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On multi-adjoint concept lattices based on heterogeneous conjunctors

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Abstract

In formal concept analysis, the sets of attributes and objects are usually different, with different meaning and, hence, it might not make sense to evaluate them on the same carrier. In this context, the operators used to obtain the concept lattice could be defined by considering different lattices associated to attributes and objects. Anyway there exist several reasons for which we need to evaluate the set of attributes and objects in the same carrier. In this direction, we present in this paper a new concept lattice, where the objects and attributes are evaluated on the same lattice L, although operators which evaluate objects and attributes in different carriers are used. Moreover, we have studied the relationship between the new concept lattice and the other one obtained directly considered different carriers to both set of attributes and objects.

Key words: Concept lattices, multi-adjoint lattices, Galois connection, implication triples

1 Introduction

Formal concept analysis was introduced by Wille in the eighties and it has become an important and appealing research topic both from the theoretical perspective [13,23,26] and from the applicative one [7, 9, 10, 12, 22].

Soon after the introduction of "classical" formal concept analysis, a number of different approaches for its generalization were introduced and, nowadays, there are works which extend the theory with ideas from fuzzy set theory [3,17,18] or fuzzy logic reasoning [2, 4, 8] or from rough set theory [16, 24, 27] or some integrated approaches such as fuzzy and rough [25], or rough and domain theory [14].

Recently, a new fuzzy framework has been introduced which is more general and flexible than other fuzzy extensions, see [20]. In this framework, we can evaluate the set of objects and attributes on different lattices L_1, L_2, because it might not make sense to evaluate objects and attributes on the same carrier.

It is convenient to recall that, sometimes, it could be interesting to weaken this framework. For instance, given a group of experts that need to evaluate a knowledge system, they could believe that the carriers associated to the set of objects and attributes should not be different, or some of them believe that the attributes should be evaluated on L_1 and some others believe that they should be evaluated on L_2 and, once the evaluation is finished, the results should be homogenized. An interesting possibility is to embed both L_1 and L_2 into a set L, and to obtain a new concept lattice $\mathcal{M}_L$, in which the set of objects and attributes are evaluated in the *same* lattice, albeit using the operators which evaluate objects and attributes in different carriers.

Firstly, we will introduce the notion of P-connected poset, which will be used to define the concept lattice $\mathcal{M}_L$, when the set of attributes and objects are evaluated in L_1 and L_2, respectively, and L_1, L_2 are L-connected. Later, the new concept lattice is related with the concept lattice introduced in [20]. Finally, some conclusions and future work are presented.

2 P-connected posets

The main notion in this contribution refers to the notion of P-connection between two complete lattices. As we will see later, this condition will allow to somehow conciliate the different values generated by the consideration of a non-commutative conjunctor in the construction of a concept lattice.

Definition 1 *Given the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P, \leq)$, we say that P_1 and P_2 are* P-connected *if there exist increasing mappings $i_1 \colon P_1 \to P$, $\phi_1 \colon P \to P_1$, $i_2 \colon P_2 \to P$ and $\phi_2 \colon P \to P_2$ verifying that $\phi_1(i_1(x)) = x$, and $\phi_2(i_2(y)) = y$, for all $x \in P_1$, $y \in P_2$.*

Example 1 *Any pair of posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ with top elements $\top_1$ and $\top_2$, respectively, are $P_1 \times P_2$-connected, with the pairwise ordering, where $P_1 \times P_2$ is the Cartesian product, and by considering the mappings ϕ_i as the projections π_i, and i_1, i_2 as the inclusions defined as $i_1(x) = (x, \top_2)$, $i_2(y) = (\top_1, y)$, for all $x \in P_1$, $y \in P_2$.*

A more complex example is presented below:

Example 2 *Assume that, in order to perform an evaluation of a product, for which we have to assign one value out of four possible ones. We ask two experts to collaborate in this task and, only when collecting the feedback from each expert, we notice that one expert has considered the ordering of values as in Fig. 1, whereas the other has considered that in Fig. 2. In both cases, the expert has used a suitable poset in order to obtain the final result of the evaluation.*

In order to unify both evaluations, we want to embed the posets in Figs. 1 and 2 into another one, for example, we might consider that given in Fig. 3.

