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A multimodal logic for closeness

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We introduce a multimodal logic for order of magnitude reasoning which considers a new logic-based alternative to the notion of closeness, we provide an axiom system and prove its soundness and completeness.

Keywords: multimodal logic, qualitative reasoning, closeness, completeness

1. Introduction

There are some multimodal logics for order of magnitude reasoning dealing with the relations of negligibility and comparability, see for instance [Balbiani, 2016, Burrieza and Ojeda-Aciego, 2005, Golińska-Pilarek and Muñoz-Velasco, 2009]; however, as far as we know, the only published reference on the notion of closeness in a logic-based context is [Burrieza et al., 2010], where the notions of closeness and distance are treated using Propositional Dynamic Logic, and their definitions are based on the concept of qualitative sum; specifically, in [Burrieza et al., 2010] two values are assumed to be close if one of them can be obtained from the other by adding a small number, and small numbers are defined as those belonging to a fixed interval.

In this work, we consider a new logic-based alternative to the notion of closeness in the context of multimodal logics. Our notion of closeness stems from the idea that two values are considered to be *close* if they are inside a prescribed area or *proximity interval*. This idea applies to the situations described in the previous paragraph, although it may differ from other intuitions based on distances since it leads to an equivalence relation, particularly, transitivity holds. Neither reflexivity nor symmetry of closeness generate any discussion among the different authors, but transitivity does. The original notion of closeness given in [Raiman, 1991] allows a certain form of transitivity which he had to tame by using a number of arbitrary limitations to avoid an unrestricted application of chaining. This arbitrariness was criticized in [Ali et al., 2003], in which a fuzzy set-based approach for handling relative orders of magnitude was introduced. It is remarkable to note that the criticism was made against the arbitrary limitations on chaining the relation, or the impossibility of considering suitable modified versions of transitivity, but not on transitivity per se.

The limitations stated above do not apply to our approach, which can be seen as founded on the notion of granularity as given in [Dubois et al., 2002], which was already

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suggested in [Zadeh, 1996]. The main difficulties in accepting closeness as a transitive relation arise in a distance-based interpretation because, then, its unrestricted use would collapse the relation since all the elements would be close. As stated above, our notion will be based not on distance but on membership to a certain element of a given set of proximity intervals, since our driving force is to define an abstract framework for dealing with natural or artificial barriers, so that the closeness relation is considered "locally." In order to clarify this idea, assume the speed limit is 100 km/h and somebody is detected by the radar driving at a higher speed v; then the driver is fined with an amount which depends on whether v belongs to, say, $(105, 120], (120, 140]$ or $(140, +\infty)$; in this example, our approach reflects the fact that two velocities can be considered to be close if and only if they are subject to the same fine, in other words, transitivity of the closeness relation is confined to elements in the same proximity interval.

On the other hand, the negligibility notion provided in this paper is a slight generalization of the one given in [Burrieza et al., 2006] where, following the line of other classical approaches, for instance [Travé-Massuyès et al., 2005], the class of 0 is considered to be just a singleton. This choice makes little sense in a qualitative approach, since considering the class of 0 to be just a singleton would require to have measures with infinite precision. Instead, we consider the qualitative class INF of *infinitesimals* which, of course, will be all close to each other. Note that these infinitesimals will be interpreted as numbers indistinguishable from 0 in the sense that their difference cannot be measured, not in the sense of hyperreal numbers.

In this work, we introduce a multimodal logic for order of magnitude reasoning which manages the notions of closeness and negligibility, then an axiom system is introduced which is proved to be sound and complete.

2. Preliminary definitions

We will consider a subset of real numbers $(\mathbb{S}, <)$ divided into the following qualitative classes:

$$\begin{array}{lll} \text{NL} = (-\infty, -\gamma) & & \text{PS} = (+\alpha, +\beta] \\ \text{NM} = [-\gamma, -\beta) & \text{INF} = [-\alpha, +\alpha] & \text{PM} = (+\beta, +\gamma] \\ \text{NS} = [-\beta, -\alpha) & & \text{PL} = (+\gamma, +\infty) \end{array}$$

Note that all the intervals are defined in terms of the landmarks $\alpha, \beta, \gamma \in \mathbb{S}$ and are considered relative to $\mathbb{S}$.

The labels correspond to "negative large" (NL), "negative medium"(NM), "negative small"(NS), "infinitesimals"(INF), "positive small" (PS), "positive medium" (PM) and "positive large" (PL). It is worth to note that this classification is slightly more general than the standard one [Travé-Massuyès et al., 2005], since the qualitative class containing the element 0, i.e. INF, needs not be a singleton; this allows for considering values very close to zero as null values in practice, which is more in line with a qualitative approach where accurate measurements are not always possible.

We will consider each qualitative class to be divided into disjoint intervals called *proximity intervals*, as shown in Figure 1. The qualitative class INF is itself one proximity interval.

Definition 1. *Let* $(\mathbb{S}, <)$ *be the set of numbers introduced above.*

- *An* r-proximity structure *is a finite set* $\mathcal{I}(\mathbb{S}) = \{I_1, I_2, \dots, I_r\}$ *of intervals in* $\mathbb{S}$*, such that:*

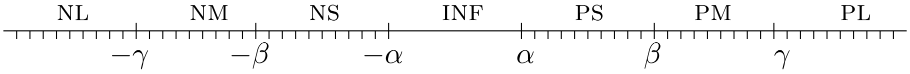

Figure 1. Proximity intervals.

