Formal concept analysis via multi-adjoint concept lattices

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Abstract

Several fuzzifications of formal concept analysis have been proposed to deal with uncertain information. In this paper, we focus on concept lattices under a multi-adjoint paradigm, which enriches the language providing greater flexibility to the user in that she can choose from a number of different connectives. Multi-adjoint concept lattices are shown to embed different fuzzy extensions of concept lattices found in the literature, the main results of the paper being the representation theorem of this paradigm and the embedding of other well-known approaches.

Keywords: concept lattices, multi-adjoint lattices, Galois connection, implication triples.

1 Introduction

Handling uncertainty, imprecise data or incomplete information has become an important research topic in the recent years. Developing reasoning methods under this kind of, so to say, 'imperfect' information is more a must than a simply need; just consider the enormous amount of information available in the web. Most of the current research areas have received this message, one frequent solution being to develop *fuzzified* versions of several well-known standard structures. In this paper, we focus on the area of formal concept analysis and, specifically, on the different generalisations of the classical definition of concept lattice to the fuzzy case.

A number of different approaches have been proposed which generalise the classical concept lattices given by Ganter and Wille [10, 24] by allowing some

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uncertainty in data. One of these approaches was proposed by Burusco and Fuentes-González [7] where fuzzy concept lattices were first presented, although they did not use residuated implications in their work and this is the reason why they could not advance much beyond the basic definitions.

Another approach was proposed independently by Pollandt [23] and Bělohlávek [2] who use complete residuated lattices as structures for the truth degrees; for this approach, a representation theorem was proved directly in a fuzzy framework in [3], setting the basis of most of the subsequent direct proofs. Bělohlávek, in [5], later extended this to the case when a fuzzy partial order is considered on a fuzzy concept lattice instead of on an ordinary partial order. Georgescu and Popescu extended this framework to non-commutative logic and similarity in a series of papers [11–14]; in a different direction, it was also extended in an asymmetric way, although only for the case of classical equality ($L = \{0, 1\}$), by Krajči, which introduced the so-called generalised concept lattices in [17, 18].

In the context of general logical frameworks, the recently introduced multi-adjoint approach is receiving considerable attention [16, 21]. The multi-adjoint framework originated as a generalisation of several non-classical logic programming frameworks whose semantic structure is the multi-adjoint lattice, in which a lattice is considered together with several conjunctors and implications making up adjoint pairs. The particular details of the different approaches were abstracted away, retaining only the minimal mathematical requirements guaranteeing operability. In particular, conjunctors were required to be neither commutative nor associative.

A new general approach to formal concept analysis has been recently proposed in [19, 22] where the multi-adjoint concept lattices were introduced, applying the philosophy of the multi-adjoint framework to formal concept analysis. Non-commutative conjunctors have been used in topics such as fuzzy concept lattices and fuzzy logic programming [11, 12, 20], and have been studied on their own, for instance in [1]. In this paper, we focus on non-commutative conjunctors and on the consequences that its use generates in the setting of formal concept analysis.

With the idea of providing a general framework in which the different approaches stated above could be conveniently accommodated, the authors worked in a general non-commutative environment; and this naturally leads to the consideration of adjoint triples, also called implication triples [1] or bi-residuated structures [20] as the main building blocks of a multi-adjoint concept lattice.

The main result introduced in this paper, apart from the introduction of multi-adjoint concept lattices, is the representation theorem, which gives conditions for a complete lattice in order to be isomorphic to a multi-adjoint concept lattice. The proof of this theorem follows the line of that given in [3] but the presentation is given in a more structured and readable way. In addition, we also show the embedding of several of the paradigms stated above into the multi-adjoint concept lattice framework. The paper finishes with a detailed example on which all the capabilities of the proposed framework are shown.

2 Multi-adjoint concept lattices

The basic building blocks of the multi-adjoint concept lattices are the adjoint triples, which are generalisations of the notion of adjoint pair under the hypothesis of non-commutative conjunctors.

Before presenting the formal definition, let us recall the notion of adjoint pair:

An *adjoint pair* on a poset $(P, \leq)$ is a pair of binary operations in P $(\&, \leftarrow)$ such that:

(1) Operation $\&$ is order-preserving in both arguments;
(2) Operation $\leftarrow$ is order-preserving in the first argument (the consequent) and order-reversing in the second argument (the antecedent);
(3) For any $x, y, z \in P$, we have that

$$x \leq (z \leftarrow y) \qquad \text{if and only if} \qquad (x \,\&\, y) \leq z$$

This last property is related to the fuzzy *modus ponens* rule, see [15], in that it can be recovered from natural requirements on the fuzzy MP.

The lack of commutativity of the conjunctor directly provides two different ways of generalising the adjoint property above, depending on which argument of the conjunction is fixed. This would lead to two different implications $\swarrow$ and $\nwarrow$ satisfying the following chain of equivalences:

$$x \leq z \swarrow y \qquad \text{iff} \qquad x \,\&\, y \leq z \qquad \text{iff} \qquad y \leq z \nwarrow x$$

Furthermore, we can be even more general and consider conjunctors whose domains are formed by different sorts, thus providing a more flexible language to a potential user. This leads to the definition of adjoint triple given below:

Definition 1 *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$, is an* adjoint triple *with respect to P_1, P_2, P_3 if:*

- $\&$ *is order-preserving in both arguments.*
- $\swarrow$ *and* $\nwarrow$ *are order-preserving in the consequent and order-reversing in the antecedent.*

- $x \leq_1 z \swarrow y$ *iff* $x \,\&\, y \leq_3 z$ *iff* $y \leq_2 z \nwarrow x$, *where* $x \in P_1$, $y \in P_2$ *and* $z \in P_3$.

The last property, which will be referred to as the *adjoint property*, can be seen as related to the fuzzy *modus ponens* rule for non-necessarily commutative conjunctors. Notice that no boundary condition is required, in difference to the usual definition of multi-adjoint lattice [21] or implication triples [1].

Some interesting consequences which will be used later, and whose proof is straightforward from the adjoint property, are stated in the following lemma.

Lemma 2 *If* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *have bottom element,* $(P_1, \leq_1)$ *and* $(P_2, \leq_2)$ *have top element, and* $(\&, \swarrow, \nwarrow)$ *is an adjoint triple, then for all* $x \in P_1, y \in P_2$ *and* $z \in P_3$ *the following properties hold:*

(1) $\perp_1 \& \, y = \perp_3$, $\quad x \,\&\, \perp_2 = \perp_3$.
(2) $z \nwarrow \perp_1 = \top_2$, $\quad z \swarrow \perp_2 = \top_1$.

It is worth to note that the occurrence of non-commutative or non-associative connectives is not completely unusual. For instance, consider that a variable represented by x can be observed with $m+1$ different values, then surely we should be working with a regular partition of $[0,1]$ into m pieces, denoted $[0,1]_m$. This means that a given value x should be fitted to this "observation" scale as the least upper bound with the form k/m (analytically, this corresponds to $\lceil m \cdot x \rceil / m$ where $\lceil \_ \rceil$ is the ceiling function). A similar consideration can be applied to both, variable y and the resulting conjunction; furthermore, it might be possible that each variable has different granularity. Formally, assume in x-axis we have a partition into n pieces, in y-axis into m pieces, and in z-axis into k pieces. Then the approximation of the product conjunction is given in the following example.

