On Adjunctions between Fuzzy Preordered Sets: Necessary Conditions ⋆

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Abstract. There exists a direct relation between fuzzy rough sets and fuzzy preorders. On the other hand, it is well known the existing parallelism between Formal Concept Analysis and Rough Set Theory. In both cases, Galois connections play a central role. In this work, we focus on adjunctions (also named isotone Galois connections) between fuzzy preordered sets; specifically, we study necessary conditions that have to be fulfilled in order such an adjunction to exist.

Keywords: Galois connection, Adjunction, Preorder, Fuzzy sets

1 Introduction

Adjunctions, together with their antitone counterparts (also called Galois connections), have played an important role in computer science because its many applications, both theoretical and practical, and in mathematics because of its ability to link apparently very disparate worlds; this is why Denecke, Erné, and Wismath stated in their monograph [12] that *Galois connections provide the structure-preserving passage between two worlds of our imagination.*

Finding an adjunction (or Galois connection) between two fields is extremely useful, since it provides a strong link between both theories allowing for mutual synergistic advantages. The algebraic study of complexity of valued constraints, for instance, has been studied in terms of establishing a Galois connection [10].

This work is focused on the study of adjunctions between fuzzy (pre-)ordered structures. Both research topics are related to, on the one hand, the theory of formal concept analysis (FCA) and, on the other hand, to rough set theory. For instance, in[22] Pawlak's information systems are studied in terms of Galois connections and functional dependencies; there are also papers which develop rough extensions of FCA by using rough Galois connections, see for instance [25]; there are works which study whether certain extensions of the upper and lower approximation operators form a Galois connection [11].

There is a number of papers which study Galois connections from the abstract algebraic standpoint [1, 2, 8, 9, 14, 15, 20] and also focusing on its applications [12, 13, 24, 26–29]. In previous works [18, 19], the authors studied the

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problem of defining a right adjoint for a mapping $f\colon (A, \leq_A) \to B$ from a partially (pre)ordered set A to an unstructured set B. The natural extension of that approach is to consider a fuzzy preordered set (A, ρ_A).

In this paper, we start the study of conditions which guarantee the existence of adjunctions between sets with a fuzzy preorder. Specifically, we provide here a set of necessary conditions for an adjunction exists between (A, ρ_A) and (B, ρ_B).

2 Preliminary Definitions and Results

The most usual underlying structure for considering fuzzy extensions of Galois connections is that of residuated lattice, $\mathbb{L} = (L, \vee, \wedge, \top, \bot, \otimes, \to)$. An $\mathbb{L}$-fuzzy set is a mapping from the universe set to the membership values structure $X\colon U \to L$ where $X(u)$ means the degree in which u belongs to X. Given X and Y two $\mathbb{L}$-fuzzy sets, X is said to *be included in* Y, denoted as $X \subseteq Y$, if $X(u) \leq Y(u)$ for all $u \in U$.

An $\mathbb{L}$-fuzzy binary relation on U is an $\mathbb{L}$-fuzzy subset of $U \times U$, that is $\rho_U\colon U \times U \to L$, and it is said to be:

- *Reflexive* if $\rho_U(a, a) = \top$ for all $a \in U$.
- *Transitive* if $\rho_U(a, b) \otimes \rho_U(b, c) \leq \rho_U(a, c)$ for all $a, b, c \in U$.
- *Symmetric* if $\rho_U(a, b) = \rho_U(b, a)$ for all $a, b \in U$.
- *Antisymmetric* if $\rho_U(a, b) = \rho_U(b, a) = \top$ implies $a = b$, for all $a, b \in U$.

Definition 1 (Fuzzy poset).
An $\mathbb{L}$-fuzzy partially ordered set *is a pair* $\mathbb{U} = (U, \rho_U)$ *in which* ρ_U *is a reflexive, antisymmetric and transitive* $\mathbb{L}$*-fuzzy relation on* U*.*

A crisp ordering can be given in U *by* $a \leq_U b$ *if and only if* $\rho_U(a, b) = \top$*.*

From now on, when no confusion arises, we will omit the prefix "$\mathbb{L}$-".