- *(1) For all $I_i, I_j \in \mathcal{I}(\mathbb{S})$, if $i \neq j$, then $I_i \cap I_j = \varnothing$.*
- *(2) $I_1 \cup I_2 \cup \cdots \cup I_r = \mathbb{S}$.*
- *(3) For all $x, y \in \mathbb{S}$ and $I_i \in \mathcal{I}(\mathbb{S})$, if $x, y \in I_i$, then x, y belong to the same qualitative class.*
- *(4) $\textsc{inf} \in \mathcal{I}(\mathbb{S})$.*

- *Given a proximity structure $\mathcal{I}(\mathbb{S})$, the binary relation of closeness $\mathfrak{c}$ is defined, for all $x, y \in \mathbb{S}$, as follows: $x\,\mathfrak{c}\,y$ if and only if there exists $I_i \in \mathcal{I}(\mathbb{S})$ such that $x, y \in I_i$.*

Notice that, by definition, the number of proximity intervals is finite, regardless of the cardinality of the set $\mathbb{S}$. This choice is justified by the applications (the number of values we can consider is always finite) and the nature of the measuring devices that after reaching a certain limit, they do not distinguish among nearly equal amounts; for instance, consider the limits to represent numbers in a pocket calculator, thermometer, speedometer, etc.

The informal notion of negligibility we will use in this paper is the following: x is said to be *negligible* with respect to y if and only if either (i) x is infinitesimal and y is not, or (ii) x is small (but not infinitesimal) and y is *sufficiently large*. Formally:

Definition 2. *The binary relation of negligibility $\mathfrak{n}$ is defined on $(\mathbb{S}, <)$ as $x\,\mathfrak{n}\,y$ if and only if one of the following situations holds:*

(i) $x \in \textsc{inf}$ *and* $y \notin \textsc{inf}$,
(ii) $x \in \textsc{ns} \cup \textsc{ps}$ *and* $y \in \textsc{nl} \cup \textsc{pl}$.

3. A logic for closeness

In this section, we will use as special modal connectives $\overrightarrow{\Box}$ and $\overleftarrow{\Box}$ to deal with the usual ordering $<$, so $\overrightarrow{\Box}A$ and $\overleftarrow{\Box}A$ have the informal readings: A *is true for all numbers greater than the current one* and A *is true for all numbers less than the current one*, respectively. Two other modal operators will be used, $\boxed{\mathsf{c}}$ for closeness, where the informal reading of $\boxed{\mathsf{c}}A$ is: A *is true for all numbers close to the current one*, and $\boxed{\mathsf{n}}$ for negligibility, where $\boxed{\mathsf{n}}A$ means A *is true for all numbers that are negligible with respect to the current onee.*

The alphabet of the language $\mathcal{L}(MQ)^{\mathcal{P}}$ is defined by using a stock of atoms or propositional variables, $\mathcal{V}$, the classical connectives $\neg, \wedge, \vee$ and $\rightarrow$; the constants for milestones $\alpha^-, \alpha^+, \beta^-, \beta^+, \gamma^-, \gamma^+$; a finite set $\mathcal{C}$ of constants for proximity intervals, $\mathcal{C} = \{c_1, \dots, c_r\}$ [1]; the unary modal connectives $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\boxed{\mathsf{n}}$, $\boxed{\mathsf{c}}$, and the parentheses '(' and ')'. We define the formulas of $\mathcal{L}(MQ)^{\mathcal{P}}$ as follows:

$$A = p \mid \xi \mid c_i \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid \overrightarrow{\Box}A \mid \overleftarrow{\Box}A \mid \boxed{\mathsf{n}}A \mid \boxed{\mathsf{c}}A$$

where $p \in \mathcal{V}$, $\xi \in \{\alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^+, \gamma^-\}$ and $c_i \in \mathcal{C}$. In order to refer to any constant for positive milestones as α^+ we will use ξ^+ and for negative ones as β^- we will use ξ^-.

[1]There are at least as many elements in $\mathcal{C}$ as qualitative classes.

The *mirror image* of a formula A is the result of replacing in A each occurrence of $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, α^+, β^+ and γ^+ respectively by $\overleftarrow{\Box}$, $\overrightarrow{\Box}$, α^-, β^- and γ^- and reciprocally. We will use the symbols $\overrightarrow{\Diamond}$, $\overleftarrow{\Diamond}$, $\Diamond_{\mathfrak{c}}$, $\Diamond_{\mathfrak{n}}$ as abbreviations, respectively, of $\neg\overrightarrow{\Box}\neg$, $\neg\overleftarrow{\Box}\neg$, $\neg\Box_{\mathfrak{c}}\neg$ and $\neg\Box_{\mathfrak{n}}\neg$. Moreover, we will introduce $\mathsf{nl}, \ldots \mathsf{pl}$ as abbreviations for qualitative classes, for instance, ps for $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}\beta^+) \vee \beta^+$. By means of qc we denote any element of the set $\{\mathsf{nl}, \mathsf{nm}, \mathsf{ns}, \mathsf{inf}, \mathsf{ps}, \mathsf{pm}, \mathsf{pl}\}$.

The cardinality r of the set $\mathcal{C}$ of constants for proximity intervals will play an important role since it, somehow, encodes the granularity of the underlying logic. This implies that, actually, *we are introducing a family of logics which depend parametrically on* r.

Definition 3. *A* multimodal qualitative frame *for* $\mathcal{L}(MQ)^{\mathcal{P}}$ *(a* frame*, for short) is a tuple* $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$*, where:*

(1) $(\mathbb{S}, <)$ *is an ordered subset of real numbers.*
(2) $\mathcal{D} = \{+\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$ *is a set of designated points in* $\mathbb{S}$ *satisfying* $-\gamma < -\beta < -\alpha < +\alpha < +\beta < +\gamma$.
(3) $\mathcal{I}(\mathbb{S})$ *is an r-proximity structure.*
(4) $\mathcal{P}$ *is a bijection (called* proximity function*),* $\mathcal{P}\colon \mathcal{C} \longrightarrow \mathcal{I}(\mathbb{S})$*, that assigns to each proximity constant c a proximity interval.*

