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An embedding of ChuCors in L-ChuCors

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Abstract

An L-fuzzy generalization of the so-called Chu correspondences between formal contexts forms a category called L-ChuCors. In this work we show that this category naturally embeds ChuCors.

Key words: Formal Concept Analysis, Category theory, L-fuzzy logic

1 Preliminaries

Formal concept analysis (FCA) introduced by Ganter and Wille [6] has become an extremely useful theoretical and practical tool for formally describing structural and hierarchical properties of data with "object-attribute" character. Bělohlávek in [1, 2] provided an L-fuzzy extension of the main notions of FCA, such as context and concept, by extending its underlying interpretation on classical logic to the more general framework of L-fuzzy logic [7].

In this work, we aim at formally describing some structural properties of inter-contextual relationships [5, 11] of L-fuzzy formal contexts by using category theory [3], following the results in [12, 13]. The category L-ChuCors is formed by considering the class of L-fuzzy formal contexts as objects and the L-fuzzy Chu correspondences as arrows between objects.

The main result here is that L-ChuCors embeds the category ChuCors. This result is illustrated by showing different categories L-ChuCors built on different underlying truth-values sets L.

In order to make this contribution as self-contained as possible, we proceed now with the preliminary definitions of complete residuated lattice, L-fuzzy context, L-fuzzy concept and L-Chu correspondence.

Definition 1 *An algebra* $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ *is said to be a* ***complete residuated lattice*** *if*

1. *$\langle L, \wedge, \vee, 0, 1\rangle$ is a complete bounded lattice with least element, 0, and greatest element, 1,*

2. *$\langle L, \otimes, 1\rangle$ is a commutative monoid,*

3. *$\otimes$ and $\to$ are adjoint, i.e. $a \otimes b \leq c$ if and only if $a \leq b \to c$, for all $a, b, c \in L$, where $\leq$ is the ordering in the lattice generated from $\wedge$ and $\vee$.*

Definition 2 *Let L be a complete residuated lattice, an L-**fuzzy context** is a triple $\langle B, A, r\rangle$ consisting of a set of objects B, a set of attributes A and an L-fuzzy binary relation r, i.e. a mapping $r\colon B \times A \to L$, which can be alternatively understood as an L-fuzzy subset of $B \times A$*

We now introduce the L-fuzzy extension provided by Bělohlávek [1], where we will use the notation Y^X to refer to the set of mappings from X to Y.

Definition 3 *Consider an L-fuzzy context $\langle B, A, r\rangle$. A pair of mappings $\uparrow\colon L^B \to L^A$ and $\downarrow\colon L^A \to L^B$ can be defined for every $f \in L^B$ and $g \in L^A$ as follows:*

$$\uparrow f(a) = \bigwedge_{o \in B} (f(o) \to r(o, a)) \qquad\qquad \downarrow g(o) = \bigwedge_{a \in A} \big(g(a) \to r(o, a)\big) \tag{1}$$

Lemma 1 *Let L be a complete residuated lattice, let $r \in L^{B\times A}$ be an L-fuzzy relation between B and A. Then the pair of operators $\uparrow$ and $\downarrow$ form a Galois connection between $\langle L^B; \subseteq\rangle$ and $\langle L^A; \subseteq\rangle$, that is, $\uparrow\colon L^B \to L^A$ and $\downarrow\colon L^A \to L^B$ are anti tonic and, furthermore, for all $f \in L^B$ and $g \in L^A$ we have $f \subseteq \downarrow\uparrow f$ and $g \subseteq \uparrow\downarrow g$.*

Definition 4 *Consider an L-fuzzy context $C = \langle B, A, r\rangle$. An L-fuzzy set of objects $f \in L^B$ (resp. an L-fuzzy set of attributes $g \in L^A$) is said to be **closed in C** iff $f = \downarrow\uparrow f$ (resp. $g = \uparrow\downarrow g$).*

Lemma 2 *Under the conditions of Lemma 1, the following equalities hold for arbitrary $f \in L^B$ and $g \in L^A$, $\uparrow f = \uparrow\downarrow\uparrow f$ and $\downarrow g = \downarrow\uparrow\downarrow g$, that is, both $\downarrow\uparrow f$ and $\uparrow\downarrow g$ are closed in C.*

Definition 5 *An L-**fuzzy concept** is a pair $\langle f, g\rangle$ such that $\uparrow f = g, \downarrow g = f$. The first component f is said to be the **extent** of the concept, whereas the second component g is the **intent** of the concept.*

The set of all L-fuzzy concepts associated to a fuzzy context (B, A, r) will be denoted as L-$FCL(B, A, r)$.

An ordering between L-fuzzy concepts is defined as follows: $\langle f_1, g_1\rangle \leq \langle f_2, g_2\rangle$ if and only if $f_1 \subseteq f_2$ if and only if $g_1 \supseteq g_2$.

