Generating isotone Galois connections on an unstructured codomain

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Abstract. Given a mapping $f\colon A \to B$ from a partially ordered set A into an unstructured set B, we study the problem of defining a suitable partial ordering relation on B such that there exists a mapping $g\colon B \to A$ such that the pair of mappings (f, g) forms an isotone Galois connection between partially ordered sets.

1 Introduction

Galois connections were introduced by Ore [25] as a pair of antitone mappings satisfying certain conditions which generalize Birkhoff's theory of polarities to apply to complete lattices. Later, Kan [19] introduced the notion of *adjunction* in a categorical context which, after instantiating to partially ordered sets turned out to be the isotone version of the notion of Galois connection.

In the recent years there has been a notable increase in the number of publications concerning Galois connections, both isotone and antitone. On the one hand, one can find lots of papers on theoretical developments or theoretical applications [7, 9, 20]; on the other hand, of course, there exist as well a lot of applications to computer science, see [23] for a first survey on applications, although more specific references on certain topics can be found, for instance, to programming [24], data analysis [23], logic [12, 18], etc.

Two research topics that have benefitted recently from the use of the theory of Galois connections is that of approximate reasoning using rough sets [13, 17, 26], and Formal Concept Analysis (FCA), either theoretically [1, 3, 6, 22] or applicatively [10, 11]. It is not surprising to see so many works dealing with both Galois connections and FCA, since the derivation operators used to define the concepts form a (antitone) Galois connection.

A number of results can be found in the literature concerning sufficient or necessary conditions for a Galois connection between ordered structures to exist.

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The main result of this paper is related to the existence and construction of the adjoint pair to a given mapping f, but *in a more general framework*.

Our initial setting is to consider a mapping $f\colon A \to B$ from a partially ordered set A into an unstructured set B, and then characterize those situations in which the set B can be partially ordered and an isotone mapping $g\colon B \to A$ can be built such that the pair (f, g) is an isotone Galois connection.

The structure of the paper is as follows: in Section 2 we introduce the preliminary definitions and results; then, in Section 3, given $f\colon A \to B$ we focus on the case in which the domain A has a poset structure, the necessary and sufficient conditions for the existence of a unique ordering on B and a mapping g such that (f, g) is an adjunction are given; Finally, in Section 4, we draw some conclusions and discuss future work.

2 Preliminaries

We assume basic knowledge of the properties and constructions related to a partially ordered set. For the sake of self-completion, we include below the formal definitions of the main concepts to be used in this section.

Definition 1. *Given a partially ordered set $\mathbb{A} = (A, \leq_A)$, $X \subseteq A$, and $a \in A$.*

- *Element a is said to be the* maximum *of X, denoted* $\max X$, *if $a \in X$ and $x \leq a$ for all $x \in X$.*
- *The* downset $a^{\downarrow}$ *of a is defined as $a^{\downarrow} = \{x \in A \mid x \leq_A a\}$.*
- *The* upset $a^{\uparrow}$ *of a is defined as $a^{\uparrow} = \{x \in A \mid x \geq_A a\}$.*

A mapping $f\colon (A, \leq_A) \to (B, \leq_B)$ between partially ordered sets is said to be

- isotone *if $a_1 \leq_A a_2$ implies $f(a_1) \leq_B f(a_2)$, for all $a_1, a_2 \in A$.*
- antitone *if $a_1 \leq_A a_2$ implies $f(a_2) \leq_B f(a_1)$, for all $a_1, a_2 \in A$.*

As usual, f^{-1} is the inverse image of f, that is, $f^{-1}(b) = \{a \in A \mid f(a) = b\}$. In the particular case in which $A = B$,

- *f is* inflationary *(also called extensive) if $a \leq_A f(a)$ for all $a \in A$.*
- *f is* deflationary *if $f(a) \leq_A a$ for all $a \in A$.*

As we are including the necessary definitions for the development of the construction of isotone Galois connections (hereafter, for brevity, termed *adjunctions*) between posets, we state below the definition of adjunction we will be working with.