Definition 4. *Let* Σ *be a frame for* $\mathcal{L}(MQ)^{\mathcal{P}}$*, a* multimodal qualitative model on Σ *(a* MQ-model*, for short) is an ordered pair* $\mathcal{M} = (\Sigma, h)$*, where h is a* meaning function *(or,* interpretation*)* $h\colon \mathcal{V} \longrightarrow 2^{\mathbb{S}}$*. Any interpretation can be uniquely extended to the set of all formulas in* $\mathcal{L}(MQ)^{\mathcal{P}}$ *(also denoted by h) by means of the usual conditions for the classical Boolean connectives and the following conditions:*

$$h(\overrightarrow{\Box}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x < y\}$$

$$h(\overleftarrow{\Box}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } y < x\}$$

$$h(\Box_{\mathfrak{c}}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x\,\mathfrak{c}\,y\}$$

$$h(\Box_{\mathfrak{n}}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x\,\mathfrak{n}\,y\}$$

$$h(\alpha^+) = \{+\alpha\} \qquad h(\beta^+) = \{+\beta\} \qquad h(\gamma^+) = \{+\gamma\}$$

$$h(\alpha^-) = \{-\alpha\} \qquad h(\beta^-) = \{-\beta\} \qquad h(\gamma^-) = \{-\gamma\}$$

$$h(c_i) = \{x \in \mathbb{S} \mid x \in \mathcal{P}(c_i)\}$$

The definitions of truth*,* satisfiability *and* validity *are the usual ones.*

Now, we consider the axiom system $MQ^{\mathcal{P}}$ for the language $\mathcal{L}(MQ)^{\mathcal{P}}$, consisting of all the tautologies of classical propositional logic together with the following axiom schemata and rules of inference:

For white connectives

K1 $\overrightarrow{\Box}(A \to B) \to (\overrightarrow{\Box}A \to \overrightarrow{\Box}B)$
K2 $A \to \overrightarrow{\Box}\overleftarrow{\Diamond}A$
K3 $\overrightarrow{\Box}A \to \overrightarrow{\Box}\overrightarrow{\Box}A$
K4 $\big(\overrightarrow{\Box}(A \vee B) \wedge \overrightarrow{\Box}(\overrightarrow{\Box}A \vee B) \wedge \overrightarrow{\Box}(A \vee \overrightarrow{\Box}B)\big) \to \big(\overrightarrow{\Box}A \vee \overrightarrow{\Box}B\big)$

For constants $\xi \in \{\alpha^+, \beta^+, \gamma^+, \alpha^-, \beta^-, \gamma^-\}$

c1 $\overleftarrow{\Diamond}\xi \vee \xi \vee \overrightarrow{\Diamond}\xi$
c2 $\xi \to (\overleftarrow{\Box}\neg\xi \wedge \overrightarrow{\Box}\neg\xi)$
c3 $\gamma^- \to \overrightarrow{\Diamond}\beta^-$
c4 $\beta^- \to \overrightarrow{\Diamond}\alpha^-$
c5 $\alpha^- \to \overrightarrow{\Diamond}\alpha^+$
c6 $\alpha^+ \to \overrightarrow{\Diamond}\beta^+$

c7 $\beta^+ \rightarrow \overrightarrow{\Diamond}\gamma^+$

For proximity constants (for all $i, j \in \{1, \ldots, n\}$)

p1 $\bigvee_{i=1}^{n} c_i$
p2 $c_i \rightarrow \neg c_j$ (for $i \neq j$)
p3 $(\overleftarrow{\Diamond} c_i \wedge \overrightarrow{\Diamond} c_i) \rightarrow c_i$
p4 $\overleftarrow{\Diamond} c_i \vee c_i \vee \overrightarrow{\Diamond} c_i$

Mixed axioms (for all $i \in \{1, \ldots, n\}$)

m1 $(c_i \wedge \mathsf{qc}) \rightarrow (\overleftarrow{\Box}(c_i \rightarrow \mathsf{qc}) \wedge \overrightarrow{\Box}(c_i \rightarrow \mathsf{qc}))$
m2 $(c_i \wedge \mathsf{inf}) \rightarrow (\overleftarrow{\Box}(\mathsf{inf} \rightarrow c_i) \wedge \overrightarrow{\Box}(\mathsf{inf} \rightarrow c_i))$
m3 $\boxed{c}A \leftrightarrow \Big(A \wedge \bigvee_{i=1}^{r} \Big(c_i \wedge \overleftarrow{\Box}(c_i \rightarrow A) \wedge \overrightarrow{\Box}(c_i \rightarrow A)\Big)\Big)$
m4 $\boxed{n}A \leftrightarrow \Big(\Big(\mathsf{inf} \rightarrow (\overleftarrow{\Box}(\neg\mathsf{inf} \rightarrow A) \wedge \overrightarrow{\Box}(\neg\mathsf{inf} \rightarrow A))\Big) \wedge$
$\Big((\mathsf{ns} \vee \mathsf{ps}) \rightarrow (\overleftarrow{\Box}(\mathsf{nl} \rightarrow A) \wedge \overrightarrow{\Box}(\mathsf{pl} \rightarrow A))\Big)\Big)$

The mirror images of **K1, K2** and **K4** are also considered as axioms.

The intuitive meaning of the previous axioms is the following: **K1-K4** (and their mirror images) constitute a fragment of basic linear-time temporal logic; **c1** and **c2** state the existence and the unicity of the milestones in a frame, respectively; **c3-c7** state the relative ordering of these milestones. Axioms **p1** and **p2** state the existence and unicity, respectively, of proximity intervals; **p3** states that all points denoted by a proximity constant form an interval; **p4** states that every proximity constant denotes some proximity interval. **m1** states that the length of a qualitative class QC fully covers a given proximity interval. **m2** is specific to deal with INF, and states that this class is totally covered by a proximity interval (in combination with **m1**, this axiom implies that INF constitutes itself a proximity interval). **m3** states that $\boxed{c}A$ is true in x if and only if A is true in all the points which are close to x, namely, in all the points belonging to the same proximity interval labeled by some proximity constant c_i. **m4** specifies $\boxed{n}$ in terms of the properties of the negligibility relation in Definition 2. The fact that **m3** and **m4** enable the representation of closeness and negligibility in terms of white connectives and constants allows us to use, from now on, only white connectives and constants.