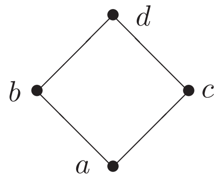

Figure 1: Poset $(P_1, \leq_1)$

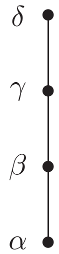

Figure 2: Poset $(P_2, \leq_2)$

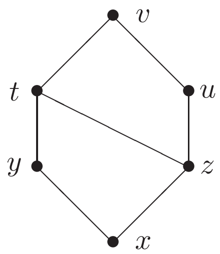

Figure 3: Poset $(P, \leq)$

	a	b	c	d
i_1	x	y	u	v

	α	β	γ	δ
i_2	x	y	t	v

Figure 4: Definition of i_1 and i_2

We can define two mappings $i_1 \colon P_1 \to P$, $i_2 \colon P_2 \to P$ as in Fig. 4; moreover, there exist several possibilities for the mappings $\phi_1 \colon P \to P_1$, $\phi_2 \colon P \to P_2$ in order to satisfy the properties in Definition 1, one of them is shown below:

	x	y	z	t	u	v
ϕ_1	a	b	c	d	c	d

	x	y	z	t	u	v
ϕ_2	α	β	γ	γ	δ	δ

As a result, P_1 and P_2 are P-connected.

Example 3 *A different example arises when we consider the posets $([0,1]_2, \leq)$ and $([0,1]_4, \leq)$, where $[0,1]_n$ is a regular partition of $[0,1]$ into n pieces, for instance $[0,1]_2 = \{0, 0.5, 1\}$, $[0,1]_4 = \{0, 0.25, 0.5, 0.75, 1\}$.*

We have that $[0,1]_2$, $[0,1]_4$ are $[0,1]$-connected, under the usual ordering, considering the mappings i_1, i_2 as the inclusions $i_1(x) = x$, $i_2(y) = y$, for all $x \in L_1$, $y \in L_2$; and ϕ_1, ϕ_2 defined as $\phi_1(t) = \lceil 2 \cdot t \rceil / 2$, $\phi_2(t) = \lceil 4 \cdot t \rceil / 4$, where $\lceil \_ \rceil$ is the ceiling function. For example, if $t = 0.55$, $\phi_1(0.55) = 1$, $\phi_2(0.55) = 0.75$,

3 Concept lattices on L-connected lattices

Firstly, we will recall the definition of adjoint triple, multi-adjoint frame and context, in order to define a new concept lattice where the objects and attributes are evaluated on the same lattice L. This new standpoint has several applications: for instance, although operators which evaluate objects and attributes in different carriers are used, we will show later that it is possible to evaluate both objects and attributes in a common lattice obtained from the original ones and, using the methods introduced in [19], to obtain a certain class of t-concepts. Originally, the condition $L_1 = L_2$ was assumed; but the construction can be extended to the cases in which $L_1 \neq L_2$, as the only requirement is that both lattices should be L-connected.

Assuming a conjunctor defined on, say $P_1 \times P_2$, directly provides two different ways of generalising the well-known adjoint property between a t-norm and its residuated implication [1, 21], depending on which argument is fixed.

Definition 2 *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$ is an* adjoint triple *with respect to P_1, P_2, P_3 if:*

1. *$\&$ is order-preserving in both arguments.*
2. *$\swarrow$ and $\nwarrow$ are order-preserving in the consequent and order-reversing in the antecedent.*
3. *$x \leq_1 z \swarrow y$ iff $x \,\&\, y \leq_3 z$ iff $y \leq_2 z \nwarrow x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$.*

The general theory formal concept analysis needs that the underlying posets have the structure of a lattice. Therefore, we will assume hereafter that we are working on lattices instead of on posets.

The multi-adjoint framework allows the existence of several adjoint triples for a given triplet of lattices.

Definition 3 *A* multi-adjoint frame *$\mathcal{L}$ is a tuple*

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \ldots, \&_n, \swarrow^n, \nwarrow_n)$$

where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are complete lattices, $(P, \leq)$ is a poset and, for all $i = 1, \ldots, n$, $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P.
Multi-adjoint frames are denoted $(L_1, L_2, L, \&_1, \ldots, \&_n)$.