Definition 2. *For every element* $a \in U$*, the extension to the fuzzy setting of the notions of upset and downset of the element* a *are defined by* $a^\uparrow, a^\downarrow\colon U \to L$ *where* $a^\downarrow(u) = \rho_U(u, a)$ *and* $a^\uparrow(u) = \rho_U(a, u)$ *for all* $u \in U$*.*

An element $a \in U$ *is a* maximum *for a fuzzy set* X *if* $X(a) = \top$ *and* $X \subseteq a^\downarrow$*. The definition of minimum is similar.*

Note that maximum and minimum elements are necessarily unique, because of antisymmetry.

Definition 3. *Let* $\mathbb{A} = (A, \rho_A)$ *and* $\mathbb{B} = (B, \rho_B)$ *be fuzzy ordered sets.*

1. *A mapping* $f\colon A \to B$ *is said to be* isotone *if* $\rho_A(a_1, a_2) \leq \rho_B(f(a_1), f(a_2))$ *for each* $a_1, a_2 \in A$*.*
2. *Moreover, a mapping* $f\colon A \to A$ *is said to be* inflationary *if* $\rho_A(a, f(a)) = \top$ *for all* $a \in A$*. Similarly. a mapping* f *is* deflationary *if* $\rho_A(f(a), a) = \top$ *for all* $a \in A$*.*

Definition 4 (Fuzzy adjunction). *Let $\mathbb{A} = (A, \rho_A)$, $\mathbb{B} = (B, \rho_B)$ be fuzzy posets, and two mappings $f: A \to B$ and $g: B \to A$. The pair (f, g) forms an adjunction between A and B, denoted $(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$ if, for all $a \in A$ and $b \in B$, the equality $\rho_A(a, g(b)) = \rho_B(f(a), b)$ holds.*

Notation 1 *From now on, we will use the following notation, for a mapping $f: A \to B$ and a fuzzy subset Y of B, the fuzzy set $f^{-1}(Y)$ is defined as $f^{-1}(Y)(a) = Y(f(a))$, for all $a \in A$.*

Finally, we recall the following theorem which states different equivalent forms to define a fuzzy adjunction.

Theorem 1 ([16]). *Let $\mathbb{A} = (A, \rho_A)$, $\mathbb{B} = (B, \rho_B)$ be fuzzy posets, and two mappings $f: A \to B$ and $g: B \to A$. The following conditions are equivalent:*

1. *$(f, g): \mathbb{A} \leftrightarrows \mathbb{B}$.*
2. *f and g are isotone, $g \circ f$ is inflationary, and $f \circ g$ is deflationary.*
3. *$f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$.*
4. *$g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$.*
5. *f is isotone and $g(b) = \max f^{-1}(b^{\downarrow})$ for all $b \in B$.*
6. *g is isotone and $f(a) = \min g^{-1}(a^{\uparrow})$ for each $a \in A$.*

Theorem 2 ([17]). *Let (A, ρ_A) be a fuzzy poset and a mapping $f: A \longrightarrow B$. Let A_f be the quotient set over the kernel relation $a \equiv_f b \iff f(a) = f(b)$. Then, there exists a fuzzy order ρ_B in B and a map $g: B \longrightarrow A$ such that $A \leftrightarrows B$ if and only if the following conditions hold:*

1. *There exists $\max[a]_f$ for all $a \in A$.*
2. *$\rho_A(a_1, a_2) \leq \rho_A(\max[a_1]_f, \max[a_2]_f)$, for all $a_1, a_2 \in A$.*

3 Building Adjunctions between Fuzzy Preordered Sets

In this section we start the generalization of Theorem 2 above to the framework of fuzzy preordered sets.

The construction will follow that given in [19] as much as possible. Therefore, we need to define a suitable fuzzy version of the p-kernel relation.

