L-fuzzy relational mathematical morphology based on adjoint triples

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Abstract

We propose an alternative to the standard structure of L-fuzzy Mathematical Morphology (MM) by, on the one hand, considering L-fuzzy relations as structuring elements and, on the other hand, by using adjoint triples to handle membership values. Those modifications lead to a framework based on set-theoretical operations where we can prove a representation theorem for algebraic morphological erosions and dilations. In addition, we also present some new results concerning duality and transformation invariance. Concerning duality, we show that duality and adjointness can coexist in this L-fuzzy relational MM. Concerning transformation invariance, we show sufficient conditions to guarantee the invariance of morphological operators under arbitrary transformations.

Keywords: Fuzzy Mathematical Morphology, Algebraic Mathematical Morphology, Fuzzy Sets, Adjoint triples.

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1. Introduction

Mathematical Morphology (MM) can be characterized as a mathematical theory for the transformation of geometrical structures by the use of set-theoretical operations. In the recent years, MM has evolved into a compendium of different but related theories since the seminal works of G. Matheron and J. Serra [28, 35, 36]. These theories proved to be useful in image processing for filtering, pattern recognition and edge detection, among others [3, 27, 35, 45]. At the beginning, only binary images were in the focus of MM but then, new tools allowed to generalize the framework to consider grayscale and color images [14, 35], graphs [33, 44], hypergraphs [7] and relations [37].

Summarizing the theoretical development of MM, we can say that it consists of

(i) a space of objects closed under a certain set of transformations, and

(ii) a set of algebraically defined operators (morphisms) that preserve the structure of the object space.

On the basis of this observation, we can say that MM tends to play a similar role as the theory of categories. The pure algebraic approach of MM called Algebraic Mathematical Morphology (AMM), supports our claim. It considers a complete lattice as the object space, and dilations and erosions as transformations that commute, respectively, with arbitrary joins and meets. In this case, dilations are precisely the sup-lattice homomorphisms, while adjoint erosions are their residua. The purpose of our contribution is to bridge the gap between the use of set-theoretical operations and AMM. To reach such a goal, we enrich the space of objects with the structure provided by an L-fuzzy relation and the use of set-theoretical operations from the L-fuzzy set theory.

L-fuzzy set theory is a generalization of the set theory allowing degrees of memberships in a complete lattice L. The set-theoretical operations used in MM were extended to the fuzzy setting by means of t-norms and implications, leading to the so called fuzzy MM [5, 8, 12, 17, 32] and subsequently, to the L-fuzzy setting leading to the so-called L-fuzzy MM [41]. In this respect, L-fuzzy MM keeps the original interpretation of erosions and dilations by means of inclusions and intersections of translated structuring elements. It is worth mentioning also the interest of some researchers about establishing links between L-fuzzy MM and other topics of the fuzzy paradigm as fuzzy concept analysis [2], bipolar fuzzy sets [6] or F-transforms [39].

In our approach, we enrich the structure of the space of objects by including an L-fuzzy relation, which can be interpreted, for instance, as the formal structure that assigns to each element in the space either the translation of a fixed structuring element (as in the original papers of MM) or a variant structuring element (as most recently in [11]). Fuzzy relations were already used in [9, 10] but the resulting framework did not provide a characterization of the class of algebraic erosions and dilations. In order to obtain such a characterization, the key point is extending the definition of fuzzy relational erosions and dilations by considering adjoint triples [15] to handle membership values. Adjoint triples were originally introduced to handle non-commutative L-fuzzy conjunctions or conjunctions with arguments in different lattices, and its components can be used to define L-fuzzy inclusions and L-fuzzy intersections; as a result, its use in our framework keeps the same set-theoretical interpretations than in L-fuzzy MM [43].

In this paper, the definitions of L-fuzzy relational erosion and dilation based on adjoint triples enable to provide a three-fold contribution. The first one is related to the invariance of the morphological operators under arbitrary transformations. The second one shows that L-fuzzy relational erosions and dilations are dual operators despite the core of those morphological operators is an adjunction (see [5] to be aware of the complexity of preserving adjunction and duality in fuzzy MM). Finally, the third contribution is a representation theorem which shows that the class of L-fuzzy relational morphological operators coincides with the class of algebraic morphological operators. The use of adjoint triples instead of complete residuated lattices turns out to be crucial for the latter result.

