On the representation theorem of multi-adjoint concept lattices

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Abstract— *Formal concept analysis has become an important and appealing research topic. There exist a number of different fuzzy extensions of formal concept analysis and of its representation theorem, which gives conditions for a complete lattice in order to be isomorphic to a concept lattice. In this paper we concentrate on the study of operational properties of the mappings α and β required in the representation theorem.*

Keywords— Formal concept analysis, multi-adjoint framework.

1 Introduction

Formal concept analysis [12] has become an important and appealing research topic both from a theoretical perspective [18, 29, 32] and from the applicative one. Regarding applications, we can find papers ranging from ontology merging [10, 27], to applications to the Semantic Web by using the notion of concept similarity [11], and from processing of medical records in the clinical domain [14] to the development of recommender systems [8].

Soon after the introduction of "classical" formal concept analysis, a number of different approaches for its generalization were introduced and, nowadays, there are works which extend the theory with ideas from fuzzy set theory [3, 21, 22] or fuzzy logic reasoning [2, 4, 9] or from rough set theory [20, 30, 33] or some integrated approaches such as fuzzy and rough [31], or rough and domain theory [19].

In this paper we concentrate on the fuzzy extensions of formal concept analysis, for which a number of different approaches have been presented. To the best of our knowledge, the first one was given in [6], although they did not advance much beyond the basic definitions, probably due to the fact that they did not use residuated implications. Later, in [3, 28] the authors independently used complete residuated lattices as structures for the truth degrees; for this approach, a representation theorem was proved directly in a fuzzy framework in [5], setting the basis of most of the subsequent direct proofs.

In [23, 24] as a new general approach to formal concept analysis multi-adjoint concept lattices were introduced, in which the philosophy of the multi-adjoint paradigm [15, 26] to formal concept analysis is applied. With the idea of providing a general framework in which the different approaches stated above could be conveniently accommodated, the authors worked in a general non-commutative environment; and this naturally led to the consideration of adjoint triples, also called implication triples [1] or bi-residuated structures [25] as the main building blocks of a multi-adjoint concept lattice.

The representation (or fundamental) theorem gives conditions for a complete lattice in order to be isomorphic to a concept lattice. This theorem is proved in the classical case [12] and in the fuzzy paradigms [5, 13, 16, 23, 28]. As a consequence, to obtain the isomorphism it is necessary to search two mappings α and β that satisfy some properties, one of these relate the mappings with the relation. In this paper, we present a characterization of this last property, which is more efficient than the actual from the computationally point of view. Moreover, some other interesting properties of mappings α and β are introduced.

The structure of the paper is as follows: in Section 2 we recall the definition of the multi-adjoint concept lattices and, in particular, the mappings α and β required in the definition of lattice representing a multi-adjoint concept lattice. Then, in Section 3, we prove some new results concerning α and β. Finally, some concluding remarks are added.

2 Multi-adjoint concept lattices

In this section we will recall the more important definitions and results from [23]. The first definition introduces the basic building blocks of the multi-adjoint concept lattices, the *adjoint triples*, which are generalisations of the notion of adjoint pair under the hypothesis of having a non-commutative conjunctor.

The lack of commutativity of the conjunctor, directly provides two different ways of generalising the well-known adjoint property between a t-norm and its residuated implication, depending on which argument is fixed in the conjunction.

Definition 1 *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$ is an* adjoint triple *with respect to P_1, P_2, P_3 if:*

1. *$\&$ is order-preserving in both arguments.*
2. *$\swarrow$ and $\nwarrow$ are order-preserving in the consequent and order-reversing in the antecedent.*
3. *$x \leq_1 z \swarrow y$ iff $x \,\&\, y \leq_3 z$ iff $y \leq_2 z \nwarrow x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$.*

Note that in the domain and codomain of the considered conjunctor we have three (in principle) different sorts, thus providing a more flexible language to a potential user. Furthermore, notice that no boundary condition is required, in difference to the usual definition of multi-adjoint lattice [26] or implication triple [1]. Nevertheless, some boundary conditions follow from the definition, specifically, from the adjoint property (condition (3) above) [23].