Given a frame, a *multi-adjoint context* is a tuple consisting of sets of objects and attributes and a fuzzy relation among them; in addition, the multi-adjoint approach also includes a function which assigns an adjoint triple to each object (or attribute).

Definition 4 *Let $(L_1, L_2, P, \&_1, \ldots, \&_n)$ be a multi-adjoint frame, a* multi-adjoint context *is a tuple (A, B, R, σ) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon B \to \{1, \ldots, n\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.*[1]

[1] A similar theory could be developed by considering a mapping $\tau\colon A \to \{1, \ldots, n\}$ which associates any element in A with some particular adjoint triple in the frame.

In order to make this contribution self-contained and since we will provide a specific construction of a Galois connection, we recall its formal definition below:

Definition 5 *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^{\downarrow}\colon P_1 \to P_2$, ${}^{\uparrow}\colon P_2 \to P_1$ mappings, the pair $({}^{\uparrow}, {}^{\downarrow})$ forms a* Galois connection *between P_1 and P_2 if and only if:*

1. *${}^{\uparrow}$ and ${}^{\downarrow}$ are order-reversing.*
2. *$x \leq_1 x^{\downarrow\uparrow}$ for all $x \in P_1$.*
3. *$y \leq_2 y^{\uparrow\downarrow}$ for all $y \in P_2$.*

In the following paragraphs, we define a suitable Galois connection on which the new concept lattice structure will be built.

Given a complete lattice $(L, \preceq)$ such that L_1 and L_2 are L-connected, a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$, and a context (A, B, R, σ), we can define the mappings[2] ${}^{\uparrow_{c\sigma}}\colon L^B \to L^A$ and ${}^{\downarrow^{c\sigma}}\colon L^A \to L^B$ defined for all $g \in L^B$ and $f \in L^A$ as follows:

$$g^{\uparrow_{c\sigma}}(a) = i_1(\inf\{R(a,b) \swarrow^{\sigma(b)} \phi_2(g(b)) \mid b \in B\}) \tag{1}$$

$$f^{\downarrow^{c\sigma}}(b) = i_2(\inf\{R(a,b) \nwarrow_{\sigma(b)} \phi_1(f(a)) \mid a \in A\}) \tag{2}$$

Note that these definitions can be related to those given in [20] in that, for each adjoint triple $(\&, \swarrow, \nwarrow)$ of the multi-adjoint frame, we can define the mappings $\&^*\colon L \times L \to P$, $\swarrow^*\colon P \times L \to L$ and $\nwarrow_*\colon P \times L \to L$ for all $x, y \in L$ and $z \in P$ as follows:

$$x \,\&^*\, y \;=\; \phi_1(x) \,\&\, \phi_2(y) \qquad\qquad \begin{aligned} z \swarrow^* y &= i_1(z \swarrow \phi_2(y)) \\ z \nwarrow_* x &= i_2(z \nwarrow \phi_1(x)) \end{aligned}$$

which, under the requirements $t \leq i_1(\phi_1(t))$ and $t \leq i_2(\phi_2(t))$, for all $t \in L$, forms another adjoint triple $(\&^*, \swarrow^*, \nwarrow^*)$. Under the additional assumption that the mappings i_j are inf-preserving, the mappings ${}^{\uparrow_{c\sigma}}\colon L^B \to L^A$ and ${}^{\downarrow^{c\sigma}}\colon L^A \to L^B$ can be written as

$$g^{\uparrow_{c\sigma}}(a) \;=\; \inf\{R(a,b) \swarrow^{*b} g(b) \mid b \in B\} \tag{3}$$

$$f^{\downarrow^{c\sigma}}(b) \;=\; \inf\{R(a,b) \nwarrow_{*b} f(a) \mid a \in A\} \tag{4}$$

and coincide with the Galois connection associated to the frame $(L_1, L_2, P, \&_1^*, \dots, \&_n^*)$ introduced in [20]. As our construction of the new concept lattice will not need either of the requirements above, the proposed framework is strictly more general than the previous one.