Example 3 *Given positive integers* $n, m, k > 0$*, let us consider the mapping* $C^k_{n,m} \colon [0,1]_n \times [0,1]_m \to [0,1]_k$*, defined for each* $x \in [0,1]_n$ *and* $y \in [0,1]_m$ *as:*

$$C^k_{n,m}(x,y) = \frac{\lceil k \cdot x \cdot y \rceil}{k} \qquad \text{where } \cdot \text{ denotes the usual product of real numbers}$$

There are connectives of the form $C^k_{n,m}$ *which are non-associative and there are connectives of the same form which are non-commutative as well, for example* $C^{10}_{10,10}$ *and* $C^4_{10,5}$ *as it is shown in [20].*

Note that $C^k_{n,m}$ *is order-preserving in both variables and generalises the classical conjunction. Now, if we define implications* $\swarrow^k_{n,m} \colon [0,1]_k \times [0,1]_m \to [0,1]_n$ *and* $\nwarrow^k_{n,m} \colon [0,1]_k \times [0,1]_n \to [0,1]_m$ *as follows:*

$$z \swarrow^k_{n,m} y = \max\{x \in [0,1]_n \mid C^k_{n,m}(x,y) \leq z\}$$
$$z \nwarrow^k_{n,m} x = \max\{y \in [0,1]_m \mid C^k_{n,m}(x,y) \leq z\}$$

then $(C^k_{n,m}, \swarrow^k_{n,m}, \nwarrow^k_{n,m})$ *is an adjoint triple, as stated in [20].*

Connectives as those in the example above can be reasonably justified as follows: If we are looking for a hotel which is close to downtown, with reasonable price and being a new building, then classical fuzzy approaches would assign a user "his" particular interpretation of "close", "reasonable" and "new". As, in practice, we can only recognize finitely many degrees of being close, reasonable, new, then the corresponding fuzzy sets have a stepwise shape. This motivates the lattice-valued approach we will assume in this paper: it is just a matter of representation that the outcome is done by means of intervals of granulation and/or indistinguishability.

Similarly to introducing several adjoint pairs in order to form a multi-adjoint lattice, we will consider several adjoint triples to introduce the notion of *multi-adjoint frame.*

Definition 4 *A* multi-adjoint frame $\mathcal{L}$ *is a tuple*

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

where $(L_i, \preceq_i)$ *are complete lattices,* $(P, \leq)$ *is a poset, and* $(\&_i, \swarrow^i, \nwarrow_i)$ *is an adjoint triple with respect to* L_1, L_2, P *for all* $i = 1, \dots, n$.

A multi-adjoint frame as above will be denoted as $(L_1, L_2, P, \&_1, \dots, \&_n)$, for short. It is convenient to note that, in principle, L_1, L_2 and P could be simply posets, the reason to consider complete lattices is that multi-adjoint frames will be used as the underlying lattice on which the operations will be made; hence, general joins and meets are required.

Given a frame, the notion of context is defined as a tuple consisting of sets of objects and attributes, a fuzzy relation among them and a function assigning an adjoint triple to each object (or attribute). Formally, the definition is the following:

Definition 5 *Given a multi-adjoint frame* $(L_1, L_2, P, \&_1, \dots, \&_n)$*, a* context *is a tuple* (A, B, R, σ) *such that* A *and* B *are non-empty sets (usually interpreted as objects and attributes),* R *is a* P*-fuzzy relation* $R\colon A \times B \to P$ *and* $\sigma\colon B \to \{1, \dots, n\}$ *is a mapping which associates any element in* B *with some particular adjoint triple in the frame.* [2]

[2] A similar theory could be developed by considering a mapping $\tau\colon A \to \{1, \dots, n\}$ which associates any element in A with some particular adjoint triple in the frame.

The fact that in a multi-adjoint context each object (or attribute) has an associated implication is interesting in that subgroups with different degrees of preference can be established in a convenient way, see the example in Section 5.

Now, given a multi-adjoint frame and a context for that frame, we can define the following mappings ${}^{\uparrow_\sigma}\colon L_2^B \longrightarrow L_1^A$ and ${}^{\downarrow^\sigma}\colon L_1^A \longrightarrow L_2^B$ which can be seen as generalisations of those given in [5, 18]:

$$
\begin{aligned}
g^{\uparrow_\sigma}(a) &= \inf\{R(a,b) \swarrow^{\sigma(b)} g(b) \mid b \in B\}\\
f^{\downarrow^\sigma}(b) &= \inf\{R(a,b) \nwarrow_{\sigma(b)} f(a) \mid a \in A\}
\end{aligned}
$$

It is worth to point out that these mappings generate a Galois connection. For sake of self-containment, this concept is defined below:

Definition 6 *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^\downarrow\colon P_1 \to P_2$, ${}^\uparrow\colon P_2 \to P_1$ mappings, the pair $({}^\uparrow, {}^\downarrow)$ forms a* Galois connection *between P_1 and P_2 if and only if:*

(1) ${}^\uparrow$ and ${}^\downarrow$ are order-reversing.
(2) $x \leq_1 x^{\downarrow\uparrow}$ for all $x \in P_1$.
(3) $y \leq_2 y^{\uparrow\downarrow}$ for all $y \in P_2$.

Proposition 7 *Given a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and a context (A, B, R, σ), the pair $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$ is a Galois connection between L_1^A and L_2^B.*

PROOF. From now on, to improve readability, we will write $({}^\uparrow, {}^\downarrow)$ instead of $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$ and $\swarrow^b$, $\nwarrow_b$ instead of $\swarrow^{\sigma(b)}$, $\nwarrow_{\sigma(b)}$.

By definition, we have to prove that:

(1) ${}^\uparrow$ and ${}^\downarrow$ are order-reversing. This is trivial since the implications are order-reversing in the second argument.
(2) $g \leq g^{\uparrow\downarrow}$ for all $g \in L_2^B$. Given $a \in A$ and $b \in B$ the next chain of inequalities holds because of the definition of $g^\uparrow(a)$ as an infimum and the adjoint property:

$$
\begin{aligned}
g^\uparrow(a) \preceq_1 R(a,b) \swarrow^b g(b) \quad &\iff \quad g^\uparrow(a) \,\&_b\, g(b) \leq R(a,b)\\
&\iff \quad g(b) \preceq_2 R(a,b) \nwarrow_b g^\uparrow(a)
\end{aligned}
$$

As these inequalities hold for all $a \in A$, by applying properties of the infimum we obtain

$$
g(b) \preceq_2 \inf\{R(a,b) \nwarrow_b g^\uparrow(a) \mid a \in A\} = g^{\uparrow\downarrow}(b)
$$

(3) $f \leq f^{\downarrow\uparrow}$ for all $f \in L_1^A$. The proof is similar. □

Now, we are in a position to define what a *concept* in our framework is. A concept is a pair $\langle g, f\rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^\uparrow = f$ and $f^\downarrow = g$; with $(^\uparrow, ^\downarrow)$ being the Galois connection defined above.

Definition 8 *The* multi-adjoint concept lattice *associated to a multi-adjoint frame* $(L_1, L_2, P, \&_1, \ldots, \&_n)$ *and a context* (A, B, R, σ) *is the set*

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^\uparrow = f, f^\downarrow = g\}$$

in which the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ *if and only if* $g_1 \preceq_2 g_2$ *(equivalently* $f_2 \preceq_1 f_1$*).*

We have just defined a poset of concepts, but recall that Proposition 7 proved that the pair of arrows $(^\uparrow, ^\downarrow)$ forms a Galois connection between the complete lattices L_1^A and L_2^B, hence the poset $(\mathcal{M}, \preceq)$ defined above is a complete lattice by the theorem below.

Theorem 9 (See [9]) *Let* $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ *be complete lattices,* $(^\uparrow, ^\downarrow)$ *a Galois connection between* L_1, L_2 *and* $\mathcal{C} = \{\langle x, y\rangle \mid x^\uparrow = y, x = y^\downarrow; x \in L_1, y \in L_2\}$ *then* $\mathcal{C}$ *is a complete lattice, where*

$$\bigwedge_{i\in I}\langle x_i, y_i\rangle = \langle \bigwedge_{i\in I} x_i, (\bigvee_{i\in I} y_i)^{\downarrow\uparrow}\rangle \quad \text{and} \quad \bigvee_{i\in I}\langle x_i, y_i\rangle = \langle (\bigvee_{i\in I} x_i)^{\uparrow\downarrow}, \bigwedge_{i\in I} y_i\rangle$$

It is convenient to note that this well-known theorem can prevent the development of *ad hoc* proofs, as that in [17], of the complete lattice structure, by a simple checking the existence of the Galois connection.