Lemma 1 *If $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ have bottom element and $(\&, \swarrow, \nwarrow)$ is an adjoint triple, then $(P_1, \leq_1)$ and $(P_2, \leq_2)$ have top element and for all $x \in P_1, y \in P_2$ and $z \in P_3$ the following properties hold:*

1. $\bot_1 \& y = \bot_3, \qquad x \& \bot_2 = \bot_3.$
2. $z \nwarrow \bot_1 = \top_2, \qquad z \swarrow \bot_2 = \top_1.$

In order to provide more flexibility into our language, we will allow the existence of several adjoint triples for a given triplet of posets. Notice, however, that since these frames will be used as the underlying structures of our generalization of concept lattice, it is reasonable to require the lattice structure on some of the posets in the definition of adjoint triple.

Definition 2 *A* multi-adjoint frame *$\mathcal{L}$ is a tuple*

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are complete lattices, $(P, \leq)$ is a poset and, for all $i = 1, \dots, n$, $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P.

For short, a multi-adjoint frame will be denoted as $(L_1, L_2, P, \&_1, \dots, \&_n)$.

Following the usual approach to formal concept analysis, given a frame, a *multi-adjoint context* is a tuple consisting of sets of objects and attributes and a fuzzy relation among them; in addition, the multi-adjoint approach also includes a function which assigns an adjoint triple to each object (or attribute). This feature is important in that it allows for defining subgroups of objects or attributes in terms of different degrees of preference, see [23]. Formally, the definition is the following:

Definition 3 *Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, a* context *is a tuple (A, B, R, σ) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon B \to \{1, \dots, n\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.*[1]

Once we have fixed a multi-adjoint frame and a context for that frame, we can define the following mappings ${}^{\uparrow_\sigma}\colon L_2^B \longrightarrow L_1^A$ and ${}^{\downarrow^\sigma}\colon L_1^A \longrightarrow L_2^B$ which can be seen as generalisations of those given in [4, 17]:

$$g^{\uparrow_\sigma}(a) = \inf\{R(a,b) \swarrow^{\sigma(b)} g(b) \mid b \in B\} \qquad (1)$$
$$f^{\downarrow^\sigma}(b) = \inf\{R(a,b) \nwarrow_{\sigma(b)} f(a) \mid a \in A\} \qquad (2)$$

These two arrows, $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$, generate a Galois connection [23]. For the sake of self-containment, this concept is defined below:

Definition 4 *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^{\downarrow}\colon P_1 \to P_2$, ${}^{\uparrow}\colon P_2 \to P_1$ mappings, the pair $({}^{\uparrow}, {}^{\downarrow})$ forms a* Galois connection *between P_1 and P_2 whenever the following conditions hold:*

1. *${}^{\uparrow}$ and ${}^{\downarrow}$ are order-reversing.*
2. *$x \leq_1 x^{\downarrow\uparrow}$ for all $x \in P_1$.*
3. *$y \leq_2 y^{\uparrow\downarrow}$ for all $y \in P_2$.*

[1] A similar theory could be developed by considering a mapping $\tau\colon A \to \{1, \dots, n\}$ which associates any element in A with some particular adjoint triple in the frame.

Proposition 1 ([23]) *Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame and (A, B, R, σ) be a context, then the pair $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$ is a Galois connection between L_1^A and L_2^B.*

As usual in the different frameworks of formal concept analysis, a *multi-adjoint concept* is a pair $\langle g, f\rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^{\uparrow_\sigma} = f$ and $f^{\downarrow^\sigma} = g$; with $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$ being the Galois connection defined above.