Rules of inference:

(MP) Modus Ponens for $\rightarrow$.
(N$\overrightarrow{\Box}$) If $\vdash A$ then $\vdash \overrightarrow{\Box}A$.
(N$\overleftarrow{\Box}$) If $\vdash A$ then $\vdash \overleftarrow{\Box}A$.

The syntactical notions of *theorem* and *proof* for $MQ^{\mathcal{P}}$ are defined as usual.

4. Completeness

Soundness is straightforward, since it is easy to check that all the axioms are valid formulas and the inference results preserve validity. In order to prove the completeness, we will follow the step-by-step method, which is a Henkin-style proof, see [Burgess, 1984]. The idea is to show that for any consistent formula A, a model for A can be built, and this is done by successive finite approximations.

It is worth to note that, although the spirit of the completeness proof is to follow a well-known method, the actual construction of the successive finite approximations has a number of particular problems, mainly related to the need of the proximity functions within a frame.

4.1 *Preliminary results and definitions*

In this section, we introduce a number of lemmas about maximal consistency which will be needed late, together with the specific notions for the actual application of the step-by-step method.

The notions of *consistency* and *maximal consistency* for $MQ^{\mathcal{P}}$ are the usual ones, and some familiarity with the basic properties of maximal consistent sets (*mc-sets*) will be assumed; $\mathcal{MC}$ will denote the set of all mc-sets of formulas. The following definition introduces two relations between mc-sets:

Definition 5. *The relations $\rhd$ and $\sim$ are defined on $\mathcal{MC}$ as follows:*

- $\Gamma_1 \rhd \Gamma_2$ *if and only if* $\{A \mid \overrightarrow{\Box} A \in \Gamma_1\} \subseteq \Gamma_2$.
- $\Gamma_1 \sim \Gamma_2$ *if and only if* $\Gamma_1 \rhd \Gamma_2$ *or* $\Gamma_1 = \Gamma_2$ *or* $\Gamma_2 \rhd \Gamma_1$.

The proofs of the following lemmas can be obtained directly from the axioms and the definitions.

Lemma 6.

(1) $\Gamma_1 \rhd \Gamma_2$ *if and only if* $\{A \mid \overleftarrow{\Box} A \in \Gamma_2\} \subseteq \Gamma_1$.
(2) $\Gamma_1 \rhd \Gamma_2$ *iff* $\{\overrightarrow{\Diamond} A \mid A \in \Gamma_2\} \subseteq \Gamma_1$ *iff* $\{\overleftarrow{\Diamond} A \mid A \in \Gamma_1\} \subseteq \Gamma_2$.
(3) $\rhd$ *is a transitive relation on* $\mathcal{MC}$.
(4) *If* $\Gamma_1 \rhd \Gamma_2$ *and* $\Gamma_1 \rhd \Gamma_3$, *then* $\Gamma_2 \sim \Gamma_3$, *for all* $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$.
(5) *If* $\Gamma_2 \rhd \Gamma_1$ *and* $\Gamma_3 \rhd \Gamma_1$, *then* $\Gamma_2 \sim \Gamma_3$, *for all* $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$.

The following lemma is specific; its proof is straightforward from the axioms for proximity constants.

Lemma 7.

(1) *Given* $\Gamma \in \mathcal{MC}$ *there is exactly one proximity constant* $c \in \mathcal{C}$ *such that* $c \in \Gamma$.
(2) *For all* $\Gamma_i \in \mathcal{MC}$ *and* $c \in \mathcal{C}$, *if* $\Gamma_1 \rhd \Gamma_2 \rhd \Gamma_3$ *and* $c \in \Gamma_1, \Gamma_3$, *then* $c \in \Gamma_2$.

Lemma 8 (Lindenbaum Lemma)**.** *Any consistent set of formulas in* $MQ^{\mathcal{P}}$ *can be extended to an mc-set in* $MQ^{\mathcal{P}}$.

Lemma 9. *Assume* $\Gamma_1 \in \mathcal{MC}$. *Then:*

(1) *If* $\overrightarrow{\Diamond} A \in \Gamma_1$, *then there exists* $\Gamma_2 \in \mathcal{MC}$ *such that* $\Gamma_1 \rhd \Gamma_2$ *and* $A \in \Gamma_2$.
(2) *If* $\overleftarrow{\Diamond} A \in \Gamma_1$, *then there exists* $\Gamma_2 \in \mathcal{MC}$ *such that* $\Gamma_2 \rhd \Gamma_1$ *and* $A \in \Gamma_2$.

The specific construction of the successive approximations of the required model for a consistent formula A forces us to consider the following weaker version of the notion of frame:

Definition 10. *Given a denumerable infinite set* $\mathcal{S}$, *a* partial frame *is a tuple* $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$ *where* $\mathbb{S}$ *is a subset of* $\mathcal{S}$, $\mathcal{D}$ *is a set of designated points in* $\mathbb{S}$, $<$ *is a total strict ordering on* $\mathbb{S}$, $\mathcal{I}(\mathbb{S})$ *is a proximity structure, and* $\mathcal{P}\colon \mathcal{C} \to \mathcal{I}(\mathbb{S})$ *is a partial* *bijective function*[1] *where $\mathcal{C}$ is the set of proximity constants.*

The set of all partial frames will be denoted by $\Xi_{\mathcal{S}}$.

Remark. *Note that partial frames satisfy the conditions in Definition 3, with the exception that $\mathcal{P}$ is a partial bijective function.*

As usual in the step-by-step method, we introduce now the definition of trace and conditional associated (in this case) to a partial frame.