The paper is structured as follows: in Section 2 we recall the basics of two theories of mathematical morphology: AMM and L-fuzzy relational MM. Then, in Section 3 we introduce the notions of L-fuzzy relational erosion and L-fuzzy relational dilation based on adjoint triples. Subsequently, Section 4 shows that basic properties of the original mathematical morphology also hold in this approach. Specifically, this section provides results about monotonicity, transformation invariance and duality. Section 5 shows the representation theorem for algebraic erosions and dilations. In Section 6 we compare our approach with others existing in the literature. Finally, in Section 7 we present the conclusions and future works.

2. Fundamentals of mathematical morphology: erosion and dilation

In this section we recall the two basic theories of MM we will use in this paper: the algebraic one [22] and the fuzzy relational one, based on [10] but provided by reformulating the more general approach of [39]. It is worth mentioning that there are other theoretical approaches to MM, for instance, the binary [28], the umbra approach [14, 35] or the fuzzy one [8, 17, 12, 20]. We focus only on erosions and dilations, but MM is not just about these operators; many other notions and operations are used as well, for instance *openings, closings*, *thickenings, thinnings*, and *hit-or-miss transformations*, among others, are also object of study in this theory. For a complete overview of MM we refer to [34].

2.1. Algebraic mathematical morphology (AMM)

The core of mathematical morphology is based on two basic operators: erosions and dilations. These operators were introduced originally on Euclidean spaces by means of translations and unions of subsets [28, 35]. However, in subsequent approaches [22, 36], such definitions were extended to apply those operators to arbitrary complete lattices. This later approach is called *algebraic mathematical morphology*. The definition of erosions and dilations in this approach is given as follows:

Definition 1. *Let $(L_1, \leq_1)$ and $(L_2, \leq)$ be two complete lattices. A mapping $\varepsilon\colon L_1 \to L_2$ is said to be an erosion if for all $X \subseteq L_1$ we have:*

$$\varepsilon(\bigwedge X) = \bigwedge_{x\in X} \varepsilon(x).$$

A mapping $\delta\colon L_2 \to L_1$ is said to be a dilation if for all $Y \subseteq L_2$ we have:

$$\delta(\bigvee Y) = \bigvee_{y\in Y} \delta(y).$$

So, roughly speaking, every erosion commutes with infima and every dilation with suprema. Note that the definition above takes into account the case where X and Y

are empty. That means that erosions assign the greatest element of L_1 to the greatest element of L_2 and dilations assign the least element of L_2 to the least element of L_1. Furthermore, it is straightforward to check that both erosions and dilations are monotonic mappings.

Perhaps the most important relation between erosions and dilations is given in terms of adjunctions, so let us begin by recalling this notion.

Definition 2. *An adjunction (ε, δ) between complete lattices $(L_1, \leq_1)$ and $(L_2, \leq)$ is pair of mappings $\varepsilon\colon L_1 \to L_2$ and $\delta\colon L_2 \to L_1$ such that for every $x \in L_1$ and $y \in L_2$ we have*

$$y \leq \varepsilon(x) \quad \textit{if and only if} \quad \delta(y) \leq x.$$

The naming ε and δ chosen in the previous definitions is not casual, since the operators introduced in Definition 1 can be related in terms of adjointness. The following two results are proven in [22], somehow rediscovering well-known facts in category theory.