Definition 5 *The* multi-adjoint concept lattice *associated to a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and a context (A, B, R, σ) is the set*

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\sigma} = f, f^{\downarrow^\sigma} = g\}$$

where the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ if and only if $g_1 \preceq_2 g_2$ (equivalently $f_2 \preceq_1 f_1$).

The ordering just defined above actually provides $\mathcal{M}$ with the structure of a complete lattice [23]. This follows from proposition 1 (the arrows $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$ forms a Galois connection) and the theorem below.

Theorem 1 ([7]) *Let $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ be complete lattices, let $({}^{\uparrow}, {}^{\downarrow})$ be a Galois connection between L_1, L_2 and consider $\mathcal{C} = \{\langle x, y\rangle \mid x^{\uparrow} = y, x = y^{\downarrow}; x \in L_1, y \in L_2\}$; then $(\mathcal{C}, \preceq)$ is a complete lattice, where*

$$\bigwedge_{i \in I}\langle x_i, y_i\rangle = \langle \bigwedge_{i \in I} x_i, (\bigvee_{i \in I} y_i)^{\downarrow\uparrow}\rangle;$$
$$\bigvee_{i \in I}\langle x_i, y_i\rangle = \langle (\bigvee_{i \in I} x_i)^{\uparrow\downarrow}, \bigwedge_{i \in I} y_i\rangle$$

and $\langle x_1, y_1\rangle \preceq \langle x_2, y_2\rangle$ if and only if $x_1 \preceq_1 x_2$.

From now on, we will fix a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and context (A, B, R, σ). Moreover, to improve readability, we will write $({}^{\uparrow}, {}^{\downarrow})$ instead of $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$ and $\swarrow^b$, $\nwarrow_b$ instead of $\swarrow^{\sigma(b)}$, $\nwarrow_{\sigma(b)}$.

In the next section, we will present some new properties about the functions α and β involved in the representation (or fundamental) theorem for the multi-adjoint framework presented in [23]. In order to do this, we will recall some necessary definitions.

Definition 6 *Given a complete lattice L, a subset $K \subseteq L$ is* infimum-dense *(resp.* supremum-dense*) if and only if for all $x \in L$ there exists $K' \subseteq K$ such that $x = \inf(K')$ (resp. $x = \sup(K')$).*

A multi-adjoint concept lattice is said to be represented by a complete lattice provided there is a pair of functions, α and β, satisfying the conditions stated in the definition below:

Definition 7 *A multi-adjoint concept lattice*[2] *$(\mathcal{M}, \preceq)$ is* represented *by a complete lattice $(V, \sqsubseteq)$ if there exists a pair of mappings $\alpha\colon A \times L_1 \to V$ and $\beta\colon B \times L_2 \to V$ such that:*

[2] Recall that we are considering a multi-adjoint concept lattice on a fixed frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and context (A, B, R, σ).

1a) $\alpha[A \times L_1]$ *is infimum-dense;*

1b) $\beta[B \times L_2]$ *is supremum-dense; and*

2) For all $a \in A$, $b \in B$, $x \in L_1$, $y \in L_2$:

$$\beta(b, y) \sqsubseteq \alpha(a, x) \quad \textit{if and only if} \quad x \,\&_b\, y \leq R(a, b)$$

From the definition of representability above the following properties follow:

Proposition 2 *Given a complete lattice* $(V, \sqsubseteq)$ *which represents a multi-adjoint concept lattice* $(\mathcal{M}, \preceq)$*, and mappings* $f \in L_1^A$ *and* $g \in L_2^B$*, we have:*

1. β *is order-preserving in the second argument.*
2. α *is order-reversing in the second argument.*
3. $g^{\uparrow}(a) = \sup\{x \in L_1 \mid v_g \sqsubseteq \alpha(a, x)\}$*, where* $v_g = \sup\{\beta(b, g(b)) \mid b \in B\}$.
4. $f^{\downarrow}(b) = \sup\{y \in L_2 \mid \beta(b, y) \sqsubseteq v_f\}$*, where* $v_f = \inf\{\alpha(a, f(a)) \mid a \in A\}$.
5. *If* $g_v(b) = \sup\{y \in L_2 \mid \beta(b, y) \sqsubseteq v\}$*, then* $\sup\{\beta(b, g_v(b)) \mid b \in B\} = v$.
6. *If* $f_v(a) = \sup\{x \in L_1 \mid v \sqsubseteq \alpha(a, x)\}$*, then* $\sup\{\alpha(a, f_v(a)) \mid a \in A\} = v$.