Definition 2. *Let* $\mathbb{A} = (A, \leq_A)$ *and* $\mathbb{B} = (B, \leq_B)$ *be posets,* $f: A \to B$ *and* $g: B \to A$ *be two mappings. The pair* (f, g) *is said to be an* ***adjunction between*** $\mathbb{A}$ ***and*** $\mathbb{B}$*, denoted by* $(f, g): \mathbb{A} \leftrightharpoons \mathbb{B}$*, whenever for all* $a \in A$ *and* $b \in B$ *we have that*

$$f(a) \leq_B b \qquad \text{if and only if} \qquad a \leq_A g(b)$$

The mapping f *is called* left adjoint *and* g *is called* right adjoint*.*

The following theorem states equivalent definitions of adjunction between posets that can be found in the literature, see for instance [5, 16].

Theorem 1. *Let* $\mathbb{A} = (A, \leq_A), \mathbb{B} = (B, \leq_B)$ *be two posets,* $f: \mathbb{A} \to \mathbb{B}$ *and* $g: \mathbb{B} \to \mathbb{A}$ *be two mappings. The following statements are equivalent:*

1. $(f, g): \mathbb{A} \leftrightharpoons \mathbb{B}$.
2. f *and* g *are isotone,* $g \circ f$ *is inflationary, and* $f \circ g$ *is deflationary.*
3. $f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ *for all* $a \in A$.
4. $g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ *for all* $b \in B$.
5. f *is isotone and* $g(b) = \max f^{-1}(b^{\downarrow})$ *for all* $b \in B$.
6. g *is isotone and* $f(a) = \min g^{-1}(a^{\uparrow})$ *for each* $a \in A$.

We introduce the technical lemma below which shows that, in some case, it is possible to get rid of the downsets (as used in item 5 of the previous theorem).

Lemma 1. *Let* $(A, \leq_A)$ *and* $(B, \leq_B)$ *be posets and* $f: A \to B$ *an isotone mapping. If* $\max f^{-1}(b^{\downarrow})$ *exists for some* $b \in f(A)$*, then* $\max f^{-1}(b)$ *exists and* $\max f^{-1}(b^{\downarrow}) = \max f^{-1}(b)$*.*

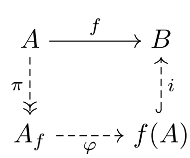
Proof. Let us denote $m = \max f^{-1}(b^{\downarrow})$ and we will prove that $a \leq_A m$, for all $a \in f^{-1}(b)$, and $m \in f^{-1}(b)$, in order to have $m = \max f^{-1}(b)$.

Consider $a \in f^{-1}(b)$, then $f(a) = b \in b^{\downarrow}$ and $a \in f^{-1}(b^{\downarrow})$, hence $a \leq_A m$.

Now, isotonicity of f shows that $f(a) = b \leq_B f(m)$. For the other inequality, simply consider that $m = \max f^{-1}(b^{\downarrow})$ implies $m \in f^{-1}(b^{\downarrow})$, which means $f(m) \leq_B b$. Therefore, $f(m) = b$ because of antisymmetry of $\leq_B$. □

3 Building adjunctions between partially ordered sets

With the general aim of finding conditions for a mapping from a poset $(A, \leq_A)$ to an unstructured set B, in order to construct an adjunction we will naturally consider the canonical decomposition of $f: A \to B$ through A_f, the quotient set of A wrt the **kernel relation** $\equiv_f$, defined as $a \equiv_f b$ if and only if $f(a) = f(b)$:



In general, given a poset $(A \leq_A)$ together with an equivalence relation $\sim$ on A, it is customary to consider the set $A_\sim = A/\sim$, the quotient set of A wrt $\sim$, and the natural projection $\pi\colon A \to A_\sim$. As usual, the equivalence class of an element $a \in A$ is denoted $[a]$ and, then, $\pi(a) = [a]$.

The following lemma provides sufficient conditions for π being the left component of an adjunction.

Lemma 2. *Let $(A, \leq_A)$ be a poset and $\sim$ an equivalence relation on A. Suppose that the following conditions hold*

1. *There exists* $\max([a])$*, for all $a \in A$.*
2. *If $a_1 \leq_A a_2$ then* $\max([a_1]) \leq_A \max([a_2])$*, for all $a_1, a_2 \in A$.*

Then, the relation $\leq_{A_\sim}$ defined by $[a_1] \leq_{A_\sim} [a_2]$ if only if $a_1 \leq_A \max([a_2])$ is an ordering in $A_\sim$ and, moreover, the pair $(\pi, \max)$ is an adjunction.