Theorem 1. *If (ε, δ) is an adjunction, then ε is an erosion and δ is a dilation. On the other hand, let $\varepsilon\colon L_1 \to L_2$ be an erosion. Then, there exists exactly one dilation $\delta_\varepsilon\colon L_2 \to L_1$ such that $(\varepsilon, \delta_\varepsilon)$ forms an adjunction. Specifically, such a dilation can be determined, for every $y \in L_2$, by the expression*

$$\delta_\varepsilon(y) = \bigwedge\{x \in L_1 \mid y \leq \varepsilon(x)\}.$$

Similarly, for every dilation $\delta\colon L_2 \to L_1$ there is exactly one erosion $\varepsilon_\delta\colon L_1 \to L_2$ such that $(\varepsilon_\delta, \delta)$ forms an adjunction. Moreover, such an erosion is determined, for every $X \in L_1$, by the expression

$$\varepsilon_\delta(x) = \bigvee\{y \in L_2 \mid \delta(y) \leq x\}.$$

2.2. Fuzzy relational mathematical morphology

Note that the algebraic approach of mathematical morphology does not provide any means for a constructive representation of erosions and/or dilations. One alternative approach is that of L-fuzzy mathematical morphology, where dilations and erosions can be efficiently represented with the help of fuzzy conjunctions (as t-norms) and fuzzy implications. In this way, L-fuzzy mathematical morphology is closer to the original approach thanks to the consideration of structuring elements and the L-fuzzy set-theoretical operations. Recall that in the original approach of mathematical morphology, erosions and dilations were defined by means of translation, inclusion and intersection of certain sets, called structuring elements. Our approach in this section can be explained in terms of the very general L-fuzzy approach of [39], defined by considering *L-fuzzy structuring functions*, instead of using the standard approaches of [17, 8].

Let $(L,\leq)$ be a complete lattice, with 1 being the top and 0 being the bottom elements. Let us recall that an L-fuzzy conjunction C is a mapping $C\colon L\times L\to L$ such that is order-preserving on both arguments and satisfies the following boundary conditions $C(1,0)=C(0,1)=C(0,0)=0$ and $C(1,1)=1$; an L-fuzzy implication I is a mapping $I\colon L\times L\to L$ such that is order-reversing on the first argument, order-preserving on the second argument and satisfies the following boundary conditions $I(1,1)=I(0,1)=I(0,0)=1$ and $I(1,0)=0$. An L-fuzzy set on a set A (called universe) is a mapping $\mu\colon A\to L$ (called membership function). Fuzzy sets arise as the special case where $L=[0,1]$. The set of L-fuzzy sets on A is denoted by L^A; note that L^A inherits the lattice structure of L. An L-fuzzy relation between two sets A and B is a L-fuzzy set on $A\times B$, i.e., a mapping $R\colon A\times B\to L$. The value assigned to $R(a,b)$ represents in which degree *"a is related to b (by R)"*. When $A=B$ we say that R is an L-fuzzy relation in A. The reader is referred to [47] for a deeper description of these notions.

The approach of [39] is based on the notion of L-fuzzy structuring function: given two sets A and B, an L-fuzzy structuring function u is a mapping $u\colon A\to B^L$. For the sake of presentation, fixed an L-fuzzy structuring function u, $a\in A$ and $b\in B$ we write u_a instead of $u(a)$ and $\overline{u}_b(a)$ instead of $u_a(b)$. Given $X\in L^A, Y\in L^B$ and an L-fuzzy structuring function u, the L-fuzzy erosion of X and the L-fuzzy dilation of Y are defined by:

$$\varepsilon_u(X)(y)=\bigwedge_{a\in A} I(u_y(a),X(a))$$

for all $y \in B$ and

$$\delta_u(Y)(x) = \bigvee_{b \in B} C(\overline{u}_x(b), Y(b))$$

for all $x \in A$, respectively.

The L-fuzzy erosion ε_u and the L-fuzzy dilation δ_u form an adjunction if and only if the L-fuzzy implication and the L-fuzzy conjunction form an adjoint pair [39, Proposition 3]. Therefore, in the context of algebraic mathematical morphology, we can reformulate the definition above in terms of L-fuzzy relations and complete residuated lattices, thus extending the approach of [10].