Finally, the fundamental theorem for multi-adjoint concept lattices presented in [23] is the following.

Theorem 2 *A complete lattice* $(V, \sqsubseteq)$ *represents a multi-adjoint concept lattice* $(\mathcal{M}, \preceq)$ *if and only if* $(V, \sqsubseteq)$ *is isomorphic to* $(\mathcal{M}, \preceq)$.

3 New results about the mappings α and β

In this section, we introduce some new interesting properties about the mappings α and β. So, let us assume a complete lattice $(V, \sqsubseteq)$ which represents a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ and the mappings $\alpha\colon A \times L_1 \to V$, $\beta\colon B \times L_2 \to V$.

We will restate below the isomorphism constructed in fundamental theorem, based on both the α and β functions, since these expressions will be used later.

Proposition 3 ([23]) *If a complete lattice* $(V, \sqsubseteq)$ *represents a multi-adjoint concept lattice* $(\mathcal{M}, \preceq)$*, then there exists an isomorphism* $\varphi\colon \mathcal{M} \to V$ *and two mappings* $\beta\colon B \times L_2 \to V$, $\alpha\colon A \times L_1 \to V$*, such that:*

$$\begin{aligned} \varphi(\langle g, f \rangle) &= \sup\{\beta(b, g(b)) \mid b \in B\} \\ &= \inf\{\alpha(a, f(a)) \mid a \in A\} \end{aligned}$$

for all concept $\langle g, f \rangle \in \mathcal{M}$.

The following result shows continuity-related properties of α and β in their second argument.

Proposition 4 *The mappings* $\beta\colon B \times L_2 \to V$ *and* $\alpha\colon A \times L_1 \to V$ *satisfy that:*

1. *For all indexed set* $Y = \{y_i\}_{i \in I} \subseteq L_2$ *and* $b \in B$:

$$\beta(b, \sup\{y_i \mid i \in I\}) = \sup\{\beta(b, y_i) \mid i \in I\}$$

2. *For all indexed set* $X = \{x_i\}_{i \in I} \subseteq L_1$ *and* $a \in A$:

$$\alpha(a, \sup\{x_i \mid i \in I\}) = \inf\{\alpha(a, x_i) \mid i \in I\}$$

Proof: 1. Consider $b \in B$ and $Y = \{y_i\}_{i \in I} \subseteq L_2$, as $\alpha[A \times L_1]$ is infimum-dense and $\beta(b, \sup Y) \in V$, there exists an indexing set Λ such that $\beta(b, \sup Y) = \inf\{\alpha(a_j, x_j) \mid j \in \Lambda\}$; as a result $\beta(b, \sup Y) \sqsubseteq \alpha(a_j, x_j)$ for every $j \in \Lambda$. From proposition 2(1), we obtain that $\beta(b, y_i) \sqsubseteq \alpha(a_j, x_j)$, for every $i \in I$ and $j \in \Lambda$, and hence $\sup\{\beta(b, y_i) \mid i \in I\} \sqsubseteq \alpha(a_j, x_j)$ for every $j \in \Lambda$, then

$$\begin{aligned} \sup\{\beta(b, y_i) \mid i \in I\} &\sqsubseteq \inf\{\alpha(a_j, x_j) \mid j \in \Lambda\} \\ &= \beta(b, \sup Y) \end{aligned}$$