Definition 11. *Let $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$ be a partial frame.*

(1) A trace *of Σ is a function $f_\Sigma : \mathbb{S} \longrightarrow \mathcal{MC}$.*

(2) A trace of Σ, f_Σ, is called:

- Coherent *if it satisfies for all $x, y \in \mathbb{S}$:*
  - *(a) $\xi \in f_\Sigma(\xi)$* [1]
  - *(b) If $x < y$, then $f_\Sigma(x) \rhd f_\Sigma(y)$*
  - *(c) Let $c_i \in \mathcal{C}$ and $I \in \mathcal{I}(\mathbb{S})$. If $c_i \in f_\Sigma(x)$ and $x \in I$, then $\mathcal{P}(c_i) = I$.*
- Full *if it is coherent and, for all formulas A, and all $x \in \mathbb{S}$, it satisfies the following conditions:*
  - *(a) if $\overrightarrow{\Diamond} A \in f_\Sigma(x)$, there exists y such that $x < y$ and $A \in f_\Sigma(y)$*
  - *(b) if $\overleftarrow{\Diamond} A \in f_\Sigma(x)$, there exists y such that $y < x$ and $A \in f_\Sigma(y)$*

The expressions (a) (resp., (b)) are called prophetic *(resp.,* historic*) conditionals for f_Σ and x. A prophetic conditional is said to be* active *if $\overrightarrow{\Diamond} A \in f_\Sigma(x)$, but there is no y such that $x < y$ and $A \in f_\Sigma(y)$; otherwise, the conditional is said to be* exhausted. *For conditionals of type historic the definitions are similar.*

The trace lemma below follows easily by induction on the complexity of A.

Lemma 12 (Trace lemma)**.** *Let f_Σ be a full trace of a frame Σ. Let h be an interpretation assigning to each propositional variable p the set $h(p) = \{x \in \mathbb{S} \mid p \in f_\Sigma(x)\}$. Then, for any formula A we have $h(A) = \{x \in \mathbb{S} \mid A \in f_\Sigma(x)\}$.*

4.2 *Step-by-step method*

We have just introduced the necessary technical terms in order to formally state the different steps in the proof. The Trace Lemma 12 will provide the model of a consistent formula A, but the application of the lemma requires a full trace of a frame Σ. Actually, it is the construction of this frame, and its corresponding full trace, what is done in a step-by-step manner, as given in the Exhausting Lemma below.

First of all, in order to start the step by step construction, we will consider an indexed denumerable infinite set $\mathcal{S} = \{x_i \mid i \in \mathbb{N}\}$ whose elements will be used to build the elements in a sequence of finite and partial frames $\{\Sigma_n\}_{n \in \mathbb{N}}$.

The initial partial frame and its trace are built upon mc-sets containing the formula A and the milestones. As, in principle, the trace in this initial step need not be full, a new step has to be considered. Each of these steps adds a new point where certain previously active conditional is exhausted. The countable union of these finite and partial frames will be the desired frame Σ, and the union of their respective traces will be the full trace f_Σ.

The following lemma inductively states how to build the next step Σ_{n+1} from Σ_n in the previously sketched construction. It is worth to state that the main technical

[1]This means that the restriction of $\mathcal{P}$ to its domain is a bijection.

[1]We are abusing the notation by using the metasymbol ξ to refer to the milestones indistinctly in the syntax and in the semantics. This convention will be applied hereafter whenever no ambiguity occurs.

difficulty in every step is to prove that the resulting construction actually satisfies the definition of partial frame given above; essentially, this amounts to checking that the properties of the proximity structure (see Definition 1) still hold in the new setting.

Lemma 13 (Exhausting lemma). *Let f_Σ be a coherent trace of a finite and partial frame Σ, and suppose that there is a conditional for f_Σ which is active. Then, there is a finite and partial frame Σ', extending Σ, and a coherent trace $f_{\Sigma'}$ extending f_Σ, such that this conditional for $f_{\Sigma'}$ is exhausted.*

Proof. Let $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$ be a frame, f_Σ a coherent trace of a finite and partial frame Σ and $x \in \mathbb{S}$. Consider an active prophetic conditional such that we have $\overrightarrow{\Diamond} A \in f_\Sigma(x)$, but there exists no y such that $x < y$ and $A \in f_\Sigma(y)$. By Lemma 9(1), there exists Γ such that $f_\Sigma(x) \rhd \Gamma$ and $A \in \Gamma$. We define an extension Σ' of Σ containing a new point $y \in \mathcal{S} \smallsetminus \mathbb{S}$ to which Γ is assigned to preserve coherence; that is, we have $\mathbb{S}' = \mathbb{S} \cup \{y\}$ and $f_{\Sigma'} = f_\Sigma \cup \{(y, \Gamma)\}$. Now we reason by induction on the number l of successors of x in $\mathbb{S}$:

Base case: If $l = 0$, then $+\gamma \le x$ and then y is introduced in PL. By Lemma 7(1) there are proximity constants c, c_x such that $c \in \Gamma$ and $c_x \in f_\Sigma(x)$ and they are unique. Given $I \in \mathcal{I}(\mathbb{S})$, consider $x \in I$. Now we have two situations:

If $c = c_x$ We define

- $\mathcal{I}(\mathbb{S}') = (\mathcal{I}(\mathbb{S}) \smallsetminus \{I\}) \cup \{I'\}$, where $I' = I \cup \{y\}$
- $<' = < \cup \{(x, y)\} \cup \{(z, y) \mid z < x\}$
- $\mathcal{P}' = (\mathcal{P} \smallsetminus \{(c, I)\}) \cup \{(c, I')\}$

Now we prove that the structure Σ' with the new definitions above belongs to the class of partial frames $\Xi_{\mathcal{S}}$.