Firstly, we need to set the corresponding fuzzy notion of transitive closure of a fuzzy relation, and this is done in the definition below:

Definition 5. *Given a fuzzy relation $S: U \times U \to L$, for all $n \in \mathbb{N}$, the iterations $S^n: U \times U \to L$ are recursively defined by the base case $S^1 = S$ and, then,*

$$S^n(a, b) = \bigvee_{x \in U} \Big(S^{n-1}(a, x) \otimes S(x, b) \Big)$$

The transitive closure of S is a fuzzy relation $S^{tr}: U \times U \to L$ defined by

$$S^{tr}(a, b) = \bigvee_{n=1}^{\infty} S^n(a, b)$$

The relation $\approx_A$ allows for gettting rid of the absence of antisymmetry, by linking together elements which are 'almost coincident'; formally, the relation $\approx_A$ is defined on a fuzzy preordered set (A, ρ_A) as follows:

$$(a_1 \approx_A a_2) = \rho_A(a_1, a_2) \otimes \rho_A(a_2, a_1) \qquad \text{for } a_1, a_2 \in A$$

The kernel equivalence relation $\equiv_f$ associated to a mapping $f\colon A \to B$ is defined as follows for $a_1, a_2 \in A$:

$$(a_1 \equiv_f a_2) = \begin{cases} \bot & \text{if } f(a_1) \neq f(a_2) \\ \top & \text{if } f(a_1) = f(a_2) \end{cases}$$

Definition 6. *Let $\mathbb{A} = (A, \rho_A)$ be a fuzzy preordered set, and $f\colon A \to B$ a mapping. The* fuzzy p-kernel *relation $\cong_A$ is the fuzzy equivalence relation obtained as the transitive closure of the union of the relations $\approx_A$ and $\equiv_f$.*

Notice that the fuzzy equivalence classes $[a]_{\cong_A} : A \to L$ are fuzzy sets defined as

$$[a]_{\cong_A}(x) = (x \cong_A a) \tag{1}$$

The notion of maximum or minimum element of a fuzzy subset X of a fuzzy preordered set is the same as in Definition 2. There is an important difference which justifies the introduction of special terminology in this context: due to the absence of antisymmetry, there exists a crisp set of maxima (resp. minima) for X, which is not necessarily a singleton, which we will denote p-max(X) (resp., p-min(X)).

The following theorem states the different equivalent characterizations of the notion of adjunction between fuzzy preordered sets. As expected, the general structure of the definitions is preserved, but those concerning the actual definition of the adjoints have to be modified by using the notions of p-maximum and p-minimum.

Theorem 3 ([16]). *Let $\mathbb{A} = (A, \rho_A), \mathbb{B} = (B, \rho_B)$ be two fuzzy preordered sets, and $f\colon \mathbb{A} \to \mathbb{B}$ and $g\colon \mathbb{B} \to \mathbb{A}$ be two mappings. The following statements are equivalent:*

1. $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$.
2. *f and g are isotone, and $g \circ f$ is inflationary, $f \circ g$ is deflationary.*
3. *$f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$.*
4. *$g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$.*
5. *f is isotone and $g(b) \in \text{p-max}\, f^{-1}(b^{\downarrow})$ for all $b \in B$.*
6. *g is isotone and $f(a) \in \text{p-min}\, g^{-1}(a^{\uparrow})$ for all $a \in A$.*

The following definitions recall the notion of Hoare ordering between crisp subsets, and then introduces an alternative statement in the subsequent lemma:

Definition 7. *Consider a fuzzy preordered set (A, ρ_A), and C, D crisp subsets of A, we define the following relations*

– $(C \sqsubseteq_W D) = \bigvee_{c \in C} \bigvee_{d \in D} \rho_A(c, d)$
– $(C \sqsubseteq_H D) = \bigwedge_{c \in C} \bigvee_{d \in D} \rho_A(c, d)$
– $(C \sqsubseteq_S D) = \bigwedge_{c \in C} \bigwedge_{d \in D} \rho_A(c, d)$