For the other inequality, let us consider $\sup\{\beta(b, y_i) \mid i \in I\}$ and, as $\alpha[A \times L_1]$ is infimum-dense, there exists an indexing set Λ' such that $\sup\{\beta(b, y_i) \mid i \in I\} = \inf\{\alpha(a_j, x_j) \mid j \in \Lambda'\}$. Now, for all $i \in I$ and $j \in \Lambda'$ we obtain that $\beta(b, y_i) \sqsubseteq \alpha(a_j, x_j)$, therefore, from Definition 7(2), $x_j \,\&_b\, y_i \leq R(a_j, b)$. Now, as $(\&_b, \swarrow^b, \nwarrow_b)$ is an adjoint triple, we have the following chain of equivalent statements:

$$\begin{aligned} x_j \,\&_b\, y_i \leq R(a_j, b) &\quad \text{for all } i \in I \\ y_i \preceq_2 R(a_j, b) \nwarrow_b x_j &\quad \text{for all } i \in I \\ \sup Y \preceq_2 R(a_j, b) \nwarrow_b x_j & \\ x_j \,\&_b \sup Y \leq R(a_j, b) & \end{aligned}$$

so, $\beta(b, \sup Y) \sqsubseteq \alpha(a_j, x_j)$ for every $j \in \Lambda'$, and thus

$$\begin{aligned} \beta(b, \sup Y) &\sqsubseteq \inf\{\alpha(a_j, x_j) \mid j \in \Lambda'\} \\ &= \sup\{\beta(b, y_i) \mid i \in I\} \end{aligned}$$

2. This proof is analogous using that $\beta[B \times L_2]$ is supremum-dense. □

We continue below by proving some boundary conditions fulfilled by α and β.

Proposition 5 *The two mappings* $\alpha\colon A \times L_1 \to V$ *and* $\beta\colon B \times L_2 \to V$ *are such that* $\alpha(a, \bot_1) = \top_V$ *and* $\beta(b, \bot_2) = \bot_V$ *for all* $b \in B$ *and* $a \in A$.

Proof: Given $a \in A$, let us prove that $\alpha(a, \bot_1) = \top_V$. Firstly, recall that lemma 1 implies that $\bot_1 \,\&_b\, y \leq R(a, b)$ for all $b \in B$ and $y \in L_2$; now, from Definition 7(2) we obtain that $\beta(b, y) \sqsubseteq \alpha(a, \bot_1)$ for all $b \in B$ and $y \in L_2$, that is, $\alpha(a, \bot_1)$ is an upper bound of the set of elements $\beta(b, y)$ for all $b \in B$ and $y \in L_2$. Now, as β is supremum-dense, there is an indexing set Λ such that $\top_V = \sup\{\beta(b_i, y_i) \mid i \in \Lambda\}$, therefore, we have that: $\top_V \sqsubseteq \alpha(a, \bot_1)$. Hence, $\top_V = \alpha(a, \bot_1)$.

The other equality follows similarly. □

From the propositions above, we obtain the following corollary which states the behaviour of α and β regarding suprema of any set (either empty or non-empty).

Corollary 1 *The mappings* $\beta\colon B \times L_2 \to V$, $\alpha\colon A \times L_1 \to V$ *satisfy that:*

1. *$\beta(b, \sup Y) = \sup\{\beta(b, y) \mid y \in Y\}$, for all $Y \subseteq L_2$ and $b \in B$.*

2. *$\alpha(a, \sup X) = \inf\{\alpha(a, x) \mid x \in X\}$, for all $X \subseteq L_1$ and $a \in A$.*

As a consequence of the property above we have the following result, which gives us a more efficient form to write Property (2) in Definition 7 to check if a lattice is isomorphic to a concept lattice, that is, in order to apply Theorem 2.