Proposition 1 *The poset $(L\text{-}FCL(B, A, r), \leq)$ is a complete lattice where*

$$\bigwedge_{j\in J} \langle f_j, g_j\rangle = \Big\langle \bigwedge_{j\in J} f_j, \uparrow \big(\bigwedge_{j\in J} f_j\big)\Big\rangle$$

$$\bigvee_{j\in J} \langle f_j, g_j\rangle = \Big\langle \downarrow \big(\bigwedge_{j\in J} g_j\big), \bigwedge_{j\in J} g_j\Big\rangle$$

Finally, we proceed with the definition of L-Chu correspondences, for which we need the notion of L-multifunction.

Definition 6 *An L-**multifunction** from X to Y is a mapping $\varphi\colon X \to L^Y$. The set L-Mfn(X, Y) of all the L-multifunctions from X to Y can be endowed with a poset structure by defining the ordering $\varphi_1 \leq \varphi_2$ as $\varphi_1(x)(y) \leq \varphi_2(x)(y)$ for all $x \in X$ and $y \in Y$.*

Definition 7 *Consider two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i\rangle, (i = 1, 2)$, then the pair $\varphi = (\varphi_l, \varphi_r)$ is called a **correspondence** from C_1 to C_2 if φ_l and φ_r are L-multifunctions, respectively, from B_1 to B_2 and from A_2 to A_1 (that is, $\varphi_l : B_1 \to L^{B_2}$ and $\varphi_r : A_2 \to L^{A_1}$).*

*The L-correspondence φ is said to be a **weak L-Chu correspondence** if the equality $\hat{r}_1(\chi_{o_1}, \varphi_r(a_2)) = \hat{r}_2(\varphi_l(o_1), \chi_{a_2})$ holds for all $o_1 \in B_1$ and $a_2 \in A_2$. By unfolding the definition of $\hat{r}_i$ this means that*

$$\bigwedge_{a_1 \in A_1} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) = \bigwedge_{o_2 \in B_2} (\varphi_l(o_1)(o_2) \to r_2(o_2, a_2)) \tag{2}$$

*A weak Chu correspondence φ is an L-**Chu correspondence** if $\varphi_l(o_1)$ is closed in C_2 and $\varphi_r(a_2)$ is closed in C_1 for all $o_1 \in B_1$ and $a_2 \in A_2$. We will denote the set of all Chu correspondences from C_1 to C_2 by L-$ChuCors(C_1, C_2)$.*

In the following definition and lemma, we introduce some connections between the right and the left sides of L-Chu correspondences.

Definition 8 *Given a mapping $\varpi : X \to L^Y$ we consider the following associated mappings $\varpi_*\colon L^X \to L^Y$ and $\varpi^* : L^Y \to L^X$, defined for all $f \in L^X$ and $g \in L^Y$ by*

1. $\varpi_*(f)(y) = \bigvee_{x \in X}(f(x) \otimes \varpi(x)(y))$
2. $\varpi^*(g)(x) = \bigwedge_{y \in Y} \varpi(x)(y) \to g(y)$

Lemma 3 *Let $C_i = \langle B_i, A_i, r_i\rangle$ for $i = 1, 2$ be L-fuzzy contexts. Let $\varphi = (\varphi_l, \varphi_r) \in L$-*ChuCors$(C_1, C_2)$. *Then*

- *for all $f \in L^{B_1}$ and $g \in L^{A_2}$, the following equalities hold*

$$\uparrow_2 (\varphi_{l*}(f)) = \varphi_r^*(\uparrow_1 (f)) \quad \text{and} \quad \downarrow_1 (\varphi_{r*}(g)) = \varphi_l^*(\downarrow_2 (g))$$

- *for all $o_1 \in B_1$ and $a_2 \in A_2$, the following equalities hold*

$$\varphi_l(o_1) = \downarrow_2 (\varphi_r^*(\uparrow_1 (\chi_{o_1}))) \quad \text{and} \quad \varphi_r(a_2) = \uparrow_1 (\varphi_l^*(\downarrow_2 (\chi_{a_2})))$$

2 The category L-ChuCors

We introduce now the category of L-Chu correspondences between L-fuzzy formal contexts as follows:

- **objects** L-fuzzy formal contexts
- **arrows** L-Chu correspondences
- **composition** $\varphi_2 \circ \varphi_1 : C_1 \to C_3$ **of arrows** $\varphi_1 : C_1 \to C_2$, $\varphi_2 : C_2 \to C_3$ ($C_i = \langle B_i, A_i, r_i \rangle$, $i \in \{1, 2\}$)
  - $(\varphi_2 \circ \varphi_1)_l : B_1 \to L^{B_3}$ and $(\varphi_2 \circ \varphi_1)_r : A_3 \to L^{A_1}$
  - $(\varphi_2 \circ \varphi_1)_l(o_1) = \downarrow_3\uparrow_3 (\varphi_{2l*}(\varphi_{1l}(o_1)))$, where
  $$\varphi_{2l*}(\varphi_{1l}(o_1))(o_3) = \bigvee_{o_2 \in B_2} \varphi_{1l}(o_1)(o_2) \otimes \varphi_{2l}(o_2)(o_3)$$
  - $(\varphi_2 \circ \varphi_1)_r(a_3) = \uparrow_1\downarrow_1 (\varphi_{1r*}(\varphi_{2r}(a_3)))$, where
  $$\varphi_{1r*}(\varphi_{2r}(a_3))(a_1) = \bigvee_{a_2 \in A_2} \varphi_{2r}(a_3)(a_2) \otimes \varphi_{1r}(a_2)(a_1)$$

Theorem 1 *L-fuzzy Chu correspondences between L-fuzzy formal contexts form a category with the composition defined above.*

Proof: We just have to check the existence of identity arrows and the associativity of composition. The latter is just a matter of straightforward calculation, the identity arrows $\iota : C \to C$ are defined as follows for any given L-fuzzy context $C = \langle B, A, r \rangle$:

- $\iota_l(o) = \downarrow\uparrow (\chi_o)$, for all $o \in B$
- $\iota_r(a) = \uparrow\downarrow (\chi_a)$, for all $a \in A$. □

3 L-ChuCors embeds ChuCors

In the following paragraph, we sketchily argue that ChuCors can be embedded in any of the extensions L-ChuCors where L is a complete residuated lattice.

Assume that $\langle L_1, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ and $\langle L_2, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ are two complete residuated lattices, such that L_2 is a sublattice of L_1. Any L_2-fuzzy formal context $\langle B, A, r \rangle$ satisfies that $r \in L_2^{B \times A} \subseteq L_1^{B \times A}$. This inclusion implies that the class of all objects of L_2-ChuCors is a subclass of L_1-ChuCors. Moreover, every concept constructed in $\langle B, A, r \rangle$ by using the underlying logic provided by L_2 can be seen as well as a concept under the logic of L_1. As a result, the concept lattice L_2-$FCL(B, A, r)$ is a sublattice of the concept lattice L_1-$FCL(B, A, r)$.

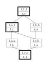
The following example illustrates the previous results on the light of two particular cases for L_i.

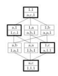
Example 1 *Consider L_1 and L_2 the lattices shown to the left of the picture below, together with the two L_2-formal contexts shown in the right.*

	a_{11}	a_{12}
o_{11}	*1*	*1*
o_{12}	a	c
o_{13}	*1*	c

	a_{21}	a_{22}	a_{23}
o_{21}	a	*1*	*1*
o_{22}	*1*	c	*1*

Consider two complete residuated lattice(s) to be consisting of the infimum on L_i, together with its residual implication defined as $k \to l = \bigvee\{m \in L \mid m \wedge k \leq l\}$, for all $k, l, m \in L_i$ where $i \in \{1, 2\}$. The concept lattices on the underlying logic of L_1 are shown in the pictures below, where the concepts in bold line are those in the frame associated to L_2.





The common L_2 and L_1-Chu correspondences are shown below:

φ_l	φ_r
1,c,c	*a,c,1*
1,c,c	*a,c,1*

,

φ_l	φ_r
1,c,c	*a,c,1*
1,a,1	*a,1,1*

The following result formally states the general relation between L_i-ChuCors.

Lemma 4 *Let C_1, C_2 be the L_2-contexts.* L_2-ChuCors$(C_1, C_2) \subseteq L_1$-ChuCors(C_1, C_2).

It is easy to see that the connection of two L_2-Chu correspondences make a new L_2-Chu correspondence. In addition, the set of L_2-Chu correspondences between two L_2-contexts is a subset of all L_1-Chu correspondences between the same contexts. L_2-Chu correspondences form a category, so the set of arrows is closed under the connections of arrows, as a result the set of L_1-Chu correspondences is closed under connections of L_2-Chu correspondences. Thus, we have just proved the following

Lemma 5 *Let C_i for $i \in \{1, 2, 3\}$ be two L_2-contexts. For every L_2-Chu correspondence $\varphi \in L_2$*-ChuCors(C_1, C_2) *and $\psi \in L_2$*-ChuCors(C_2, C_3) *holds $\psi \circ \varphi \in L_1$*-ChuCors$(C_1, C_3)$.

In consequence, we can state

Theorem 2 *Under the environment hypotheses of this section, the category* L_2-ChuCors *naturally embeds in* L_1-ChuCors*.*

As the category ChuCors of classical Chu correspondences are defined on classical, two-valued logic, which is a special case of any logic defined on complete residuated lattice, we obtain

Corollary 1 *The category* ChuCors *naturally embeds in* L-ChuCors

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