Recall that a complete residuated lattice is a 6-tuple $(L, \leq, *, \rightarrow, 0, 1)$ such that:

- $(L, \leq)$ is a complete lattice, with 1 being the top and 0 being the bottom element.
- $(L, *, 1)$ is a commutative monoid with unit element 1.
- $(\rightarrow, *)$ forms an adjoint pair, i.e.

$$z \leq (y \rightarrow x) \quad \text{if and only if} \quad y * z \leq x$$

Definition 3. *Let A and B be two sets, let $(L, \leq, *, \rightarrow, 0, 1)$ be a complete residuated lattice and let $R\colon A \times B \rightarrow L$ be an L-fuzzy relation between A and B. The L-fuzzy relational erosion of $X \in L^A$ and the L-fuzzy relational dilation of $Y \in L^B$ by the L-fuzzy relation R are defined by:*

$$\varepsilon_R(X)(y) = \bigwedge_{a \in A} R(a, y) \rightarrow X(a) \tag{1}$$

for all $y \in B$ and

$$\delta_R(Y)(x) = \bigvee_{b \in B} R(x, b) * Y(b) \tag{2}$$

for all $x \in A$, respectively.

3. Relational L-fuzzy mathematical morphology based on adjoint triples

Inspired by the definition of L-fuzzy relational dilations and erosions (Definition 3), we provide a modification intended to increase their expressive power. Our approach is based on adjoint triples, which are a generalization of the pairs of adjoint operators in complete residuated lattices.

Definition 4 ([30])**.** *Let* $(P_1,\leq_1)$, $(P_2,\leq_2)$, $(P_3,\leq_3)$ *be three posets. We say that the mappings* $\&\colon P_1\times P_2\to P_3$, $\searrow\colon P_2\times P_3\to P_1$, *and* $\nearrow\colon P_1\times P_3\to P_2$ *form an adjoint triple among* P_1,P_2 *and* P_3 *whenever:*

$$x\leq_1 y\searrow z \quad \text{if and only if} \quad x\,\&\,y\leq_3 z \quad \text{if and only if} \quad y\leq_2 x\nearrow z \tag{3}$$

for all $x\in P_1$, $y\in P_2$ *and* $z\in P_3$.

Before showing some examples, it is worth noticing that mappings in adjoint triples have a similar interpretation to those in residuated lattices. Hence, the mappings $\nearrow$ (resp. $\searrow$) and $\&$ can be used to define inclusions and intersections between fuzzy sets, respectively (see [13, 17, 25, 46, 47]). The main differences between the mappings in residuated lattices and those in adjoint triples is that, firstly, the conjunction $\&$ in adjoint triples may be non-commutative and, secondly, mappings $\nearrow,\searrow$ and $\&$ in adjoint triples allow to define intersections and conjunctions between L-fuzzy sets with different underlying sets of truth-values.

Example 1. *The operators* $\&\colon[0,1]\times[0,1]\to[0,1]$, $\searrow\colon[0,1]\times[0,1]\to[0,1]$ *and* $\nearrow\colon[0,1]\times[0,1]\to[0,1]$ *given by*

$$x\&y=x^2\cdot y$$
$$x\searrow y=\begin{cases}1 & \text{if } x=0\\ \min\left\{\sqrt{\frac{y}{x}},1\right\} & \text{otherwise.}\end{cases}$$
$$x\nearrow y=\begin{cases}1 & \text{if } x=0\\ \min\left\{\frac{y}{x^2},1\right\} & \text{otherwise.}\end{cases}$$

for all $x,y\in[0,1]$, *form an adjoint triple between* $[0,1]$, $[0,1]$ *and* $[0,1]$. *It is worth noticing that the operator & can be viewed as a non-commutative fuzzy conjunction.*

The following example shows a direct relationship between MM and adjoint triples.