Lemma 1. *Consider a fuzzy preordered set* (A, ρ_A)*, and* $X, Y \subseteq A$ *such that* $\text{p-min}\, X \neq \varnothing \neq \text{p-min}\, Y$*, then*

$$\big(\text{p-min}\, X \sqsubseteq_W \text{p-min}\, Y\big) = \big(\text{p-min}\, X \sqsubseteq_H \text{p-min}\, Y\big) = \big(\text{p-min}\, X \sqsubseteq_S \text{p-min}\, Y\big)$$

and their value coincides with $\rho_A(x, y)$ *for any* $x \in \text{p-min}\, X$ *and* $y \in \text{p-min}\, Y$

Proof. Firstly, notice that if $u_1, u_2 \in \text{p-min}\, X$, then $\rho_A(u_1, u_2) = \top$, by the definition of $\text{p-min}\, X$.

Secondly, $\rho_A(x_1, y_1) = \rho_A(x_2, y_2)$ for all $x_1, x_2 \in \text{p-min}\, X$, $y_1, y_2 \in \text{p-min}\, Y$. Indeed, $\rho_A(x_1, y_1) \geq \rho_A(x_1, x_2) \otimes \rho_A(x_2, y_1) = \top \otimes \rho_A(x_2, y_1) \geq \rho_A(x_2, y_2) \otimes \rho_A(y_2, y_1) = \rho_A(x_2, y_2)$. Analogously, $\rho_A(x_2, y_2) \geq \rho_A(x_1, y_1)$. □

We can now state the main contribution of this work: some necessary conditions for the existence of fuzzy adjunctions between fuzzy preordered sets. The result obtained resembles that in the crisp case [19]:

Theorem 4. *Given fuzzy preordered sets* $\mathbb{A} = (A, \rho_A)$ *and* $\mathbb{B} = (B, \rho_B)$*, and mappings* $f\colon A \to B$ *and* $g\colon B \to A$ *such that* $(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$ *then*

1. $gf(A) \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
2. $\text{p-min}(UB[a]_{\cong_A} \cap gf(A)) \neq \varnothing$*, for all* $a \in A$.
3. $\rho_A(a_1, a_2) \;\leq\; \Big(\text{p-min}(UB[a_1]_{\cong_A} \cap gf(A)) \;\sqsubseteq\; \text{p-min}(UB[a_2]_{\cong_A} \cap gf(A))\Big)$ *for all* $a_1, a_2 \in A$.

Proof. 1. Consider $a \in A$, and let us show that $gf(a) \in \text{p-max}[gf(a)]_{\cong_A}$.
By definition of p-maximum element of a fuzzy set, we have to prove that it is an element of its core, and also an upper bound. To begin with, it is straightforward that $[gf(a)]_{\cong_A}(gf(a)) = \top$, therefore we have just to prove the inclusion $[gf(a)]_{\cong_A} \subseteq (gf(a))^{\downarrow}$ between fuzzy sets, that is, we have to prove $[gf(a)]_{\cong_A}(u) \leq \rho_A(u, gf(a))$ for all $u \in A$.
Recall that relation $\cong_A$ has been defined as the transitive closure of the join $\approx_A \cup \equiv_f$, which we will denote R hereafter. Specifically, by using the definition of transitive closure (Defn. 5) and properties of the supremum, we will prove by induction that any iteration R^n satisfies the following inequality:

$$gf(a) R^n u \leq \rho_A(u, gf(a)) \qquad \forall u \in A \tag{2}$$

– For $n = 1$ and $u \in A$, let us prove the inequality by using the definition of the relations involved we obtain

$$\begin{aligned} gf(a)Ru &= (gf(a) \approx_A u) \vee (gf(a) \equiv_f u) \\ &= (\rho_A(gf(a), u) \otimes \rho_A(u, gf(a))) \vee (gf(a) \equiv_f u) \\ &\leq \rho_A(u, gf(a)) \vee (gf(a) \equiv_f u) \end{aligned}$$