Proof. To begin with, the relation $\leq_{A_\sim}$ is well defined since, by the first hypothesis, $\max([a])$ exists for all $a \in A$.

Reflexivity Obvious, since $[a] \leq_{A_\sim} [a]$ if and only if $a \leq_A \max([a])$, and the latter holds for all $a \in A$.

Transitivity Assume $[a_1] \leq_{A_\sim} [a_2]$ and $[a_2] \leq_{A_\sim} [a_3]$.
From $[a_1] \leq_{A_\sim} [a_2]$, by definition, we have $a_1 \leq_A \max([a_2])$. Now, from $[a_2] \leq_{A_\sim} [a_3]$ we obtain, by definition of the ordering and the second hypothesis that $\max([a_2]) \leq_A \max([a_3])$. As a result, we obtain $[a_1] \leq_{A_\sim} \max([a_3])$, that is, $[a_1] \leq_{A_\sim} [a_3]$.

Antisymmetry Assume $a_1, a_2 \in A$ such that $[a_1] \leq_{A_\sim} [a_2]$ and $[a_2] \leq_{A_\sim} [a_1]$.
By hypothesis, we have that $a_1 \leq_A \max([a_2])$ then $\max([a_1]) \leq_A \max([a_2])$, and $a_2 \leq_A \max([a_1])$ then $\max([a_2]) \leq_A \max([a_1])$. Since $\leq_A$ is antisymmetric, then $\max([a_1]) = \max([a_2])$; now, we have that the intersection of the two classes $[a_1]$ and $[a_2]$ is non-empty, therefore $[a_1] = [a_2]$.

Once again by the first hypothesis, max can be seen as a mapping $A_\sim \to A$. Now, the adjunction follows by the definition of π and the ordering:

$$\begin{aligned} \pi(a_1) \leq_{A_\sim} [a_2] &\text{ if and only if } [a_1] \leq_{A_\sim} [a_2] \\ &\text{ if and only if } a_1 \leq_A \max([a_2]) \end{aligned}$$

□

The previous lemma gave sufficient conditions for π being a left adjoint; the following result states that the conditions are also necessary, and that the ordering relation and the right adjoint are uniquely defined.

Lemma 3. *Let $(A \leq_A)$ be a poset and $\sim$ an equivalence relation on A. Let $A_\sim = A/\sim$ be the quotient set of A wrt $\sim$, and $\pi\colon A \to A_\sim$ the natural projection. If there exists an ordering relation $\leq_{A_\sim}$ in $A_\sim$ and $g\colon A_\sim \to A$ such that $(\pi, g)\colon A \leftrightharpoons A_\sim$ then,*

1. $g([a]) = \max([a])$ *for all* $a \in A$.
2. $[a_1] \leq_{A_\sim} [a_2]$ *if and only if* $a_1 \leq_A \max([a_2])$ *for all* $a_1, a_2 \in A$.
3. *If* $a_1 \leq_A a_2$ *then* $\max([a_1]) \leq_A \max([a_2])$ *for all* $a_1, a_2 \in A$.

Proof.

1. By Theorem 1, we have $g([a]) = \max \pi^{-1}([a]^{\downarrow})$. Now, Lemma 1 leads to $\max \pi^{-1}([a]^{\downarrow}) = \max \pi^{-1}([a]) = \max([a])$.
   There is a slight abuse of notation in that $[a]$ is sometimes considered as a single element, i.e. one equivalence class of the quotient set, and sometimes as the set of elements of the equivalence class. The context helps to clarify which meaning is intended in each case.
2. By the adjointness of (π, g), definition of π, and the previous item we have the following chain of equivalences
$$\begin{aligned} [a_1] \leq_{A_\sim} [a_2] \ &\text{if and only if } \ \pi(a_1) \leq_{A_f} [a_2] \\ &\text{if and only if } \ a_1 \leq_A g([a_2]) \\ &\text{if and only if } \ a_1 \leq_A \max([a_2]) \end{aligned}$$
3. Finally, since π and g are isotone maps, $a_1 \leq_A a_2$ implies $[a_1] \leq_{A_f} [a_2]$, and $g([a_1]) \leq_A g([a_2])$, therefore $\max([a_1]) \leq_A \max([a_2])$ by item 1 above. □

Continuing with the analysis of the decomposition, we naturally arrive to the following result.