Example 2. *Let* $(A,+)$ *be a group and let L be a complete lattice. Given a crisp subset* $B\in 2^A$ *and an L-fuzzy subset* $F\in L^A$ *we can define dilations and erosions (based on flat structuring elements) in* L^A *as follows*

$$\delta_B(F)(x)=\bigvee_{b\in B}F(x-b) \qquad\qquad \varepsilon_B(F)(x)=\bigwedge_{b\in B}F(x+b)$$

Furthermore, let us consider the operators $\&\colon 2^A \times L^A \to L^A$, $\nearrow\colon 2^A \times L^A \to L^A$ *and* $\searrow\colon L^A \times L^A \to 2^A$ *given by*

$$B \,\&\, F = \delta_B(F)$$

$$B \nearrow F = \varepsilon_B(F)$$

$$F \searrow G = \{z \in A \mid F(y) \leq G(y+z) \text{ for all } y \in A\}$$

then $(\&, \nearrow, \searrow)$ *forms an adjoint triple between* $2^A, L^A$ *and* L^A.

Note also that every adjoint pair in a complete residuated lattice is an adjoint triple in which the operator $\&$ is commutative (i.e., $\& = *$) and, hence, both implications coincide; that is, in such a case $\nearrow = \searrow = \to$. Therefore the well-known Gödel, product and Łukasiewicz t-norms,

$$x \&_G y = \min\{x, y\} \qquad x \&_P y = x \cdot y \qquad x \&_Ł y = \max\{x + y - 1, 0\}$$

together with their residual implications,

$$x \to_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases} \qquad x \to_P y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

$$x \to_Ł y = \begin{cases} 1 & \text{if } x \leq y \\ \min\{1 - x + y, 1\} & \text{otherwise.} \end{cases}$$

can be seen as examples of adjoint triples. On the other hand, adjoint triples allow to consider different lattices for operating with the relevant arguments involved, as the following example shows.

Example 3. *Let* $[0,1]_m$ *be a uniform discretization of* $[0,1]$ *into m pieces, for example* $[0,1]_2 = \{0, 0.5, 1\}$ *divides the unit interval into two pieces. Consider the discretization of the Gödel t-norm represented by the operator* $\&_G^*\colon [0,1]_{20} \times [0,1]_8 \to [0,1]_{100}$ *defined, for each* $x \in [0,1]_{20}$ *and* $y \in [0,1]_8$, *as:*

$$x \,\&_G^*\, y = \frac{\lceil 100 \cdot \min\{x, y\} \rceil}{100}$$

where $\lceil \_ \rceil$ *is the ceiling function. For this operator, the corresponding residuated implication operators* $\searrow_G^*\colon [0,1]_8 \times [0,1]_{100} \to [0,1]_{20}$ *and* $\nearrow_G^*\colon [0,1]_{20} \times [0,1]_{100} \to [0,1]_8$

are defined as:

$$
\begin{aligned}
a \searrow_G^* b &= \frac{\lfloor 20 \cdot (a \searrow^G b) \rfloor}{20} \\
c \nearrow_G^* b &= \frac{\lfloor 8 \cdot (c \nearrow_G b) \rfloor}{8}
\end{aligned}
$$

where $\lfloor \_ \rfloor$ *is the floor function. The tuple* $(\&_G^*, \searrow_G^*, \nearrow_G^*)$ *is an adjoint triple where the operator* $\&_G^*$ *is neither commutative nor associative.* □

The lemma below is essential for our definition of L-fuzzy dilations and erosions.

Lemma 1 ([16]). *If* $(\&, \searrow, \nearrow)$ *is an adjoint triple w.r.t.* $(P_1, \leq_1), (P_2, \leq_2), (P_3, \leq_3)$, *then*

1. *If* P_2 *and* P_3 *are complete lattices then,* $\big(x \nearrow (\cdot),\ x \,\&(\cdot)\big)$ *is an adjunction for all* $x \in P_1$.
2. $\&$ *is order-preserving on both arguments, i.e. for all* $x_1, x_2, x \in P_1$, $y_1, y_2, y \in P_2$
   (a) *If* $x_1 \leq_1 x_2$*, then* $(x_1 \,\&\, y) \leq_3 (x_2 \,\&\, y)$.
   (b) *if* $y_1 \leq_2 y_2$*, then* $(x \,\&\, y_1) \leq_3 (x \,\&\, y_2)$.
3. $\searrow$, $\nearrow$ *are order-reversing on the first argument and order-preserving on the second argument.*

The new morphological operators of dilation and erosion based on P-fuzzy relations and adjoint triples are defined as follows. Let us recall that a P-fuzzy relation between two sets A and B is a mapping $R\colon A \times B \to P$.