Expressions (1), (2) do not coincide with those given in [20], because they are not defined directly from a residuated implication, although the mappings i_1, i_2, ϕ_1 and ϕ_2 are involved as well. Hence, we need to prove that these mappings form a Galois connection.

[2] The subscript c refers to the L-connection, since we are using the mappings ϕ_j and i_j; on the other hand, σ is needed to refer the particular choice of adjoint triple for a given b.

Proposition 1 *Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, where L_1 and L_2 are L-connected, and a context (A, B, R, σ), the pair $(^{\uparrow_{c\sigma}}, ^{\downarrow^{c\sigma}})$ is a Galois connection between L^A and L^B.*

The Galois connection just obtained is defined on a frame where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are L-connected. This Galois connection allows for defining a new concept lattice following the usual construction: a *concept* is a pair $\langle g^*, f^* \rangle$ satisfying $g^* \in L^B$, $f^* \in L^A$ and that $(g^*)^{\uparrow_c} = f^*$ and $(f^*)^{\downarrow^c} = g^*$; with $(^{\uparrow_c}, ^{\downarrow^c})$ being the Galois connection defined above.[3]

Definition 6 *The* multi-adjoint abelianized concept lattice *associated to a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and context (A, B, R, σ), where L_1 and L_2 are L-connected, is the set*

$$\mathcal{M}_L = \{\langle g^*, f^* \rangle \mid \langle g^*, f^* \rangle \text{ is a concept}\}$$

in which the ordering is defined by $\langle g_1^*, f_1^* \rangle \preceq \langle g_2^*, f_2^* \rangle$ if and only if $g_1^* \preceq g_2^*$ (equivalently $f_2^* \preceq f_1^*$).

Note that as $(^{\uparrow_c}, ^{\downarrow^c})$ is a Galois connection, the pair $(\mathcal{M}_L, \preceq)$ is, indeed, a complete lattice [6].

In the rest of the section, we establish a comparison between the concept lattices $\mathcal{M}_L$ (defined above) and $\mathcal{M}$ (defined in [20]). Hence, we will fix a context (A, B, R, σ), a frame $(L_1, L_2, P, \&_1, \dots, \&_n)$, where L_1 and L_2 are L-connected, and the corresponding multi-adjoint concept lattices $\mathcal{M}$ and $\mathcal{M}_L$.

Firstly we will prove, in the following result, that each concept $\langle g, f \rangle$ in $\mathcal{M}$ determines a concept in $\mathcal{M}_L$.

Proposition 2 *If $\langle g, f \rangle \in \mathcal{M}$, then the mappings $g^* \colon B \to L$, $f^* \colon A \to L$, defined as $g^* = i_2 \circ g$, $f^* = i_1 \circ f$, form a concept of the multi-adjoint concept lattice $\mathcal{M}_L$.*

Now, given a mapping $g \colon B \to L_2$, we have to possible ways to construct the smallest concept in $\mathcal{M}_L$ containing g:

- Considering the mapping $i_2 \circ g \in L^B$ and obtaining the corresponding concept in $\mathcal{M}_L$, that is, $\langle (i_2 \circ g)^{\uparrow_c \downarrow^c}, (i_2 \circ g)^{\uparrow_c} \rangle$.
- Obtaining the corresponding concept in $\mathcal{M}$ and, by Proposition 2, considering the concept $\langle i_2 \circ (g)^{\uparrow\downarrow}, i_2 \circ (g)^{\uparrow} \rangle$ in $\mathcal{M}_L$.

The following proposition states that the two constructions given above coincide.

Proposition 3 *Given a mapping $g \colon B \to L_2$, the concepts $\langle (i_2 \circ g)^{\uparrow_c \downarrow^c}, (i_2 \circ g)^{\uparrow_c} \rangle$ and $\langle i_2 \circ (g)^{\uparrow\downarrow}, i_2 \circ (g)^{\uparrow} \rangle$ coincide.*

[3] We include $*$ as a superscript in this new construction so that we can distinguish this new approach from that in [20]. Note that, in order to simplify the notation, references to σ have been omitted.

Similarly, we obtain a concept of $\mathcal{M}$ from each concept of $\mathcal{M}_L$, and the two possible construction of the smallest concept containing $g^*\colon B \to L$ coincide.