Proposition 6 *Given $a \in A$, $b \in B$, the applications $\beta_b\colon L_2 \to V$, $\alpha_a\colon L_1 \to V$ have residuated mappings, that is, there exist $\beta'_b\colon L_2 \to V$, $\alpha'_a\colon L_1 \to V$ such that:*

$$\begin{array}{rcl} \beta_b(y) \sqsubseteq v & \textit{if and only if} & y \preceq_2 \beta'_b(v) \\ v \sqsubseteq \alpha_a(x) & \textit{if and only if} & x \preceq_1 \alpha'_a(v) \end{array}$$

for all $x \in L_1$, $y \in L_2$ and $v \in V$.

Proof: If we define $\beta'_b(v) = \sup\{y \in L_2 \mid \beta_b(y) \sqsubseteq v\}$ and, similarly, $\alpha'_a(v) = \sup\{x \in L_1 \mid v \sqsubseteq \alpha_a(x)\}$, we obtain the result straightforward from Corollary 1. □

The following proposition states a necessary and sufficient condition for the mappings α and β to fulfill the second condition in the definition of representable lattice.

Proposition 7 *The mappings α and β satisfy Property (2) in Definition 7 if and only if, for all $a \in A$, $b \in B$, $x \in L_1$, $y \in L_2$, (some of QUITAR) the following equalities hold:*

$$\begin{array}{rcl} \beta'_b(\alpha_a(x)) & = & R(a,b) \nwarrow x \\ \alpha_a(\beta'_b(y)) & = & R(a,b) \swarrow y \end{array}$$

Proof: Firstly, we assume that the mappings α and β satisfy Property (2) in Definition 7. The first equality is given from the following chain of equivalences, given $a \in A$, $b \in B$, $x \in L_1$, $y \in L_2$:

$$\begin{array}{rcl} y \preceq_2 \beta'_b(\alpha_a(x)) & \iff & \beta(b, y) \sqsubseteq \alpha(a, x) \\ & \iff & x \,\&_b\, y \leq R(a,b) \\ & \iff & y \preceq_2 R(a,b) \nwarrow x \end{array}$$

if we substitute y by $R(a,b) \nwarrow x$ in the first sentence and y by $\beta'_b(\alpha_a(x))$ in the last one. The second equality follows similarly.

Now, we assume that $\beta'_b(\alpha_a(x)) = R(a,b) \nwarrow x$, hence

$$\begin{array}{rcl} \beta(b, y) \sqsubseteq \alpha(a, x) & \iff & y \preceq_2 \beta'_b(\alpha_a(x)) \\ & \stackrel{(*)}{\iff} & y \preceq_2 R(a,b) \nwarrow x \\ & \iff & x \,\&_b\, y \leq R(a,b) \end{array}$$

where $(*)$ is given from the hypothesis. □

As a result of the previous proposition, we obtain a straightforward mechanism to obtain the mappings α and β in order to check whether a lattice is isomorphic to a concept lattice.

Finally, the following property shows that any subset of $A \times L_1$ or of $B \times L_2$ is related to a concept via α and φ, or β and φ, respectively.

Proposition 8 *Consider a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ represented by a complete lattice $(V, \sqsubseteq)$ and the mappings $\alpha\colon A \times L_1 \to V$, $\beta\colon B \times L_2 \to V$, then for each $K \subseteq A \times L_1$, there exists a unique concept $\langle g, f\rangle \in \mathcal{M}$ such that*

$$\inf\{\alpha(a, x) \mid (a, x) \in K\} \quad = \quad \varphi(\langle g, f\rangle)$$

Analogously, for each $K' \subseteq B \times L_2$, there exists a unique concept $\langle g, f\rangle \in \mathcal{M}$ such that

$$\sup\{\beta(b, y) \mid (b, y) \in K'\} \quad = \quad \varphi(\langle g, f\rangle)$$

Proof: Given $K \subseteq A \times L_1$, let us consider the sets $K_a = \{x \mid (a, x) \in K\}$, and the function $h\colon A \to L_1$ defined as $h(a) = \sup K_a$.