Depending on the value of $gf(a) \equiv_f u$, which is a crisp relation, there are just two possible cases to consider, and both are straightforward:
If $(gf(a) \equiv_f u) = \bot$, there is nothing to prove, as the previous inequality collapses to inequality (2).
If $(gf(a) \equiv_f u) = \top$, inequality (2) degenerates to a tautology since the upper bound turns out to be $\top$. In effect, we have $fgf(a) = f(u)$ by definition of the kernel relation $\equiv_f$, in addition, using the hypothesis $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$, we have that

$$\begin{aligned} \rho_A(u, gf(a)) &= \rho_B(f(u), f(a)) \\ &= \rho_B(fgf(a), f(a)) = \rho_A(gf(a), gf(a)) = \top \end{aligned}$$

– Assume inequality (2) holds for $n-1$. By definition of the n-th iteration of a fuzzy relation, and the induction hypothesis, we have that

$$\begin{aligned} gf(a)R^n u &= \bigvee_{x \in A} \Big(gf(a)R^{n-1}x \otimes xRu \Big) \\ &\leq \bigvee_{x \in A} \Big(\rho_A(x, gf(a)) \otimes \big((x \approx_A u) \vee (x \equiv_f u) \big) \Big) \\ &= \bigvee_{x \in A} \Big(\rho_A(x, gf(a)) \otimes \big((\rho_A(x, u) \otimes \rho_A(u, x)) \vee (x \equiv_f u) \big) \Big) \\ &\leq \bigvee_{x \in A} \Big(\rho_A(x, gf(a)) \otimes \big(\rho_A(u, x) \vee (x \equiv_f u) \big) \Big). \end{aligned}$$

Now, similarly to case $n = 1$, for every disjunct above there are two cases depending on the outcome of the kernel relation:
If $(x \equiv_f u) = \bot$, by commutativity of $\otimes$ and transitivity of ρ_A, then the corresponding disjunct simplifies to $\rho_A(u, gf(a))$.
If $(x \equiv_f u) = \top$, then the disjunct simplifies to $\rho_A(x, gf(a))$; but, moreover, using the fact that $f(x) = f(u)$ and the hypothesis $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$, we have that

$$\rho_A(x, gf(a)) = \rho_B(f(x), f(a)) = \rho_B(f(u), f(a)) = \rho_A(u, gf(a))$$

Summarizing, inequation (2) holds for all n and, by definition of the transitive closure, we have $[gf(a)]_{\cong_A}(u) \leq \rho_A(u, gf(a))$ for all $u \in A$.

2. Note that the set of upper bounds and the image involved in this condition are crisp sets. Specifically, we will prove that $gf(a)$ belongs to the intersection p-min$(UB[a]_{\cong_A} \cap g(f(A)))$.

To begin with, we have to check that $gf(a) \in UB[a]_{\cong_A} \cap gf(A)$. As it is obvious that $gf(a) \in gf(A)$, we have just to show $gf(a) \in UB[a]_{\cong_A}$, that is, $gf(a)$ is an upper bound of the fuzzy set $[a]_{\cong_A}$. We have to prove that $(a \cong_A u) \le \rho_A(u, gf(a))$ holds for all $u \in A$. Again, by using the definition of $\cong_A$ as transitive closure, and properties of the supremum, it is sufficient to show that

$$aR^n u \le \rho_A(u, gf(a)) \qquad \forall u \in A \tag{3}$$

From now on, the proof follows the line of the previous item.