3 Comparison with other approaches

In this section we will consider $(\mathcal{M}, \preceq)$ to be the multi-adjoint concept lattice associated to fixed multi-adjoint frames and contexts.

3.1 Pollandt concept lattices

In this section we will show how Pollandt's first approach to concept lattices can be embedded into the framework of multi-adjoint concept lattices.

The following description of Pollandt's original approach is taken from [8]. Let us consider the unit interval $[0, 1]$ with its usual ordering as the underlying lattice, together with two sets A and B representing the sets of attributes and objects respectively. Then the fuzzy subsets on A and B are considered, that is the sets $[0,1]^A$ and $[0,1]^B$ and, finally, a fuzzy relation $R \in [0,1]^{A\times B}$; so the context used is the following $([0,1], A, B, R)$.

Then, a pair of mappings ${}^{\uparrow}\colon [0,1]^B \to [0,1]^A$ and ${}^{\downarrow}\colon [0,1]^A \to [0,1]^B$ is defined as follows:

$$g^{\uparrow}(a) = \inf_{b\in B}\{R(a,b) \leftarrow^L g(b)\}\ , \qquad f^{\downarrow}(b) = \inf_{a\in A}\{R(a,b) \leftarrow^L f(a)\}$$

where $\leftarrow^L$ is the Łukasiewicz implication, i. e., $y \leftarrow^L x = \min\{1, 1-x+y\}$. Pollandt shows that the set of fixed points of the composition φ of these two mappings, defined as $\varphi(g) = g^{\uparrow\downarrow}$, forms a complete lattice. Thus, a *concept* is defined as a pair $\langle g, f\rangle$ such that $g^{\uparrow} = f$ and $f^{\downarrow} = g$, that is, the fixed points of φ, and the concept lattice is the set $\mathcal{C} = \{\langle g, f\rangle \mid g^{\uparrow} = f \text{ and } f^{\downarrow} = g\}$.

Now we will show how this concept lattice can be embedded into the multi-adjoint framework. For the Łukasiewicz implication $\leftarrow^L$ it is well-known that the pair $(\&_L, \leftarrow^L)$ forms an adjoint pair, where $\&_L$ is the Łukasiewicz conjunction $x \,\&_L\, y = \max(0, x+y-1)$. Moreover, as $\&_L$ is obviously commutative we have that considering $\nwarrow_L \;=\; \leftarrow^L \;=\; \swarrow^L$, then $(\&_L, \swarrow^L, \nwarrow_L)$ is an adjoint triple.

Pollandt's fuzzy concept lattice $\mathcal{C}$ can be seen as a multi-adjoint concept lattice just considering the frame $([0,1],[0,1],[0,1],\le,\le,\le,\&_L,\swarrow^L,\nwarrow_L)$ and context (A, B, R, σ) where σ associates to each object the unique conjunctor $\&_L$, i.e., $\sigma(b) = \&_L$ for every $b \in B$.

It is worth to note that Pollandt generalized her approach in [23] to a general framework by considering a complete residuated lattice instead of the unit interval; such a generalization was independently obtained by Bělohlávek [2] (see [4] for an overview). The embedding of this more general approach into the multi-adjoint concept lattices can be obtained similarly as above.

3.2 L-fuzzy concept lattices by Burusco and Fuentes-González

We will first recall the L-fuzzy concept lattices given by Burusco and Fuentes-González in [7,8]. The essential components form a tuple $\mathcal{L} = (L, \preceq, {}^{-}, \oplus)$ such that $(L, \preceq)$ is a complete lattice, ${}^{-}$ is a complementation operator on L, and $\oplus$ is a t-conorm on L.

Let A, B be two sets representing the sets of attributes and objects and consider a fuzzy relation $R \in L^{A\times B}$, the fuzzy sets L^A and L^B and the mappings $({}^{\uparrow}, {}^{\downarrow})$ defined as follows for $g \in L^B$ and $f \in L^A$:

$$g^{\uparrow}(a) = \inf_{b\in B}\{R(a,b) \oplus \overline{g(b)}\} \qquad f^{\downarrow}(b) = \inf_{a\in A}\{R(a,b) \oplus \overline{f(a)}\}$$

The authors show, by using Knaster-Tarski theorem, that set of fixed points of the mapping $\varphi\colon L^B \to L^B$, defined as $\varphi(g) = g^{\uparrow\downarrow}$, forms a complete lattice; therefore, the set of L-fuzzy concepts, called the L-fuzzy concept lattice, is defined as $\mathcal{C} = \{\langle g, f\rangle \mid g^{\uparrow} = f \text{ and } f^{\downarrow} = g\}$.

The L-fuzzy concept lattice generalises that by Pollandt, since, in the particular case of $L = [0, 1]$ the Łukasiewicz implication satisfies $y \leftarrow^L x = y \oplus \overline{x}$ where $\oplus$ is the t-conorm given by $x \oplus y = \min(1, x + y)$ and $^{-}$ is Zadeh's negation.

In the general case, given a L-fuzzy concept lattice it is obvious that we can consider the construction $y \oplus \overline{x}$ as an implication operator $y \leftarrow x$, since it increases in the consequent and decreases in the antecedent. However, this is not sufficient for $\leftarrow$ having an associated conjunctor, &, such that $(\&, \leftarrow)$ is an adjoint pair.

It is not difficult to prove that if $\leftarrow$ is inf-preserving in the first argument, i.e., $(\inf\{z \in Z\}) \leftarrow y = \inf\{z \leftarrow y \mid z \in Z\}$, then the following definition $x \,\&\, y = \inf\{z \in L \mid x \preceq z \leftarrow y\}$ provides a conjunctor such that $(\&, \leftarrow)$ is an adjoint pair. Under the additional hypothesis of commutativity of this conjunctor we can define $\nwarrow \;=\; \leftarrow \;=\; \swarrow$ so that $(\&, \swarrow, \nwarrow)$ is an adjoint triple. In this case, the L-fuzzy concept lattice can be seen as a multi-adjoint concept lattice just by considering the frame $(L, L, L, \preceq, \preceq, \preceq, \&, \swarrow, \nwarrow)$ and the context (A, B, R, σ) where $\sigma(b) = \&$ for every $b \in B$.

A first difficulty arises from the fact that, unfortunately, the properties of $\leftarrow$ shown above need not imply the commutativity of the conjunctor, as shown in the following example, thus showing that, in general, L-fuzzy concept lattices cannot be seen as particular cases of multi-adjoint concept lattices.

Example 10 *If we consider the lattice* $[0, 1]$ *with the usual ordering, the maximum operator* max *as t-conorm, and* $\overline{x} = 1 - x$ *as complementation, what we get is Kleene-Dienes implication, that is,* $y \leftarrow x = \max\{y, 1 - x\}$. *It is easy to check that it is inf-preserving in its first argument, hence the equation* $x \,\&\, y = \inf\{z \in [0, 1] \mid x \preceq z \leftarrow y\}$ *defines a conjunctor such that* $(\&, \leftarrow)$ *is an adjoint pair. However,* & *is not commutative because*

$$1 \,\&\, \tfrac{1}{2} = \inf\{z \in [0,1] \mid 1 \leq z \leftarrow \tfrac{1}{2}\} = \inf\{z \in [0,1] \mid 1 \leq \max\{\tfrac{1}{2}, z\}\} = 1$$
$$\tfrac{1}{2} \,\&\, 1 = \inf\{z \in [0,1] \mid \tfrac{1}{2} \leq z \leftarrow 1\} = \inf\{z \in [0,1] \mid \tfrac{1}{2} \leq \max\{0, z\}\} = \tfrac{1}{2}$$

Moreover, even assuming the existence of an adjoint triple $(\&, \swarrow, \nwarrow)$ for the resulting non-commutative conjunctor, the definition of $^{\downarrow}$ in the framework of L-fuzzy concept lattices needs not match that given in the case of multi-adjoint concept lattices.

