)$ (therefore, we can choose w to be x_b above) or, otherwise, $w = a_0$. In either case, $g(b) \in \varphi_S(w)$ by construction (namely, (C1) and (C2)). Thus,

$$\rho_B^{a_0}(f(a), b) = \rho_A(x, g(b)) \text{ for any } x \in \varphi_S(a). \tag{8}$$

The proof will be complete if we can show that, fixing $x \in \varphi_S(a)$, the equality $\rho_A(x, g(b)) = \rho_A(a, g(b))$ holds.

Firstly, from the definition of φ_S (see (2)) it follows that $x \in \varphi_S(a)$ implies $\rho_A(a, x) = \top$ and, hence, we have that

$$\rho_A\big(x, g(b)\big) = \rho_A(a, x) \otimes \rho_A\big(x, g(b)\big) \leq \rho_A\big(a, g(b)\big). \tag{9}$$

Furthermore, using hypothesis (7), it follows that

$$\rho_A\big(a, g(b)\big) \leq \rho_A\big(\varphi_S(a), \varphi_S\big(g(b)\big)\big) = \rho_A(x, y) \tag{10}$$

for any $x \in \varphi_S(a)$ and $y \in \varphi_S(g(b))$. Since $y \in \varphi_S\big(g(b)\big)$, we have that $\rho_A(y, \alpha) = \top$ for all $\alpha \in \mathrm{Up}([g(b)]_{\cong_{Af}}) \cap S$.

On the other hand, since $g(b) \in S$, by (5) we have that $g(b) \in \text{p-max}[a]_{\cong_{Af}}$ for some $a \in A$, therefore

$$\top = [a]_{\cong_{Af}}\big(g(b)\big) = \big(a \cong_{Af} g(b)\big)$$

as a result, by Lemma 1 the fuzzy equivalence classes $[a]_{\cong_{Af}}$ and $[g(b)]_{\cong_{Af}}$ coincide and, thus, $g(b) \in \text{p-max}[g(b)]_{\cong_{Af}}$. In particular $g(b) \in \mathrm{Up}([g(b)]_{\cong_{Af}})$, hence $g(b) \in \mathrm{Up}([g(b)]_{\cong_{Af}}) \cap S$. As a result, we obtain $\rho_A\big(y, g(b)\big) = \top$. Now, connecting expression (10) with the transitivity of ρ_A, we find that

$$\rho_A\big(a, g(b)\big) \leq \rho_A(x, y) = \rho_A(x, y) \otimes \rho_A\big(y, g(b)\big) \leq \rho_A\big(x, g(b)\big) \tag{11}$$

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for all $x \in \varphi_S(a)$. Joining Eqs. (9) and (11) we obtain, $\rho_A(x, g(b)) = \rho_A(a, g(b))$ and, finally, Eq. (8) leads to

$$\rho_B^{a_0}\big(f(a), b\big) = \rho_A(a, g(b)).$$

$\square$

We can now conclude this section by stating necessary and sufficient conditions for the existence of a right adjoint from a fuzzy preposet to an unstructured set. In this statement, for the sake of readability, we do not use the syntactically sugared version of the previous lemma (namely, φ_S) but, instead, state the conditions directly in their low level appearance.

Theorem 5. *Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preposet, and consider a mapping $f \colon A \to B$, then there exist a fuzzy preorder relation ρ_B on B and a mapping $g \colon B \to A$ such that $(f, g) \colon \mathbb{A} \leftrightarrows \mathbb{B}$ if and only if there exists a subset $S \subseteq A$ such that, for all $a, a_1, a_2 \in A$:*

(1) $S \subseteq \bigcup\limits_{a \in A} \text{p-max}[a]_{\cong_{Af}}$.

(2) $\text{p-min}(\text{Up}([a]_{\cong_{Af}}) \cap S) \neq \varnothing$.

(3) $\rho_A(a_1, a_2) \leq \Big(\text{p-min}(\text{Up}([a_1]_{\cong_{Af}}) \cap S) \sqsubseteq \text{p-min}(\text{Up}([a_2]_{\cong_{Af}}) \cap S)\Big)$.

PROOF. The necessity follows from Corollary 1, considering $S = gf(A)$; the sufficiency follows from Lemma 5. $\square$

Example 3. For the mapping $f \colon A \to B$ given in Example 1 and the fuzzy preorder relation ρ_B^c given in Example 2, the right adjoint $g \colon B \to A$ is defined as $g(p) = g(q) = g(s) = g(t) = c$ and $g(r) = \top$. $\square$

Observe that Theorem 5 differs from Theorem 2 in a number of points. In the case of preposets, assuming (f, g) is an adjunction, $gf(a)$ is not necessarily included in $[a]_{\cong_{Af}}$ (as in the case of posets) but, in fact, it is a minimal upper bound of $[a]_{\cong_{Af}}$ and also a p-maximum of its own equivalence class. Both first and second conditions of Theorem 5 guarantee that the construction of the right adjoint in this context satisfies the required properties described before.

4. Conclusions

Given a mapping $f\colon \langle A, \rho_A \rangle \to B$ from a fuzzy preposet A into an unstructured set B, we have obtained necessary and sufficient conditions to define a suitable fuzzy preorder relation ρ_B on B such that there exists a right adjoint $g\colon \langle B, \rho_B \rangle \to A$.

It should be stressed that the right adjoint, in general, is not unique. In fact, there is a number of degrees of freedom in its construction: just note that the parameterized construction of g has been given (in the case of a non-surjective f) in terms of an element $a_0 \in A$. We chose a convenient construction to extend the induced fuzzy order relation on the image of f to the whole set B, but it is worth to stress that our results do not imply that every right adjoint can be constructed in this way, and there may exist other constructions that are adequate as well. This is an interesting topic for future work.

When focusing on fuzzy generalizations of order relations one can find some interesting developments in the study of both fuzzy order and fuzzy preorder relations, see [4, 5] for instance. In these works, it is argued that the versions of reflexivity and antisymmetry commonly used are too strong and, as a consequence, the resulting fuzzy order relations are very close to the classical case. Accordingly, one interesting line of future work will be the adaptation of the current results to these alternative weaker definitions.

Another topic of future work could be the study of alternative interpretations of the notion of adjunction between multivalued functions (i.e., relations) both in crisp and fuzzy frameworks, with the aim of building a right adjoint for a given multivalued function.

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