We start with the conditions 1-4 of Definition 1 to show that $\mathcal{I}(\mathbb{S}')$ is a proximity structure. The proof of conditions 1 and 2 is immediate. With respect to condition 3, we only have to show that y belongs to the same qualitative class than x; if $+\gamma < x$, since $x <' y$, then both $x, y \in$ PL; on the other hand, the case $x = +\gamma$ cannot hold since, by coherence of f_Σ, we have $\gamma^+ \in f_\Sigma(x)$ and as $f_\Sigma(x) \rhd \Gamma$, then $\overleftarrow{\Diamond} \gamma^+ \in \Gamma$; now, we have $\gamma^+ \wedge c \in f_\Sigma(x)$, thus by axiom m1 we have $\overrightarrow{\Box}(c \to \mathsf{pm}) \in f_\Sigma(x)$ and, taking into account $f_\Sigma(x) \rhd \Gamma$ once again, we obtain $\mathsf{pm} \in \Gamma$ reaching a contradiction with $\overleftarrow{\Diamond} \gamma^+ \in \Gamma$. For condition 4, just note that INF is the same class in Σ and Σ', hence the result. Finally, the set $\mathcal{D}$ is the same in Σ and Σ', $<'$ is clearly an strict order relation on $\mathbb{S}'$ so defined, and $\mathcal{P}'$ is a partial bijection by its construction.

Property (b) of the coherence of $f_{\Sigma'}$ follows from Lemma 6(3), properties (a) and (c) are immediate. This same observation applies to the coherence of all the following cases.

If $c \neq c_x$, the extended structure is the following:

- $\mathcal{I}(\mathbb{S}') = \mathcal{I}(\mathbb{S}) \cup \{I'\}$, where I' is the singleton interval $[y]$
- $<' = < \cup \{(x, y)\} \cup \{(z, y) \mid z < x\}$
- $\mathcal{P}' = \mathcal{P} \cup \{(c, I)\}$.

Inductive case: Assume the result is true for $l - 1$.

Let x^s be the immediate successor of x. If $\overrightarrow{\Diamond} A \in f_\Sigma(x^s)$ we reduce to the induction hypothesis $l-1$ by replacing x by x^s; otherwise, $\neg A \wedge \overrightarrow{\Box} \neg A \in f_\Sigma(x^s)$, hence $\Gamma \rhd f_\Sigma(x^s)$. Moreover, by Lemma 7(1) there are proximity constants c, c_x, c_{x^s} such that $c \in \Gamma$, $c_x \in f_\Sigma(x)$ and $c_{x^s} \in f_\Sigma(x^s)$, and they are unique.

Now, we have to consider several cases:

(A) $x, x^s \in I$ for some $I \in \mathcal{I}(\mathbb{S})$. Then, by coherence of f_Σ (Definition 11), we obtain

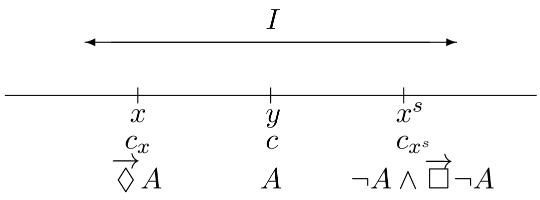

Figure 2. Inductive Case (A)

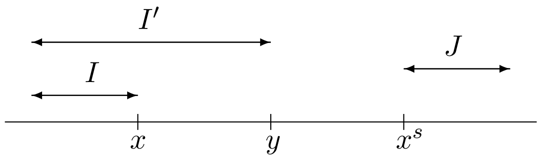

Figure 3. Inductive Case ($b1$)

$\mathcal{P}(c_x) = \mathcal{P}(c_{x^s}) = I$ and, as $\mathcal{P}$ is a bijection, $c_x = c_{x^s}$. Now, since $f_\Sigma(x) \rhd \Gamma \rhd f_\Sigma(x^s)$, we obtain $c_x = c_{x^s} = c$ by Lemma 7(2). Thus (see Fig. 2):

- $\mathcal{I}(\mathbb{S}') = (\mathcal{I}(\mathbb{S}) \smallsetminus \{I\}) \cup \{I'\}$, where $I' = I \cup \{y\}$
- $<' = < \cup \{(x, y), (y, x^s)\} \cup \{(z, y) \mid z < x\} \cup \{(y, z) \mid x^s < z\}$
- $\mathcal{P}' = (\mathcal{P} \smallsetminus \{(c, I)\}) \cup \{(c, I')\}$

In order to prove that Σ' belongs to the class $\Xi_{\mathcal{S}}$ we have just to prove the conditions of Definition 1 for $\mathcal{I}(\mathbb{S}')$. Conditions 1 and 2 are obviously satisfied; for condition 3, as $x, x^s \in I$, then x, x^s belong to the same qualitative class since Σ is in $\Xi_{\mathcal{S}}$. Now y is inserted between them, so it is easy to see by the construction that y belongs to the same qualitative class as x and x^s. For condition 4, provided that $I' = I \cup \{y\}$, it is obvious that INF is a proximity interval.

(B) $x \in I$ and $x^s \in J$ for some $I, J \in \mathcal{I}^{\mathbb{S}}$ with $I \neq J$. This means, by coherence of f_Σ and bijectivity of $\mathcal{P}$, that $c_x \neq c_{x^s}$. Now, we have three subcases:

($b1$) $c_x = c \neq c_{x^s}$
($b2$) $c_x \neq c = c_{x^s}$
($b3$) $c_x \neq c \neq c_{x^s}$

The cases ($b1$) and ($b2$) are analogous, so we will just prove case ($b1$) (see Fig. 3). Firstly, let us define Σ' as in case (A) above; we just check that Σ satisfies the conditions in Definition 1. Conditions 1 and 2 are immediate; for Condition 3, given a qualitative class QC such that $x \in$ QC, as $x \in I' = I \cup \{y\}$, it is enough to show that $y \in$ QC, and we will consider the different the possibilities for x:

- It cannot be case that $x = -\alpha$, since its successor x^s would be less than or equal to $+\alpha$ and $I = J$ (contradiction).
- Moreover, x cannot be ξ, where $\xi \in \{+\alpha, +\beta, +\gamma\}$: in effect, as $c_x = c$, $f_\Sigma(\xi) = f_\Sigma(x)$ and $\xi^+ \in f_\Sigma(\xi)$, then $c \wedge \xi^+ \in f_\Sigma(x)$ and by theorem $(c \wedge \xi^+) \to \overrightarrow{\Box}\neg c$ we have $\overrightarrow{\Box}\neg c \in f_\Sigma(x)$.
  Now, as $f_\Sigma(x) \rhd \Gamma$ we finally obtain $c \notin \Gamma$ reaching a contradiction.
- If any other situation, obviously, y belongs to QC.