Last but not least, another important difference between L-fuzzy concept lattices and multi-adjoint concept lattices is that, in the latter case, the pair $(^{\uparrow}, ^{\downarrow})$ forms a Galois connection; as a result, φ is a closure operator, and the concepts, that is the fixed points of $\varphi(g) = g^{\uparrow\downarrow}$, are obtained after just two iterations of φ. In the former case, the pair $(^{\uparrow}, ^{\downarrow})$ is not necessarily a Galois connection, as note, hence the number of iterations needed in order to obtain the fixed points is not known in advance.

3.3 Krajči's generalised concept lattices

The purpose of this section is to compare the multi-adjoint framework for concept analysis with that introduced by Krajči. To begin with, let us recall the following definition of left continuity, introduced in [18].

Definition 11 *Let $(P, \leq)$ be a poset and $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ complete lattices:*

- $\&\colon L_1 \times L_2 \to P$ *is* left-continuous in the first argument *if given $y \in L_2$, $z \in P$ and a non-empty subset $X \subseteq L_1$, the condition "$x \mathbin{\&} y \leq z$ holds for all $x \in X$" implies that* $(\sup X) \mathbin{\&} y \leq z$.
- $\&\colon L_1 \times L_2 \to P$ *is* left-continuous in the second argument *if given $x \in L_1$, $z \in P$ and a non-empty subset $Y \subseteq L_2$, the condition "$x \mathbin{\&} y \leq z$ holds for all $y \in Y$" implies $x \mathbin{\&} (\sup Y) \leq z$.*
- $\&\colon L_1 \times L_2 \to P$ *is* left-continuous *if it is left-continuous in both arguments.*

The following proposition generalises the existence of residuated implication for continuous t-norms to the context of adjoint triples.

Proposition 12 *Let $\&\colon L_1 \times L_2 \to P$ be an order-preserving operator in both arguments where P has a bottom element, then the following two conditions are equivalent:*

1. *$\&$ is left-continuous and $\bot_1 \mathbin{\&} y = \bot$, $x \mathbin{\&} \bot_2 = \bot$, for all $x \in L_1$, $y \in L_2$.*
2. *There exist two functions, $\swarrow$ and $\nwarrow$, such that $(\&, \swarrow, \nwarrow)$ is an adjoint triple.*

PROOF. (1 implies 2)

The function $\swarrow\colon P \times L_2 \to L_1$ is defined as expected: given $y \in L_2$ and $z \in P$ consider the set $X = \{x \in L_1 \mid x \mathbin{\&} y \leq z\}$, now

$$z \swarrow y = \sup X = \sup\{x \in L_1 \mid x \mathbin{\&} y \leq z\}$$

we will now prove that it satisfies the adjoint property with respect to $\&$.

Consider elements $x \in L_1$, $y \in L_2$, $z \in P$ such that $x \mathbin{\&} y \leq z$. Obviously, we have that $x \in X$ and $x \preceq_1 \sup X$, hence $x \preceq_1 z \swarrow y = \sup X$.

Conversely, assume that we have $x \preceq_1 z \swarrow y$. By the boundary conditions, the set X is non-empty for it contains the bottom element; therefore by left-continuity we have $(\sup X) \mathbin{\&} y \leq z$, that is, $(z \swarrow y) \mathbin{\&} y \leq z$. Finally, by the assumption and the monotonicity of $\&$ in the first argument we obtain

$$x \mathbin{\&} y \leq z$$

The rest of this part concerns the definition of $\nwarrow$ and checking its adjoint properties. The definition of the function $\nwarrow : P \times L_1 \to L_2$, for all $x \in L_1$ and $z \in P$, is given as

$$z \nwarrow x = \sup\{y \in L_2 \mid x \mathbin{\&} y \leq z\}$$

the proof of the adjoint property is similar to the previous one, as a result $(\&, \swarrow, \nwarrow)$ is an adjoint triple.

(2 implies 1)

Let us assume the adjoint property, and consider $y \in L_2$, $z \in P$, and a non-empty subset $X \subseteq L_1$ such that $x \mathbin{\&} y \leq z$, for all $x \in X$.

By the adjoint property, for all $x \in X$ the inequality $x \mathbin{\&} y \leq z$ implies that $x \preceq_1 z \swarrow y$ and, by definition of supremum, $(\sup X) \preceq_1 z \swarrow y$. Using the adjoint property again, we obtain $(\sup X) \mathbin{\&} y \leq z$, and $\&$ is left-continuous in the first argument. The proof of left-continuity in the second argument is similar just using $\nwarrow$.

The boundary conditions follow directly from Lemma 2. $\square$

The requirement of the boundary conditions is essential to construct the adjoint triple. It is not difficult to show an example of a left-continuous order-preserving operator which does not fulfill them, as a result the two-sided implications do not allow to form an adjoint triple.

Example 13 *Let* $\& \colon [0,1] \times [0,1] \longrightarrow [0,1]$ *be the constant operator defined as* $x \mathbin{\&} y = 0.5$*, for all* $x, y \in [0,1]$*, hence* $\&$ *is order-preserving.*

Left-continuity of $\&$ *is straightforward: In the first argument, given* $y \in [0,1]$ *and* $X \subseteq [0,1]$*, if* $0.5 = x \mathbin{\&} y \leq z$ *holds for* $z \in [0,1]$ *and all* $x \in X$*, then* $(\sup X) \mathbin{\&} y = 0.5 \leq z$*; analogously in the second argument. However, obviously it doesn't verifies the required boundary condition with the bottom element, since* $0 \mathbin{\&} y = 0.5 \neq 0$*.*

A version of Proposition 12 above is stated without proof in [6, theorem 5] without the additional requirement of the boundary conditions for $\&$; however, the boundary conditions are necessary as stated above. It seems that the authors overlooked that left-continuity does not imply the boundary conditions; moreover, Krajči explicitly requires boundary conditions in his statement of the basic theorem of generalised concept lattices [18].

Now, in order to formally prove the embedding of Krajči's approach into the multi-adjoint framework, let us introduce the definition of generalised concept lattices.

Consider non-empty sets A and B, a P-fuzzy relation on their Cartesian product $R\colon A \times B \to P$, and a monotone left-continuous operator $\&\colon L_1 \times L_2 \to P$.

The mappings ${}^{\uparrow}\colon L_2^B \to L_1^A$ and ${}^{\downarrow}\colon L_1^A \to L_2^B$ are defined as follows:

$$g^{\uparrow}(a) = \sup\{x \in L_1 \mid (\forall b \in B) x \,\&\, g(b) \leq R(a,b)\}$$
$$f^{\downarrow}(b) = \sup\{y \in L_2 \mid (\forall a \in A) f(a) \,\&\, y \leq R(a,b)\}$$

The set $\mathcal{G} = \{(g,f) \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow} = f, f^{\downarrow} = g\}$ with the following order: $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ iff $g_1 \preceq_2 g_2$ is called a *generalised concept lattice*.

Now, we introduce the main result which relates both frameworks (a similar result under different terminology is stated without proof in [6, Thm 6]).

Theorem 14 *Given a generalised concept lattice $(\mathcal{G}, \preceq)$, where the conjunctor operator $\&\colon L_1 \times L_2 \to P$ satisfies $\bot_1 \,\&\, y = \bot$ and $x \,\&\, \bot_2 = \bot$, for all $x \in L_1$, $y \in L_2$, then there exist a multi-adjoint frame and a context such that the corresponding multi-adjoint concept lattice equals $(\mathcal{G}, \preceq)$.*

PROOF. By Proposition 12 we have that there exist two functions, $\swarrow$ and $\nwarrow$, such that $(\&, \swarrow, \nwarrow)$ is an adjoint triple.