Proposition 4 *If $\langle g^*, f^*\rangle \in \mathcal{M}_L$, then the mappings $g\colon B \to L_2$, $f\colon A \to L_1$, defined as: $g = \phi_2 \circ g^*$, $f = \phi_1 \circ f^*$, form a concept of the multi-adjoint concept lattice $\mathcal{M}$. Moreover, given a mapping $g^*\colon B \to L$, the concepts $\langle(\phi_2 \circ g^*)^{\uparrow\downarrow}, (\phi_2 \circ g^*)^{\uparrow}\rangle$ and $\langle \phi_2 \circ (g^*)^{\uparrow_c \downarrow^c}, \phi_2 \circ (g^*)^{\uparrow_c}\rangle$ coincide.*

It is worth to take into account that the result above can be given analogously for any $f\colon A \to L_1$ as well.

Finally, as a consequence of the definition of L-connection and the above result, we have that the following theorem.

Theorem 1 *The mappings $\Phi\colon \mathcal{M}_L \to \mathcal{M}$ and $\mathcal{I}\colon \mathcal{M} \to \mathcal{M}_L$ defined, for each $\langle g, f\rangle \in \mathcal{M}$ and $\langle g^*, f^*\rangle \in \mathcal{M}_L$, as follows*

$$\begin{aligned}\Phi(\langle g^*, f^*\rangle) &= \langle \phi_2 \circ g^*, \phi_1 \circ f^*\rangle \\ \mathcal{I}(\langle g, f\rangle) &= \langle i_2 \circ g, i_1 \circ f\rangle\end{aligned}$$

are well-defined and $\Phi \circ \mathcal{I}\colon \mathcal{M} \to \mathcal{M}$ is the identity mapping. However, in general, $\mathcal{I} \circ \Phi\colon \mathcal{M}_L \to \mathcal{M}_L$ is not the identity mapping, but a closure operator.

Let $Fp(\mathcal{M}_L)$ be the subset of $\mathcal{M}_L$ consisting of all the fix-points of the $\mathcal{I} \circ \Phi$, i.e.

$$Fp(\mathcal{M}_L) = \{\langle g^*, f^*\rangle \in \mathcal{M}_L \mid \mathcal{I} \circ \Phi(\langle g^*, f^*\rangle) = \langle g^*, f^*\rangle\}$$

With this notation, the theorem above guarantees the following result:

Corollary 1 *The concept lattices $\mathcal{M}$ and $Fp(\mathcal{M}_L)$ are isomorphic.*

As a consequence of the previous isomorphism, several existing algorithms developed to obtain concept lattices where the conjunctors have the same carrier for both arguments can be applied; for instance, Lindig's algorithm [15], or its extension for graded attributes [5]. In order to obtain the concept lattice $\mathcal{M}$, we firstly use a fast algorithm to build the concept lattice $\mathcal{M}_L$ and then, compute the set $Fp(\mathcal{M}_L)$ of all fix-points of $\mathcal{I} \circ \Phi$, perhaps applying the algorithm once more. Finally, we apply Φ to obtain $\mathcal{M}$.

As the complexity of the algorithm used depends on the size of L, we should find, whenever possible, the least lattice L such that L_1 and L_2 are L-connected.

4 Conclusion

Usually, in formal concept analysis, the sets of attributes and objects are different, with different meaning and, hence, it might not make sense to evaluate them on the same carrier. In this context, the operators used to obtain the concept lattice could be defined considering different lattices associated to attributes and objects, see [20].

Anyway there exist several reasons for which we need to evaluate the set of attributes and objects in the same carrier. In this direction, a new concept lattice, where the objects and attributes are evaluated on the same lattice L, has been introduced, although operators which evaluate objects and attributes in different carriers are used.

Moreover, we have studied the relationship between the new concept lattice and the other one obtained directly considered different carriers to both set of attributes and objects, introduced in [20].

As future work, we want to study how the theory presented here can be applied to obtain t-concepts [11,19] when, originally, the set of attributes and objects are evaluated in different lattices. Another aim is to obtain mechanisms to find the least lattice L such that L_1 and L_2 are L-connected.

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