Lemma 4. *Consider a poset $(A, \leq_A)$ and a bijective mapping $\varphi\colon A \to B$, then there exists a unique ordering relation in B, which is defined as $b \leq_B b'$ if and only if $\varphi^{-1}(b) \leq_A \varphi^{-1}(b')$, such that $(\varphi, \varphi^{-1})\colon A \leftrightharpoons B$.*

Proof. Straightforward. □

As a consequence of the previous results, we have established necessary and sufficient conditions ensuring the existence and uniqueness of right adjoint for any surjective mapping f from a poset A to an unstructured set B.

The third part of this section is devoted to considering the case in which f is not surjective. In this case, in general, there are several possible orderings on B which allow to define the right adjoint. The crux of the construction is related to the definition of an order-embedding of the image into the codomain set.

More generally, the idea is to extend an ordering defined just on a subset of a set to the whole set.

Definition 3. *Given a subset $X \subseteq B$, and a fixed element $m \in X$, any preordering $\leq_X$ in X can be extended to a preordering $\leq_m$ on B, defined as the reflexive and transitive closure of the relation $\leq_X \cup \{(m, y) \mid y \notin X\}$.*

Note that the relation above can be described as, for all $x, y \in B$, $x \leq_m y$ if and only if some of the following holds:

(a) $x, y \in X$ and $x \leq_X y$
(b) $x \in X, y \notin X$ and $x \leq_X m$
(c) $x, y \notin X$ and $x = y$

It is not difficult to check that if the initial relation $\leq_X$ is an ordering relation, then $\leq_m$ is an ordering as well. Formally, we have

Lemma 5. *Given a subset $X \subseteq B$, and a fixed element $m \in X$, then $\leq_X$ is an ordering in X if and only if $\leq_m$ is an ordering on B.*

Proof. Just some routine computations are needed to check that $\leq_m$ is antisymmetric using the properties of $\leq_X$.

Conversely, if $\leq_m$ is an ordering, then $\leq_X$ is an ordering as well, since it is a restriction of $\leq_m$. □

Lemma 6. *Let X be a subset of B, consider a fixed element $m \in X$, and an ordering $\leq_X$ in X. Define the mapping $j_m \colon (B, \leq_m) \to (X, \leq_X)$ as*

$$j_m(x) = \begin{cases} x & \text{if } x \in X \\ m & \text{if } x \notin X \end{cases}$$

Then, $(i, j_m) \colon (X, \leq_X) \leftrightharpoons (B, \leq_m)$ where i denotes the inclusion $X \hookrightarrow B$.

Proof. It follows easily by routine computation. □

Theorem 2. *Given a poset $(A, \leq_A)$ and a map $f \colon A \to B$, let $\equiv_f$ be the kernel relation. Then, there exists an ordering $\leq_B$ in B and a map $g \colon B \to A$ such that $(f, g) \colon A \leftrightharpoons B$ if and only if*

1. *There exists* $\max([a])$ *for all $a \in A$.*
2. *For all $a_1, a_2 \in A$, $a_1 \leq_A a_2$ implies* $\max([a_1]) \leq_A \max([a_2])$.

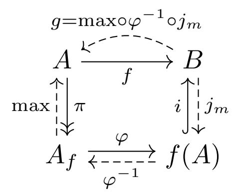
Proof. Assume that there exists an adjunction $(f, g) \colon A \leftrightharpoons B$ and let us prove items 1 and 2.