We easily obtain that

$$\sup\{x \in L_1 \mid (\forall b \in B) x \,\&\, g(b) \leq R(a,b)\}$$

is equal to

$$\sup\{x \in L_1 \mid (\forall b \in B) x \preceq_1 R(a,b) \swarrow g(b)\}$$

because both sets are equal by the adjoint property for $(\&, \swarrow)$. Furthermore, by the characterization of the infimum as the supremum of the lower bounds, the latter turns out to be equal to

$$\inf\{R(a,b) \swarrow g(b) \mid b \in B\}$$

As a result we obtain

$$\sup\{x \in L_1 \mid (\forall b \in B) x \,\&\, g(b) \leq R(a,b)\} = \inf\{R(a,b) \swarrow g(b) \mid b \in B\} \quad (1)$$

and a similar argument allows to prove that

$$\sup\{y \in L_2 \mid (\forall a \in A) f(a) \,\&\, y \leq R(a,b)\} = \inf\{R(a,b) \nwarrow f(a) \mid a \in A\} \quad (2)$$

Now, consider the multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ defined from the frame $(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&, \swarrow, \nwarrow)$, and the context (A, B, R, σ), where σ assigns the operator $\&$ to every $b \in B$.

Equalities (1) and (2) show that the Galois connections used to build the generalised concept lattice and the multi-adjoint concept lattice coincide; therefore, both lattices coincide as well. $\square$

4 The representation theorem

An extension of the representation (or fundamental) theorem on the classical concept lattice [10] for the multi-adjoint framework is presented below. In some sense, the result is similar to those given in previous extensions of the classical concept lattices to the fuzzy case, but the presentation has been simplified.

To begin with, we need to introduce some definitions and preliminary results. We start by introducing the notions of infimum-dense, supremum-dense, and representability, which will be used later in the statement of Proposition 17.

Firstly, an infimum-dense (resp. supremum-dense) subset $K \subseteq L$ is such that the set of the infima (resp. suprema) of all its subsets coincides with L. Formally, we have:

Definition 15 *Given a complete lattice L, a subset $K \subseteq L$ is* infimum-dense *(resp.* supremum-dense*) if and only if for all $x \in L$ there exists $K' \subseteq K$ such that $x = \inf(K')$ (resp. $x = \sup(K')$).*

A multi-adjoint concept lattice is said to be represented by a complete lattice provided there is a pair of functions satisfying the conditions stated in the definition below:

Definition 16 *A multi-adjoint concept lattice*[3] *$(\mathcal{M}, \preceq)$ is* represented *by a complete lattice $(V, \sqsubseteq)$ if there exists a pair of mappings $\alpha\colon A \times L_1 \to V$ and $\beta\colon B \times L_2 \to V$ such that:*

1a) $\alpha[A \times L_1]$ is infimum-dense;
1b) $\beta[B \times L_2]$ is supremum-dense; and

[3] Recall that we are considering a multi-adjoint concept lattice on a fixed frame $(L_1, L_2, P, \&_1, \dots, \&_n)$, and the context (A, B, R, σ).

2) For each $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$:

$$\beta(b, y) \sqsubseteq \alpha(a, x) \quad \text{if and only if} \quad x \&_b y \leq R(a, b)$$

The following proposition presents some consequences which can be obtained from the definition of representability.

Proposition 17 *Given a complete lattice $(V, \sqsubseteq)$ which represents a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$, and mappings $f \in L_1^A$ and $g \in L_2^B$, we have:*

(1) β is order-preserving in the second argument.
(2) α is order-reversing in the second argument.
(3) $g^{\uparrow}(a) = \sup\{x \in L_1 \mid v_g \sqsubseteq \alpha(a, x)\}$, where $v_g = \sup\{\beta(b, g(b)) \mid b \in B\}$.
(4) $f^{\downarrow}(b) = \sup\{y \in L_2 \mid \beta(b, y) \sqsubseteq v_f\}$, where $v_f = \inf\{\alpha(a, f(a)) \mid a \in A\}$.
(5) If $g_v(b) = \sup\{y \in L_2 \mid \beta(b, y) \sqsubseteq v\}$, then $\sup\{\beta(b, g_v(b)) \mid b \in B\} = v$.
(6) If $f_v(a) = \sup\{x \in L_1 \mid v \sqsubseteq \alpha(a, x)\}$, then $\inf\{\alpha(a, f_v(a)) \mid a \in A\} = v$.

PROOF. We give the proofs for items 1, 3 and 5, since the other are similar.

(1). Let $y_1 \preceq_2 y_2 \in L_2$, and let us check that $\beta(b, y_1) \sqsubseteq \beta(b, y_2)$ for all $b \in B$.

As $\alpha[A \times L_1]$ is infimum-dense, by considering $\beta(b, y_2) \in V$ there exists a set $K \subseteq A \times L_1$ such that $\beta(b, y_2) = \inf \alpha[K]$; hence, in particular $\beta(b, y_2) \sqsubseteq \alpha(a, x)$ for all $(a, x) \in K$.

Now, by Definition 16(2), for all $(a, x) \in K$ it follows that $x \&_b y_2 \leq R(a, b)$ and, as $y_1 \preceq_2 y_2$, by monotonicity

$$x \&_b y_1 \leq x \&_b y_2 \leq R(a, b) \text{ for all } (a, x) \in K$$

This, again by Definition 16(2), implies that $\beta(b, y_1)$ is a lower bound of the set $\alpha[K]$, that is $\beta(b, y_1) \sqsubseteq \alpha(a, x)$ for all $(a, x) \in K$. Finally, as $\beta(b, y_2) = \inf \alpha[K]$, the inequality $\beta(b, y_1) \sqsubseteq \beta(b, y_2)$ follows, and β is order-preserving in the second argument.

(3). Recall that $g^{\uparrow}(a) = \inf\{R(a, b) \swarrow^b g(b) \mid b \in B\}$.

Now, given $x \in L_1$, by the adjoint property, $x \preceq_1 R(a, b) \swarrow^b g(b)$ is equivalent to $x \&_b g(b) \leq R(a, b)$ which, in turn, is also equivalent, by Definition 16(2), to $\beta(b, g(b)) \sqsubseteq \alpha(a, x)$ for all $b \in B$, and by taking the supremum, is equivalent to $v_g = \sup\{\beta(b, g(b)) \mid b \in B\} \sqsubseteq \alpha(a, x)$. As a result, we obtain the equality of the sets:

$$\{x \in L_1 \mid x \preceq_1 R(a, b) \swarrow^b g(b) \text{ for all } b \in B\} = \{x \in L_1 \mid v_g \sqsubseteq \alpha(a, x)\}$$

Therefore:

$$
\begin{aligned}
g^{\uparrow}(a) &= \inf\{R(a,b) \swarrow^{b} g(b) \mid b \in B\} \\
&= \sup\{x \in L_1 \mid x \preceq_1 R(a,b) \swarrow^{b} g(b) \text{ for all } b \in B\} \\
&= \sup\{x \in L_1 \mid v_g \sqsubseteq \alpha(a,x)\}
\end{aligned}
$$

(5). In order to prove the equality, we will firstly show that for any $v \in V$, the inequality $\sup\{\beta(b, g_v(b)) \mid b \in B\} \sqsubseteq v$ holds.

Consider $v \in V$, as the set $\alpha[A \times L_1]$ is infimum-dense, there is a set $K \subseteq A \times L_1$ such that $v = \inf\{\alpha(a,x) \mid (a,x) \in K\}$. As a result, in order to prove

$$\sup\{\beta(b, g_v(b)) \mid b \in B\} \sqsubseteq \inf\{\alpha(a,x) \mid (a,x) \in K\} \quad (= v)$$

it is enough to show that $\beta(b, g_v(b)) \sqsubseteq \alpha(a,x)$ for all $b \in B$ and $(a,x) \in K$.