For condition 4, once again provided that $I' = I \cup \{y\}$, it is obvious that INF is a proximity interval.

In case ($b3$) (see Fig. 4), we introduce a new proximity interval K between I and J

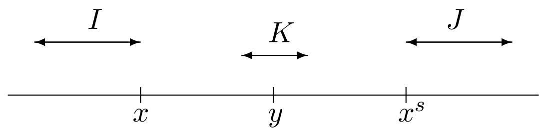

Figure 4. Inductive Case ($b3$)

consisting of a unique point y between x and x^s as above. In this case, x is the upper bound of the interval I and x^s is the lower bound of the interval J. Thus, we define $\mathcal{I}(\mathbb{S}') = \mathcal{I}(\mathbb{S}) \cup \{K\}$, where $K = [y]$ (a singleton interval) and $\mathcal{P}' = \mathcal{P} \cup \{(c, K)\}$.

It is easy to verify that Σ' as defined is a partial frame. The proof of conditions 1, 2 and 3 of Definition 1 is immediate. With respect to condition 4, note that INF is the same class in Σ and Σ', because $y \notin$ INF, since $x \in I$ and $x^s \in J$ and $I \neq J$, so these points cannot be the extreme points of the interval $[-\alpha, +\alpha]$.

The case for an active historic conditional is treated in a similar way.

□

Theorem 14 (Completeness)**.** *If A is valid formula of $\mathcal{L}(MQ)^{\mathcal{P}}$, then A is a theorem of $MQ^{\mathcal{P}}$.*

Proof. We prove that any consistent formula A of $\mathcal{L}(MQ)^{\mathcal{P}}$ is satisfiable.

In order to carry out the construction of a model for A, let us first consider the class of frames $\Xi_{\mathcal{S}}$ of Definition 10.

Given an enumeration of formulas $A_0, A_1, \ldots, A_n, \ldots$ of the language $\mathcal{L}(MQ)^{\mathcal{P}}$, and taking into account the set $\mathcal{S} = \{x_i \mid i \in \mathbb{N}\}$, we can also give an enumeration to each prophetic (historic) conditional.

Now, since A is consistent, there exists a mc-set Γ containing A. We will start with a finite and partial frame $\Sigma_0 = (\mathbb{S}_0, \mathcal{D}, <_0, \mathcal{I}(\mathbb{S}_0), \mathcal{P}_0)$ which is the basis of the construction of the required frame, and a corresponding trace f_{Σ_0}. By Lemma 7(1), we have that Γ contains at most one of the constants $\alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^+, \gamma^-$. Now, the construction of Σ_0 depends on whether Γ contains one constant or not:

- If none of the milestones are in Γ, we define Σ_0 in a coherence-preserving way by using the theorem $\mathsf{nl} \vee \mathsf{nm} \vee \mathsf{ns} \vee \mathsf{inf} \vee \mathsf{ps} \vee \mathsf{pm} \vee \mathsf{pl}$.

  Taking into account that Γ does not contain milestones, the previous disjunction reduces in Γ to one of the following cases:

Case	Reduces to	Case	Reduces to
nl	$\overrightarrow{\Diamond}\gamma^-$	ps	$\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}\beta^+$
nm	$\overleftarrow{\Diamond}\gamma^- \wedge \overrightarrow{\Diamond}\beta^-$	pm	$\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}\beta^+$
ns	$\overleftarrow{\Diamond}\beta^- \wedge \overrightarrow{\Diamond}\alpha^-$	pl	$\overleftarrow{\Diamond}\gamma^+$
inf	$\overleftarrow{\Diamond}\alpha^- \wedge \overrightarrow{\Diamond}\alpha^+$		

  For instance, if $\overrightarrow{\Diamond}\gamma^- \in \Gamma$, then by using Lemma 9(1) and axioms c3-c7, there exist $\Gamma_1, \ldots, \Gamma_6 \in \mathcal{MC}$ such that $\gamma^- \in \Gamma_1$, $\beta^- \in \Gamma_2$, $\alpha^- \in \Gamma_3$, $\alpha^+ \in \Gamma_4$, $\beta^+ \in \Gamma_5$, $\gamma^+ \in \Gamma_6$, and $\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \Gamma_3 \rhd \Gamma_4 \rhd \Gamma_5 \rhd \Gamma_6$. Moreover, by Lemma 7(1) and theorem $\mathsf{qc} \to \neg\mathsf{qc}'$ (for $\mathsf{qc} \neq \mathsf{qc}'$) there are constants $c_0 \in \Gamma$, $c_1 \in \Gamma_1$, $c_2 \in \Gamma_2$, $c_3 \in \Gamma_3, c_3 \in \Gamma_4$, $c_4 \in \Gamma_5$ and $c_5 \in \Gamma_6$, which are all different from each other. We define Σ_0 as follows:

  – $\mathbb{S}_0 = \{x_0, +\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$, where $x_0 \in \mathcal{S}$;

– $\mathcal{D} = \{+\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$;
– $<_0$ is the transitive closure of
$\{((x_0, -\gamma), (-\gamma, -\beta), (-\beta, -\alpha), (-\alpha, +\alpha), (+\alpha, +\beta), (+\beta, +\gamma)\}$;
– $\mathcal{I}(\mathbb{S}_0) = \{[x_0], [-\gamma], [-\beta], [-\alpha, +\alpha], [+\beta], [+\gamma]\}$;
– $\mathcal{P}_0 = \{(c_0, [x_0]), (c_1, [-\gamma]), (c_2, [-\beta]), (c_3, [-\alpha, +\alpha]), (c_4, [+\beta]), (c_5, [+\gamma])\}$;

and the corresponding trace f_{Σ_0} is defined by

$$f_{\Sigma_0} = \{(x_0, \Gamma)\}, (-\gamma, \Gamma_1), (-\beta, \Gamma_2), (-\alpha, \Gamma_3), (+\alpha, \Gamma_4), (+\beta, \Gamma_5), (+\gamma, \Gamma_6)\},$$

which is clearly coherent.