- For $n = 1$, and $u \in A$, we have that

$$\begin{aligned} aRu &= (a \approx_A u) \vee (a \equiv_f u) \\ &= \big(\rho_A(a, u) \otimes \rho_A(u, a)\big) \vee (a \equiv_f u) \\ &\le \rho_A(u, a) \vee (a \equiv_f u). \end{aligned}$$

  Considering the two possible values of $a \equiv_f u$:
  If $(a \equiv_f u) = \bot$, by monotonicity of f and the adjunction property, we have that

$$\rho_A(u, a) \le \rho_B(f(u), f(a)) = \rho_A(u, gf(a)).$$

  If $(a \equiv_f u) = \top$, inequality (3) once again degenerates to a tautology. Specifically, using $f(a) = f(u)$ and the adjunction property, we have

$$\rho_A(u, gf(a)) = \rho_B(f(u), f(a)) = \rho_B(f(a), f(a)) = \top$$

- Assume the inequality (3) holds for $n-1$, and let us prove it for n. For this, consider $x \in A$,

$$\begin{aligned} aR^n u &= \bigvee_{x \in A} aR^{n-1}x \otimes xRu \\ &\le \bigvee_{x \in A} \rho_A(x, gf(a)) \otimes \big((x \approx_A u) \vee (x \equiv_f u)\big) \\ &= \bigvee_{x \in A} \rho_A(x, gf(a)) \otimes \big((\rho_A(x, u) \otimes \rho_A(u, x)) \vee (x \equiv_f u)\big) \\ &\le \bigvee_{x \in A} \rho_A(x, gf(a)) \otimes \big(\rho_A(u, x) \vee (x \equiv_f u)\big). \end{aligned}$$

  Once again, we reason on each disjunct separately, considering the possible results of $x \equiv_f a$, and using monotonicity of f and the hypothesis $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ when necessary:
  If $(x \equiv_f u) = \bot$, then the result follows by commutativity of $\otimes$ and transitivity of ρ_A.
  If $(x \equiv_f u) = \top$, from $f(x) = f(u)$, then we have

$$\rho_A(x, gf(a)) = \rho_B(f(x), f(a)) = \rho_B(f(u), f(a)) = \rho_A(u, gf(a))$$

Summarizing, we have proved that $gf(a)$ is an upper bound of the fuzzy set $[a]_{\cong_A}$.
Finally, for the minimality, we have to check that $\rho_A(gf(a), x) = \top$ for all $x \in UB[a]_{\cong_A} \cap g(f(A)))$.
Consider $x \in UB[a]_{\cong_A} \cap g(f(A))$; then there exists $a_1 \in A$ such that $x = gf(a_1)$ and $(a \cong_A u) \le \rho_A(u, x)$ for all $u \in A$. Particularly, considering $u = a$ and using the monotonicity of g and the adjunction property, we have that,

$$\begin{aligned}\top = (a \cong_A a) \le \rho_A(a, x) &= \rho_A(a, gf(a_1)) \\ &= \rho_B(f(a), f(a_1)) \\ &\le \rho_A(gf(a), gf(a_1)) = \rho_A(gf(a), x).\end{aligned}$$

3. Consider $a_1, a_2 \in A$, as f and g are isotone maps, then we have

$$\rho_A(a_1, a_2) \le \rho_A(g(f(a_1)), g(f(a_2)))$$

From the inequality above, we directly obtain the required condition

$$\rho_A(a_1, a_2) \le \Big(\text{p-min}(UB[a_1]_{\cong_A} \cap g(f(A))) \sqsubseteq \text{p-min}(UB[a_2]_{\cong_A} \cap g(f(A)))\Big)$$

since we have just proved above that $g(f(a)) \in \text{p-min}(UB[a]_{\cong_A} \cap gf(A))$ for all $a \in A$. □

Corollary 1. *Let $\mathbb{A} = (A, \rho_A)$ be a fuzzy preordered set, let B be an unstructured set and $f\colon A \to B$ be a mapping. If f is the left adjoint for an adjunction, then there exists a subset $S \subseteq A$ such that*

(1) $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
(2) $\text{p-min}(UB[a]_{\cong_A} \cap S) \neq \varnothing$, *for all* $a \in A$.
(3) $\rho_A(a_1, a_2) \le \Big(\text{p-min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq \text{p-min}(UB[a_2]_{\cong_A} \cap S)\Big)$ *for all* $a_1, a_2 \in A$.