Fix elements $(a,x) \in K$ and $b \in B$, and assume the existence of an element $y \in L_2$ such that $\beta(b,y) \sqsubseteq v$. Then, by the representation of v as an infimum, we get $\beta(b,y) \sqsubseteq v \sqsubseteq \alpha(a,x)$. Now, we can apply the chain of equivalences

$$\beta(b,y) \sqsubseteq \alpha(a,x) \quad \Longleftrightarrow \quad x \,\&_b\, y \leq R(a,b) \quad \Longleftrightarrow \quad y \preceq_2 R(a,b) \nwarrow_b x$$

and compute the supremum on y to obtain $g_v(b) \preceq_2 R(a,b) \nwarrow_b x$. Note that if there is no $y \in L_2$ such that $\beta(b,y) \sqsubseteq v$, then $g_v(b) = \bot$ and we obtain $g_v(b) \preceq_2 R(a,b) \nwarrow_b x$ as well. Applying back the equivalences above, we finally get $\beta(b, g_v(b)) \sqsubseteq \alpha(a,x)$.

For the other inequality, we use that $\beta[B \times L_2]$ is supremum-dense in order to write $v = \sup\{\beta(b,y) \mid (b,y) \in K'\}$ for some subset $K' \subseteq B \times L_2$. This means, in particular, that given $(b,y) \in K'$, we have that $\beta(b,y) \sqsubseteq v$ and, moreover, $y \preceq_2 \sup\{y \in L_2 \mid \beta(b,y) \sqsubseteq v\} = g_v(b)$.

As β is order-preserving in the second argument, we obtain for all $(b,y) \in K'$:

$$
\begin{aligned}
\beta(b,y) &\sqsubseteq \beta(b, g_v(b)) \\
&\sqsubseteq \sup\{\beta(b, g_v(b)) \mid (b,y) \in K'\} \\
&\sqsubseteq \sup\{\beta(b, g_v(b)) \mid b \in B\}
\end{aligned}
$$

Finally, applying that v is the supremum on $(b,y) \in K'$, we get the inequality $v \sqsubseteq \sup\{\beta(b, g_v(b)) \mid b \in B\}$. $\square$

The last notion we need is not related to the statement of the representation theorem, but to its proof: the characteristic mappings.

Definition 18 *Given a set A, a poset P with bottom element $\bot$, and elements $a \in A$, $x \in P$, the characteristic mapping $@^x_a \colon A \to P$, read "at point a the* *value is x", is defined as:*

$$@_a^x(a') = \begin{cases} x, & \textit{if } \ a' = a \\ \bot, & \textit{otherwise} \end{cases}$$

Lemma 19 *In the multi-adjoint concept lattice $(\mathcal{M}, \preceq)$, given $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$, the following equalities hold:*

$$\begin{array}{lll} @_a^{x\downarrow}(b') = R(a,b') \nwarrow_{b'} x & \textit{for all} & b' \in B \\ @_b^{y\uparrow}(a') = R(a',b) \swarrow^{b} y & \textit{for all} & a' \in A \end{array}$$

PROOF. By definition of $^\downarrow$ on the mapping $@_a^x$, we get

$$@_a^{x\downarrow}(b') = \inf\{R(a',b') \nwarrow_{b'} @_a^x(a') \mid a' \in A\} = R(a,b') \nwarrow_{b'} x$$

where the last inequality follows because $R(a,b') \nwarrow_{b'} \bot_1 = \top_2$ (this fact is a consequence of the adjoint property, since $\bot_1 \preceq_1 R(a,b') \swarrow^{b'} \top_2$).

The other equality follows similarly. $\square$

We can now state and prove the fundamental theorem for multi-adjoint concept lattices.

Theorem 20 *A complete lattice $(V, \sqsubseteq)$ represents a multi-adjoint concept lattice $(\mathcal{M}, \preceq)$ if and only if $(V, \sqsubseteq)$ is isomorphic to $(\mathcal{M}, \preceq)$.*

PROOF. Assume that $(V, \sqsubseteq)$ represents $(\mathcal{M}, \preceq)$, then we have the existence of the mappings $\alpha\colon A \times L_1 \to V$, $\beta\colon B \times L_2 \to V$, these mappings will be used to construct an isomorphism $\varphi\colon \mathcal{M} \to V$.

For every concept $\langle g, f\rangle \in \mathcal{M}$ the mapping φ is defined as follows:

$$\varphi(\langle g, f\rangle) = \sup\{\beta(b, g(b)) \mid b \in B\}$$

Firstly, let us introduce another mapping $\psi\colon V \to \mathcal{M}$, which will be proven to be the inverse of φ. This ψ is defined for each $v \in V$ as

$$\psi(v) = \langle g_v, f_v\rangle$$

where the functions g_v and f_v are defined, for each $b \in B$ and $a \in A$, as in Proposition 17(items 5 and 6). This proposition will be used to show that ψ is well-defined, that is, $\langle g_v, f_v\rangle$ is a concept. We have only to take into account that by items 3 and 5, v_{g_v} and v coincide; therefore

$$\begin{aligned}
{g_v}^{\uparrow}(a) &= \sup\{x \in L_1 \mid v_{g_v} \sqsubseteq \alpha(a,x)\} && \text{(Proposition 17(3))}\\
&= \sup\{x \in L_1 \mid v \sqsubseteq \alpha(a,x)\} && (v_{g_v} = v)\\
&= f_v(a) && \text{(Proposition 17(5))}
\end{aligned}$$

The equality ${f_v}^{\downarrow} = g_v$ is proved analogously.

In order to prove $\psi \circ \varphi = id$, note that given a concept $\langle g, f\rangle$, the equality $\psi(\varphi(\langle g, f\rangle)) = \langle g, f\rangle$ holds if $f = f_{\varphi(\langle g,f\rangle)}$. But this is obvious, since by definition of φ and Proposition 17(3) we have that $v_g = \sup\{\beta(b, g(b)) \mid b \in B\} = \varphi(\langle g, f\rangle)$; moreover, taking into account that $g^{\uparrow} = f$, we can write

$$\begin{aligned}
f(a) = g^{\uparrow}(a) &= \sup\{x \in L_1 \mid v_g \sqsubseteq \alpha(a,x)\}\\
&= \sup\{x \in L_1 \mid \varphi(\langle g, f\rangle) \sqsubseteq \alpha(a,x)\}\\
&= f_{\varphi(\langle g,f\rangle)}(a)
\end{aligned}$$

Proposition 17(5) directly implies that the other composition gives the identity, since $v = \sup\{\beta(b, g_v(b)) \mid b \in B\} = \varphi(\langle g_v, f_v\rangle) = \varphi(\psi(v))$ for all $v \in V$.

Once we have that φ is a bijection, it is sufficient to prove that it preserves and reflects the ordering, see [9, Thm. 2.19], in order to prove that it is a lattice isomorphism.

The proof of φ being order-preserving is a straightforward consequence of its definition and Proposition 17(1). Consider $\langle g_1, f_1\rangle$, $\langle g_2, f_2\rangle$ in $\mathcal{M}$ such that $\langle g_1, f_1\rangle \le \langle g_2, f_2\rangle$, we have that $g_1 \le g_2$ and therefore $\beta(b, g_1(b)) \sqsubseteq \beta(b, g_2(b))$ for all $b \in B$, since β is order-preserving in the second argument. Thus, by definition of φ, we obtain that:

$$\varphi(\langle g_1, f_1\rangle) \sqsubseteq \varphi(\langle g_2, f_2\rangle)$$

To prove that φ reflects the ordering, we directly show that its inverse mapping ψ is order-preserving as well. Consider $v_1 \sqsubseteq v_2$, and let us show that $g_{v_1} \le g_{v_2}$.