The other cases are treated similarly.

- If Γ contains a milestone, we proceed similarly as in the previous case, obtaining $\gamma^- \in \Gamma_1$, $\beta^- \in \Gamma_2$, $\alpha^- \in \Gamma_3$, $\alpha^+ \in \Gamma_4$, $\beta^+ \in \Gamma_5$, $\gamma^+ \in \Gamma_6$, where Γ coincides with some Γ_i $(1 \leq i \leq 6)$, and defining Σ_0 as follows
  – $\mathbb{S}_0 = \mathcal{D} = \{+\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$;
  – $<_0$ is the transitive closure of
  $\{(-\gamma, -\beta), (-\beta, -\alpha), (-\alpha, +\alpha), (+\alpha, +\beta), (+\beta, +\gamma)\}$;
  – $\mathcal{I}(\mathbb{S}_0) = \{[-\gamma], [-\beta], [-\alpha, +\alpha], [+\beta], [+\gamma]\}$;
  – $\mathcal{P}_0 = \{(c_1, [-\gamma]), (c_2, [-\beta]), (c_3, [-\alpha, +\alpha]), (c_4, [+\beta]), (c_5, [+\gamma])\}$;

  and the corresponding trace f_{Σ_0} is defined as

$$f_{\Sigma_0} = \{(-\gamma, \Gamma_1), (-\beta, \Gamma_2), (-\alpha, \Gamma_3), (+\alpha, \Gamma_4), (+\beta, \Gamma_5), (+\gamma, \Gamma_6)\},$$

which is coherent.

The base case of the construction is the frame Σ_0 defined above. Now, assume that $\Sigma_n = (\mathbb{S}_n, \mathcal{D}, <_n, \mathcal{I}(\mathbb{S}_n), \mathcal{P}_n)$ and f_{Σ_n} are defined. Then, we choose the first active conditional in the enumeration and, by the Exhausting Lemma (Lemma 13), an extension $\Sigma_{n+1} = (\mathbb{S}_{n+1}, \mathcal{D}, <_{n+1}, \mathcal{I}(\mathbb{S})_{n+1}, \mathcal{P}_{n+1}) \in \Xi_{\mathcal{S}}$ of Σ_n and an extension $f_{\Sigma_{n+1}}$ of f_{Σ_n} are constructed such that this conditional for $f_{\Sigma_{n+1}}$ is exhausted.

Now, consider $\Sigma = \bigcup \Sigma_i$ to be the union of the countable sequence of finite and partial frames Σ_i, and f_Σ its trace defined as the union of the corresponding traces f_{Σ_i}. Following the ideas in [Burgess, 1984], the construction of Σ can be proved to be a frame (we have just to take into account that $\mathcal{P}$ is a total bijective function by axioms p1 and p4, and the Exhausting Lemma) and f_Σ is full. Thus, the consistent formula A is satisfiable by applying the Trace Lemma (Lemma 12). □

5. Conclusions and future work

We have presented a multimodal logic for qualitative reasoning to deal with the concept of closeness. The idea is based on the so called proximity intervals. We proved that this logic in sound and complete. In [Burrieza et al., 2016], we have shown some examples of its expressivity in order to denote particular positions of the proximity intervals. As a future work, we consider the study of the decidability and complexity of this logic and to apply it to real scenarios.

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REFERENCES

Ali, A., Dubois, D., and Prade, H. (2003). Qualitative reasoning based on fuzzy relative orders of magnitude. *IEEE transactions on Fuzzy Systems*, 11(1):9–23.

Balbiani, P. (2016). Reasoning about negligibility and proximity in the set of all hyperreals. *Journal of Applied Non-Classical Logics*, 16:14–36.

Burgess, J. P. (1984). Basic tense logic. In *Handbook of Philosophical Logic*, pages 89–133. Springer.

Burrieza, A., Muñoz-Velasco, E., and Ojeda-Aciego, M. (2006). Order of magnitude qualitative reasoning with bidirectional negligibility. *Lecture Notes in Artificial Intelligence*, 4177:370–378.

Burrieza, A., Muñoz-Velasco, E., and Ojeda-Aciego, M. (2010). Closeness and distance in order of magnitude qualitative reasoning via PDL. *Lecture Notes in Artificial Intelligence*, 5988:71–80.

Burrieza, A., Muñoz-Velasco, E., and Ojeda-Aciego, M. (2016). A hybrid approach to closeness in the framework of order of magnitude qualitative reasoning. *Lecture Notes in Artificial Intelligence*, 9648:721–729.

Burrieza, A. and Ojeda-Aciego, M. (2005). A multimodal logic approach to order of magnitude qualitative reasoning with comparability and negligibility relations. *Fundamenta Informaticae*, 68(1):21–46.

Dubois, D., Hadj-Ali, A., and Prade, H. (2002). Granular computing with closeness and negligibility relations. In Lin, T. Y., Yao, Y. Y., and Zadeh, L. A., editors, *Data Mining, Rough Sets and Granular Computing*, pages 290–307. Physica-Verlag.

Golińska-Pilarek, J. and Muñoz-Velasco, E. (2009). Relational approach for a logic for order of magnitude qualitative reasoning with negligibility, non-closeness and distance. *Logic Journal of IGPL*, 17(4):375–394.

Raiman, O. (1991). Order of magnitude reasoning. *Artificial Intelligence*, 51:11–38.

Travé-Massuyès, L., Prats, F., Sánchez, M., and Agell, N. (2005). Relative and absolute order-of-magnitude models unified. *Annals of Mathematics and Artificial Intelligence*, 45:323–341.

Zadeh, L. A. (1996). Fuzzy sets and information granularity. In Klir, G. J. and Yuan, B., editors, *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems*, pages 433–448. World Scientific Publishing Co., Inc., River Edge, NJ, USA.