By Corollary 1, we have that, for all $a' \in A$, the following equality holds

$$\alpha(a', h(a')) = \inf\{\alpha(a', x) \mid x \in K_a\}$$

Therefore:

$$\begin{array}{rcl} \multicolumn{3}{l}{\inf\{\alpha(a', h(a')) \mid a' \in A\} =} \\ & = & \inf\{\inf\{\alpha(a', x) \mid x \in K_{a'}\} \mid a' \in A\} \\ & = & \inf\{\alpha(a', x) \mid (a', x) \in K\} \end{array}$$

Finally, we obtain the following chain of equalities:

$$\begin{array}{rcl} \inf\{\alpha(a, x) \mid (a, x) \in K\} & = & \inf\{\alpha(a', h(a')) \mid a' \in A\} \\ & \stackrel{(1)}{=} & \sup\{\beta(b', h^{\downarrow}(b')) \mid b' \in B\} \\ & \stackrel{(2)}{=} & \varphi(\langle h^{\downarrow}, h^{\downarrow\uparrow}\rangle) \end{array}$$

where (1) follows by Proposition 2, (2) by Proposition 3. This means that the concept whose existence is postulated in the statement is $\langle h^{\downarrow}, h^{\downarrow\uparrow}\rangle$.

Now, the uniqueness follows from the isomorphism φ:
If there would exist another concept $\langle g, f\rangle$ such that $\sup\{\beta(b, g(b)) \mid b \in B\} = \inf\{\alpha(a, x) \mid (a, x) \in K\}$, we would have:

$$\begin{array}{rcl} \varphi(\langle h^{\downarrow}, h^{\downarrow\uparrow}\rangle) & = & \sup\{\beta(b, h^{\downarrow}(b)) \mid b \in B\} \\ & = & \inf\{\alpha(a, x) \mid (a, x) \in K\} \\ & = & \sup\{\beta(b, g(b)) \mid b \in B\} \\ & = & \varphi(\langle g, f\rangle) \end{array}$$

Thus, $\langle h^{\downarrow}, h^{\downarrow\uparrow}\rangle = \langle g, f\rangle$.

The second statement follows similarly. □

4 Conclusions

The representation theorem is one of the most important results in the theory of formal concept analysis, since it provides conditions in order to determine whether a given lattice is isomorphic to some concept lattice. In this paper, an analytic expression for the mappings α and β involved in the representation theorem of t-concept lattices is provided, together with some interesting properties.

References

[1] A. Abdel-Hamid and N. Morsi. Associatively tied implicacions. *Fuzzy Sets and Systems*, 136(3):291–311, 2003.

[2] C. Alcalde, A. Burusco, R. Fuentes-González, and I. Zubia. Treatment of L-fuzzy contexts with absent values. *Information Sciences*, 179:1–15, 2009.

[3] R. Belohlávek. Lattice generated by binary fuzzy relations (extended abstract). In *4th Intl. Conf. on Fuzzy Sets Theory and Applications*, page page 11, 1998.

[4] R. Belohlávek. Concept lattices and order in fuzzy logic. *Annals of Pure and Applied Logic*, 128:277–298, 2004.

[5] R. Belohlávek. Lattices of fixed points of fuzzy Galois connections. *Mathematical Logic Quartely*, 47(1):111–116, 2004.

[6] A. Burusco and R. Fuentes-González. The study of L-fuzzy concept lattice. *Mathware & Soft Computing*, 3:209–218, 1994.

[7] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, second edition, 2002.

[8] P. du Boucher-Ryana and D. Bridge. Collaborative recommending using formal concept analysis. *Knowledge-Based Systems*, 19(5):309–315, 2006.

[9] S.-Q. Fan, W.-X. Zhang, and W. Xu. Fuzzy inference based on fuzzy concept lattice. *Fuzzy Sets and Systems*, 157(24):3177–3187, 2006.

[10] A. Formica. Ontology-based concept similarity in formal concept analysis. *Information Sciences*, 176(18):2624–2641, 2006.