Given $b \in B$, we obviously have that

$$\{y \in L_2 \mid \beta(b, y) \sqsubseteq v_1\} \subseteq \{y \in L_2 \mid \beta(b, y) \sqsubseteq v_2\}$$

now, applying suprema

$$\begin{aligned}
g_{v_1}(b) &= \sup\{y \in L_2 \mid \beta(b, y) \sqsubseteq v_1\}\\
&\le \sup\{y \in L_2 \mid \beta(b, y) \sqsubseteq v_2\} = g_{v_2}(b)
\end{aligned}$$

This finishes the proof that $\mathcal{M}$ and V are isomorphic.

Conversely, given an isomorphism $\varphi\colon \mathcal{M} \to V$, let us show that V represents $\mathcal{M}$.

To begin with, the mappings $\alpha\colon A \times L_1 \to V$ and $\beta\colon B \times L_2 \to V$ can be naturally defined, for every $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$, as follows:

$$\alpha(a,x) = \varphi(\langle @_a^{x\downarrow}, @_a^{x\downarrow\uparrow}\rangle) \qquad \beta(b,y) = \varphi(\langle @_b^{y\uparrow\downarrow}, @_b^{y\uparrow}\rangle)$$

Firstly, let us show that $\alpha[A \times L_1]$ is infimum-dense. By definition, we have to prove that given $v \in V$ there exists $K \subseteq A \times L_1$ such that $v = \inf(\alpha[K])$.

Since φ is an isomorphism, we will prove the corresponding statement on $\mathcal{M}$. Consider $\langle g, f\rangle = \varphi^{-1}(v) \in \mathcal{M}$, and define $K = \{(a, f(a)) \mid a \in A\} \subseteq A \times L_1$, then it is sufficient to prove that

$$\langle g, f\rangle = \inf\{\langle @_a^{f(a)\downarrow}, @_a^{f(a)\downarrow\uparrow}\rangle \mid a \in A\}$$

which, moreover, reduces to prove the corresponding statement on one of the components of the concept. We will prove that $g(b) = \inf\{@_a^{f(a)\downarrow}(b) \mid a \in A\}$.

By Lemma 19, we have that $@_a^{f(a)\downarrow}(b) = R(a,b) \nwarrow_b f(a)$, thus

$$\inf\{@_a^{f(a)\downarrow}(b) \mid a \in A\} = \inf\{R(a,b) \nwarrow_b f(a) \mid a \in A\} = f^{\downarrow}(b) = g(b)$$

Similarly, we can prove that $\beta[B \times L_2]$ is supremum-dense.

It only remains to prove that given $a \in A$, $b \in B$, $x \in L_1$ and $y \in L_2$, we have that $\beta(b,y) \sqsubseteq \alpha(a,x)$ if and only if $x \,\&_b\, y \le R(a,b)$.

By the definition of α and β above, and the fact that φ is order-isomorphism, we have that $\beta(b,y) \sqsubseteq \alpha(a,x)$ is equivalent to $\langle @_b^{y\uparrow\downarrow}, @_b^{y\uparrow}\rangle \le \langle @_a^{x\downarrow}, @_a^{x\downarrow\uparrow}\rangle$ and, in particular, to $@_b^{y\uparrow\downarrow} \preceq_2 @_a^{x\downarrow}$. From the properties of Galois connection and Lemma 19 we obtain the following chain of inequalities

$$y = @_b^{y}(b) \preceq_2 @_b^{y\uparrow\downarrow}(b) \preceq_2 @_a^{x\downarrow}(b) = R(a,b) \nwarrow_b x$$

now, from the properties of adjoint triple we obtain

$$x \,\&_b\, y \le R(a,b)$$

For the other implication, assume $x \,\&_b\, y \le R(a,b)$ and let us prove $@_a^{x} \preceq_1 @_b^{y\uparrow}$, since this implies $@_b^{y\uparrow\downarrow} \preceq_2 @_a^{x\downarrow}$ which turns out to be equivalent to the inequality $\beta(b,y) \sqsubseteq \alpha(a,x)$.

Consider $a' \in A$ with $a' \neq a$, then $@_a^x(a') = \bot_1$ and therefore $@_a^x(a') \preceq_1 @_b^{y\uparrow}(a')$ holds. Otherwise, if $a' = a$, as $x \,\&_b\, y \leq R(a,b)$ applying the adjoint property and Lemma 19 we obtain that:

$$@_a^x(a) = x \preceq_1 R(a,b) \swarrow^b y = @_b^{y\uparrow}(a) \qquad \square$$

Just a quick note regarding an improvement of a previous representation theorem: let us notice that, in Proposition 17 it is proved directly that the function α is order-reversing and β is order-preserving in their second argument, hence these hypotheses, which are explicitly required for the representation theorem of [17], can be dropped.

Let us finish this section with a further proposition, stated without proof because it is just an easy calculation, which relates the behaviour of the mappings α and β, and shows that the construction based on β done in the proof of the fundamental theorem could have been done essentially in the same terms using α.

Proposition 21 *Given a multi-adjoint concept lattice* $(\mathcal{M}, \preceq)$*, and a concept* $\langle g, f\rangle \in \mathcal{M}$ *and two mappings* $\beta\colon B \times L_2 \to \mathcal{M}$*,* $\alpha\colon A \times L_1 \to \mathcal{M}$*, where* β *is* $(\mathcal{M}, R)$*-related to* α*, we have that:*

$$\sup\{\beta(b, g(b)) \mid b \in B\} = \inf\{\alpha(a, f(a)) \mid a \in A\}$$

5 A worked example

Let us consider that we have written a scientific paper and we still have to decide which journal the paper will be submitted to. According to the main topics of the paper, a number of journals are considered as potential target. The target journal will be chosen according to several parameters appearing in the ISI Journal Citation Report.

The sets of attributes and objects are the following:

$$A = \{\text{Impact Factor}, \text{Immediacy Index}, \text{Cited Half-Life}, \text{Best Position}\}$$
$$B = \{\text{AMC}, \text{CAMWA}, \text{FSS}, \text{IEEE-FS}, \text{IJGS}, \text{IJUFKS}, \text{JIFS}\}$$

where the "best position" means the best quartile of the different categories under which the journal is included, and the journals considered are Applied Mathematics and Computation (AMC), Computer and Mathematics with Applications (CAMWA), Fuzzy Sets and Systems (FSS), IEEE transactions on Fuzzy Systems (IEEE-FS), International Journal of General Systems (IJGS), International Journal of Uncertainty Fuzziness and Knowledge-based Systems (IJUFKS), Journal of Intelligent and Fuzzy Systems (JIFS).

We will consider a multi-adjoint frame with three different lattices: one for handling the information taken from the JCR, which is rounded to the second decimal digit; a second one to handle information about the attributes, in which we estimate steps of 0.05 in order to distinguish to appreciate a qualitative difference; and a third one, used to set the different levels of preference of the journal, which is considered to be of 0.125 (hence the unit interval is divided into eight equal pieces)

Let $([0,1]_{20}, [0,1]_8, [0,1]_{100}, \leq, \leq, \leq, \&_P^*, \&_L^*)$ be a multi-adjoint frame where [4] $\&_P^*$ and $\&_L^*$ are the discretisations of the product and Łukasiewicz conjunctors respectively, defined as in Example 3.

The corresponding residuated implications $\swarrow_P^*, \swarrow_L^* \colon [0,1]_{100} \times [0,1]_8 \to [0,1]_{20}$ and $\nwarrow_P^*, \nwarrow_L^* \colon [0,1]_{100} \times [0,1]_{20} \to [0,1]_8$ are defined as:

$$b \swarrow_P^* a = \frac{\lfloor 20 \cdot \min\{1, b/a\} \rfloor}{20} \qquad b \nwarrow_P^* c = \frac{\lfloor 8 \cdot \min\{1, b/c\} \rfloor}{8}$$
$$b \swarrow_L^* a = \frac{\lfloor 20 \cdot \min\{1, 1+b-a\} \rfloor}{20} \qquad b \nwarrow_L^* c = \frac{\lfloor 8 \cdot \min\{1, 1+b-c\} \rfloor}{8}$$

where $\lfloor \_ \rfloor$ is the floor function.