Given $a \in A$, item 1 holds because of the following chain of equalities, where the first equality follows from Theorem 1, the second one follows from Lemma 1, and the third because of the definition of $[a]$:

$$g(f(a)) = \max f^{-1}(f(a)^{\downarrow}) = \max f^{-1}(f(a)) = \max([a]) \tag{1}$$

Now, item 2 is straightforward, because if $a_1 \leq_A a_2$ then, by isotonicity, $f(a_1) \leq_B f(a_2)$ and $g(f(a_1)) \leq_A g(f(a_2))$. Therefore, by Equation (1) above, $\max([a_1]) \leq_A \max([a_2])$.

Conversely, given $(A, \leq_A)$ and $f\colon A \to B$ and items 1 and 2, let us prove that f is the left adjoint of a mapping $g\colon B \to A$. To begin with, consider the canonical decomposition of f through the quotient set A_f of A wrt $\equiv_f$, see below, where $\pi\colon A \to A_f$ is the natural projection, $\pi(a) = [a]$, $\varphi([a]) = f(a)$, and $i(b) = b$ is the inclusion mapping.



Firstly, by Lemma 2, using conditions 1 and 2, and the fact that $[a] = \pi(a)$, we obtain that $(\pi, \max)\colon A \leftrightharpoons A_f$.

Moreover, since the mapping $\varphi\colon A_f \to f(A)$ is bijective, we can apply Lemma 4 in order to induce an ordering $\leq_{f(A)}$ on $f(A)$ such that we have another adjunction, the pair $(\varphi, \varphi^{-1})\colon A_f \leftrightharpoons f(A)$.

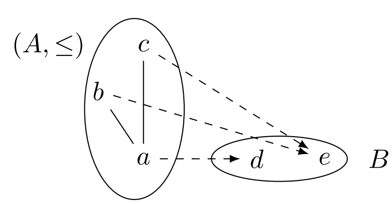
Then, considering an arbitrary element $m \in f(A)$, the ordering $\leq_{f(A)}$ also induces an ordering $\leq_m$ on B, as stated in Lemma 5, and a map $j_m\colon B \to f(A)$ such that $(i, j_m)\colon f(A) \leftrightharpoons B$.

Finally, the composition $g = \max \circ \varphi^{-1} \circ j_m\colon B \to A$ is such that (f, g) is an adjunction. □

We end this section with two counterexamples showing that the conditions in the theorem cannot be removed.

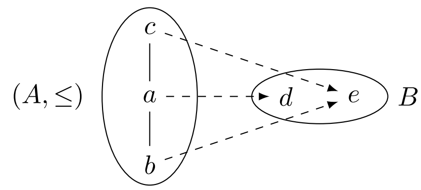
Let $A = \{a, b, c\}$ and $B = \{d, e\}$ be two sets and $f\colon A \to B$ defined as $f(a) = d$ and $f(b) = f(c) = e$.

Condition 1 cannot be removed: Consider $(A, \leq)$ where $a \leq b, a \leq c$ and b, c not related. Then $[b] = \{b, c\}$ and there does not exist $\max([b])$.



The right adjoint does not exist because $\max f^{-1}(e^{\downarrow})$ would not be defined for any ordering in B.

Condition 2 cannot be removed: Consider $(A, \leq)$, where $b \leq a \leq c$.



In this case, Condition 1 holds, since there exist both $\max[a] = a$ and $\max[b] = c$, but Condition 2 clearly does not. Again, the right adjoint does not exist because f will never be isotone in any possible ordering defined in B.

4 Conclusions

Given a mapping $f : A \to B$ from a partially ordered set A into an unstructured set B, we have obtained necessary and sufficient conditions which allow us for defining a suitable partial ordering relation on B such that there exists a mapping $g : B \to A$ such that the pair of mappings (f, g) forms an adjunction between partially ordered sets. The results obtained in Theorem 2, regardless of the fact that the proof is not exactly straightforward, are in consonance with the intuition and the well-known facts about Galois connections.

A first source of future work is to consider A to be a preordered set, and try to find an isotone Galois connection between preorders. In this context, there are no clear candidate conditions for the existence of the preorder relation in B, since the notion of maximum is not unique in a preordered setting due to the absence of antisymmetry.

Another topic for future work is related to obtaining a fuzzy version of the obtained result, in the sense of considering either fuzzy Galois connections [2, 4, 15, 21] or considering the framework of fuzzy posets and fuzzy preorders.

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