[11] A. Formica. Concept similarity in formal concept analysis: An information content approach. *Knowledge-Based Systems*, 21(1):80–87, 2008.

[12] B. Ganter and R. Wille. *Formal Concept Analysis: Mathematical Foundation*. Springer Verlag, 1999.

[13] G. Georgescu and A. Popescu. Concept lattices and similarity in non-commutative fuzzy logic. *Fundamenta Informaticae*, 53(1):23–54, 2002.

[14] G. Jiang, K. Ogasawara, A. Endoh, and T. Sakurai. Context-based ontology building support in clinical domains using formal concept analysis. *International Journal of Medical Informatics*, 71(1):71–81, 2003.

[15] P. Julian, G. Moreno, and J. Penabad. On fuzzy unfolding: A multi-adjoint approach. *Fuzzy Sets and Systems*, 154(1):16–33, 2005.

[16] S. Krajci. The basic theorem on generalized concept lattice. In V. Snásel and R. Belohlávek, editors, *International Workshop on Concept Lattices and their Applications, CLA 2004*, pages 25–33, 2004.

[17] S. Krajci. A generalized concept lattice. *Logic Journal of IGPL*, 13(5):543–550, 2005.

[18] S. O. Kuznetsov. Complexity of learning in concept lattices from positive and negative examples. *Discrete Applied Mathematics*, 142:111–125, 2004.

[19] Y. Lei and M. Luo. Rough concept lattices and domains. *Annals of Pure and Applied Logic*, 2009. Article in press (http://dx.doi.org/10.1016/j.apal.2008.09.028).

[20] M. Liu, M. Shao, W. Zhang, and C. Wu. Reduction method for concept lattices based on rough set theory and its application. *Computers & Mathematics with Applications*, 53(9):1390–1410, 2007.

[21] X. Liu, W. Wang, T. Chai, and W. Liu. Approaches to the representations and logic operations of fuzzy concepts in the framework of axiomatic fuzzy set theory I. *Information Sciences*, 177(4):1007–1026, 2007.

[22] X. Liu, W. Wang, T. Chai, and W. Liu. Approaches to the representations and logic operations of fuzzy concepts in the framework of axiomatic fuzzy set theory II. *Information Sciences*, 177(4):1027–1045, 2007.

[23] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.

[24] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. On multi-adjoint concept lattices: denition and representation theorem. *Lect. Notes in Artificial Intelligence*, 4390:197–209, 2007.

[25] J. Medina, M. Ojeda-Aciego, A. Valverde, and P. Vojtáš. Towards biresiduated multi-adjoint logic programming. *Lect. Notes in Artificial Intelligence*, 3040:608–617, 2004.

[26] J. Medina, M. Ojeda-Aciego, and P. Vojtáš. Similarity-based unification: a multi-adjoint approach. *Fuzzy Sets and Systems*, 146:43–62, 2004.

[27] V. Phan-Luong. A framework for integrating information sources under lattice structure. *Information Fusion*, 9:278–292, 2008.

[28] S. Pollandt. *Fuzzy Begriffe*. Springer, Berlin, 1997.

[29] K.-S. Qu and Y.-H. Zhai. Generating complete set of implications for formal contexts. *Knowledge-Based Systems*, 21:429–433, 2008.

[30] M.-W. Shao, M. Liu, and W.-X. Zhang. Set approximations in fuzzy formal concept analysis. *Fuzzy Sets and Systems*, 158(23):2627–2640, 2007.

[31] L. Wang and X. Liu. Concept analysis via rough set and afs algebra. *Information Sciences*, 178(21):4125–4137, 2008.

[32] X. Wang and W. Zhang. Relations of attribute reduction between object and property oriented concept lattices. *Knowledge-Based Systems*, 21(5):398–403, 2008.

[33] Q. Wu and Z. Liu. Real formal concept analysis based on grey-rough set theory. *Knowledge-Based Systems*, 22(1):38–45, 2009.