The fuzzy relation between them, $R \colon A \times B \to P$, is the normalization to the unit interval $[0,1]$ of the information in the JCR, and can be seen in Table 1.

Table 1
Fuzzy relation between the objects and the attributes.

R	AMC	CAMWA	FSS	IEEE-FS	IJGS	IJUFKS	JIFS
Impact Factor	0.34	0.21	0.52	0.85	0.43	0.21	0.09
Immediacy Index	0.13	0.09	0.36	0.17	0.1	0.04	0.06
Cited Half-Life	0.31	0.71	0.92	0.65	0.89	0.47	0.93
Best Position	0.75	0.5	1	1	0.5	0.25	0.25

The problem of choosing a *suitable* journal to submit depends on the definition of "suitability" we have in mind. For example, a fuzzy notion of suitability can be defined as a journal with high impact factor, relatively big immediacy index, more than 5.5 years of half-life and with not a bad position in the listing of the category. Such a notion of suitability can be defined, in the

[4] Recall that $[0,1]_m$ denotes a regular partition of $[0,1]$ into m pieces.

context (A, B, R, σ) where $\sigma(b) = \&_P$ for every $b \in B$, by the fuzzy subset $f\colon A \to [0,1]$ below:

$$
\begin{aligned}
f(\text{Impact Factor}) &= 0.75, & f(\text{Immediacy Index}) &= 0.3,\\
f(\text{Cited Half-Life}) &= 0.55, & f(\text{Best Position}) &= 0.5
\end{aligned}
$$

Now, the problem consists in finding a multi-adjoint concept which represents the suitable journal as defined by the fuzzy set f.

As any concept gets completely determined by any of its components, it is sufficient to compute the component $f^{\downarrow}$ which, in addition, will provide information about the suitability (modulo f) of every journal. As explained in the previous sections, the required computations are as follows:

$$
\begin{aligned}
f^{\downarrow}(\text{AMC}) &= \inf\{R(a, \text{AMC}) \nwarrow_P^* f(a)\colon a \in A\}\\
&= \inf\{0.34 \nwarrow_P^* 0.75, 0.13 \nwarrow_P^* 0.3, 0.31 \nwarrow_P^* 0.55, 0.75 \nwarrow_P^* 0.5\}\\
&= \frac{\lfloor 8 \cdot \min\{1, 0.13/0.3\} \rfloor}{8}\\
&= 0.375
\end{aligned}
$$

For the rest of the journals, the computation is similar, obtaining the following results

$$
\begin{aligned}
&f^{\downarrow}(\text{AMC}) = 0.375 \quad && f^{\downarrow}(\text{CAMWA}) = 0.25 \quad && f^{\downarrow}(\text{FSS}) = 0.625\\
&f^{\downarrow}(\text{JIFS}) = 0 && f^{\downarrow}(\text{IJGS}) = 0.25 && f^{\downarrow}(\text{IJUFKS}) = 0.125\\
&f^{\downarrow}(\text{IEEE-FS}) = 0.5
\end{aligned}
$$

based on which, the most suitable journal is FSS. Note that the use of this particular definition for "suitability" does not directly select the one with highest impact factor, despite being the property with the highest weight, since other attributes are taken into account as well.

One important feature of the multi-adjoint framework is that it allows to associate different adjoint triples to each object (resp. attribute). For instance, if we would like to submit preferably to a journal listed under the Artificial Intelligence category (i.e. IEEE-FS, IJUFKS, and JIFS), the multi-adjoint framework allows for modifying the underlying context in order to assign a different adjoint triple to the journals we are more interested in.

We will consider the context (A, B, R, σ'), where $\sigma'(b) = \&_P$ for every $b \in B_1$ and $\sigma'(b) = \&_L$ for every $b \in B_2$, where $B_1 = \{\text{AMC}, \text{CAMWA}, \text{FSS}, \text{IJGS}\}$ and $B_2 = \{\text{IEEE-FS}, \text{IJUFKS}, \text{JIFS}\}$.

This particular selection of σ' allows for using the Łukasiewicz implication in order to compute the values for journals in the AI category, hence the definition of $f^{\downarrow}$ is modified considering different cases:

$$
\begin{aligned}
f^{\downarrow}(b_1) &= \inf\{R(a,b_1) \nwarrow^{*}_{P} f(a) \colon a \in A\} \quad \text{for } b_1 \in B_1 \\
f^{\downarrow}(b_2) &= \inf\{R(a,b_2) \nwarrow^{*}_{L} f(a) \colon a \in A\} \quad \text{for } b_2 \in B_2
\end{aligned}
$$

The final result that we obtain in this case is

$$
\begin{array}{lll}
f^{\downarrow}(\mathrm{AMC}) = 0.375 & f^{\downarrow}(\mathrm{CAMWA}) = 0.25 & f^{\downarrow}(\mathrm{FSS}) = 0.625 \\
f^{\downarrow}(\mathrm{JIFS}) = 0.25 & f^{\downarrow}(\mathrm{IJGS}) = 0.25 & f^{\downarrow}(\mathrm{IJUFKS}) = 0.375 \\
f^{\downarrow}(\text{IEEE-FS}) = 0.75 & &
\end{array}
$$

which states that the journal that better suits our needs is IEEE-FS.

It is important to note that the mere assignment of 'greater' operators to a subset of objects does not imply that the better selection is necessarily in this subset. For instance, consider the following modification f_1 of the notion of suitability:

$$
\begin{array}{ll}
f_1(\text{Impact Factor}) = 0.65, & f_1(\text{Immediacy Index}) = 0.45, \\
f_1(\text{Cited Half-Life}) = 0.55, & f_1(\text{Best Position}) = 0.5
\end{array}
$$

The results associated to this f_1 are shown below

$$
\begin{array}{lll}
f_1^{\downarrow}(\mathrm{AMC}) = 0.25 & f_1^{\downarrow}(\mathrm{CAMWA}) = 0.125 & f_1^{\downarrow}(\mathrm{FSS}) = 0.75 \\
f_1^{\downarrow}(\mathrm{JIFS}) = 0.375 & f_1^{\downarrow}(\mathrm{IJGS}) = 0.125 & f_1^{\downarrow}(\mathrm{IJUFKS}) = 0.5 \\
f_1^{\downarrow}(\text{IEEE-FS}) = 0.625 & &
\end{array}
$$

Therefore, in spite of having increased the preference for journals in the AI category, for this particular definition of suitable journal FSS remains as the best journal, and IEEE-FS is the second best suited.

6 Conclusions and Future Work

Multi-adjoint concept lattices have been introduced as a generalisation of different existing approaches to fuzzified and/or generalised versions of the classical concept lattice. One of the interesting features is that in a multi-adjoint context each object (or attribute) has an associated implication and, thus, subgroups with different degrees of preference can be easily established.

The proof of the representation theorem for multi-adjoint concept lattices has been presented in a more structured and readable way than that given in [18] for the generalized concept lattice; the idea has been to work with our adjoint triples in the same way that [5]. Moreover, the multi-adjoint concept lattice has been shown to embed the generalised concept lattice as well as other different fuzzy extensions of the classical concept lattice [10], such as the fuzzy concepts of [7] and of [5] for the case of $\{0, 1\}$-equality and crisp ordering.

Continuing with the comparison of the multi-adjoint frame with other fuzzy approaches, one future work would be to study the relationship between the concepts given in [11]. Another point to take into account is the introduction of fuzzy orderings in order to completely embed the fuzzy concept lattice of [5].

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