It is worth to notice that the necessary conditions obtained above closely follow the characterizations one obtained in the crisp case for existence of adjunctions between preordered sets. Specifically, in [19], it was proved that given any (crisp) preordered set $\mathbb{A} = (A, \lesssim_A)$ and a mapping $f\colon \mathbb{A} \to B$, there exists a preorder $\mathbb{B} = (B, \lesssim_B)$ and $g\colon B \to A$ such that (f, g) forms a crisp adjunction between $\mathbb{A}$ and $\mathbb{B}$ *if and only if* there exists a subset S of A such that the following conditions hold:

(1) $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
(2) $\text{p-min}(UB[a]_{\cong_A} \cap S) \neq \varnothing$, for all $a \in A$.
(3) If $a_1 \lesssim_A a_2$, then $\Big(\text{p-min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq \text{p-min}(UB[a_2]_{\cong_A} \cap S)\Big)$, for $a_1, a_2 \in A$.

Obviously, although in this paper we have just proved one implication (the necessary conditions), as the obtained results are exactly the corresponding fuzzy translation of the crisp one, it seems likely that the converse should hold as well.

In order to provide some clue about the significance of the obtained conditions, it is worth to recall the characterization of the existence of adjunctions from a crisp poset to an unstructured set, which somehow unifies some well-known facts about adjunctions in a categorical sense, i.e. if g is a right adjoint then it preserves limits.

In [18] it was proved that given a poset $(A, \leq_A)$ and a map $f\colon A \to B$, there exists an ordering $\leq_B$ in B and a map $g\colon B \to A$ such that (f, g) is a crisp adjunction between posets from $(A, \leq_A)$ to $(B, \leq_B)$ *if and only if*

(i) There exists $\max([a]_{\equiv_f})$ for all $a \in A$.
(ii) $a_1 \leq_A a_2$ implies $\max([a_1]_{\equiv_f}) \leq_A \max([a_2]_{\equiv_f})$, for all $a_1, a_2 \in A$.

where $\equiv_f$ is the kernel relation wrt f.

These two conditions are closely related to the different characterizations of the notion of adjunction, as stated in Theorem 1 (items 5 and 6); specifically, condition (i) above states that if $b \in B$ and $f(a) = b$, then necessarily $g(b) = \max([a]_{\cong_f})$, whereas condition (ii) is related to the isotonicity of both f and g.

In some sense, the necessary conditions (1), (2), (3) obtained in Corollary 1 reflect the considerations given in the previous paragraph, but the different underlying ordered structure leads to a different formalization. Formally, condition (i) above is split into (1) and (2), since in a preordered setting, if $b \in B$ and $f(a) = b$, $g(b)$ needs not be in the same class as a but being maximum in its class (1). However, the latter condition is too weak and (2) provides exactly the remaining requirements needed in order to adequately reproduce the desired properties for g. Now, condition (3) it just the rephrasing of (ii) in terms of the properties described in (2).

4 Future Work

We have provided a set of necessary conditions for the existence of right adjunction to a mapping $f\colon (A, \rho_A) \to B$. The immediate future task is to study the other implication in order to find a set of necessary and sufficient conditions so that it is possible to define a fuzzy preorder on B such that f is a left adjoint.

Several papers on *fuzzy* Galois connections have been written since its introduction in [1]; consider for instance [3, 14, 23] for some recent ones. Another source of future work will be to study possible generalizations of the previously obtained results to the existence of fuzzy adjunctions within appropriate structures, and study the potential relationship to other approaches based on adequate versions of fuzzy closure systems [21].

Last but not least, in the recent years there has been some interesting developments on the study of both fuzzy partial orders and fuzzy preorders, see [4–7] for instance. In these works, it is noticed that versions antisymmetry and reflexivity commonly used are too strong and, as a consequence, the resulting fuzzy partial orders are very close to the classical case. Accordingly, another line of future work will be the adaptation of the current results to these alternative weaker definitions.

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