Definition 5. *Let* P *be a poset, let* A *and* B *be two arbitrary sets, let* L_1 *and* L_2 *be two complete lattices, let* $(\&, \searrow, \nearrow)$ *be an adjoint triple among* P, L_2 *and* L_1*, and let* R *be a* P*-fuzzy relation on* $A \times B$*. The* L*-fuzzy relational erosion with respect to* R, $\varepsilon_R\colon L_1^A \to L_2^B$ *is defined, for all* $X \in {L_1}^A$ *and* $b \in B$*, as*

$$
\varepsilon_R(X)(b) = \bigwedge_{a \in A} R(a,b) \nearrow X(a)
$$

and the L*-fuzzy relational dilation with respect to* R, $\delta_R\colon L_2^B \to L_1^A$*, for all* $Y \in {L_2}^B$ *and* $a \in A$*, is defined as*

$$
\delta_R(Y)(a) = \bigvee_{b \in B} R(a,b) \& Y(b).
$$

We remark that similar operators can be used in the framework of formal concept analysis, in particular, as the concept-forming operators when considering its (rough) object-oriented extension [1, 29]. By considering the implication $\searrow$ we obtain another approach which is elaborated in the following paragraphs.

An alternative definition of Definition 5 arises when the pair $(\&, \searrow)$ is considered. This new definition is related to the so-called property-oriented extension in formal concept analysis. Hence, the *L-fuzzy relational property-oriented erosion with respect to R*, $\varepsilon_R\colon L_2^B \to L_1^A$, is defined as:

$$\varepsilon_{R_p}(Y)(a) \quad = \quad \bigwedge_{b \in B} R(a,b) \searrow Y(b)$$

for all $Y \in {L_2}^B$ and $a \in A$, and the *L-fuzzy relational property-oriented dilation with respect to R*, $\delta_R\colon L_2^B \to L_1^A$, is defined for all $X \in {L_1}^A$ and $b \in B$ as

$$\delta_{R_p}(X)(b) \quad = \quad \bigvee_{a \in A} X(a) \& R(a,b)$$

A third possibility could be to consider the pair of operators $(\nearrow, \searrow)$, but from them we can obtain not an adjunction, but an antitone Galois connection and, thus this pair cannot be used to define an erosion and a dilation in a natural way.

As a result, by using an adjoint triple, the erosion and dilation operators defined in Equation (1) and (2) can be generalized in two ways either as $(\varepsilon_R, \delta_R)$ or as $(\varepsilon_{R_p}, \delta_{R_p})$ on the same context. Hence, depending on the set in which the erosion and the dilation need to be applied we can use either one or the other. This fact considerably increases the flexibility of the proposed methodology.

From now on, we will just study the pair $(\varepsilon_R, \delta_R)$, since similar results can be obtained for the other one by applying the general properties of these operators [15, 16]. Thus, hereafter, all *L*-fuzzy relational erosions and dilations will be considered to be defined by using adjoint triples as in Definition 5.

Since *L*-fuzzy relational erosions and dilations can be defined between two families of fuzzy sets on different universes and with different sets of truth-values, an interesting consequence of adding adjoint triples in the definition of *L*-fuzzy relational erosions is that thresholding can be considered as a specific case of the newly introduced *L*-fuzzy relational morphology. Note that although thresholding is clearly an algebraic erosion, it is not an erosion in most of the constructive approaches of mathematical morphology based on set-theoretical operations.

Example 4. *Let L be a complete lattice and let us define the operator*

$$\begin{aligned} \nearrow &\colon L \times L \to \{0,1\} \\ (\alpha,\beta) &\mapsto \alpha \nearrow \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

Note that $\{0,1\}$ with the order given by $0 \leq 1$ has structure of a complete lattice. In fact, the operator $\nearrow$ is part of an adjoint triple where $\&\colon L \times \{0,1\} \to L$ and $\searrow\colon \{0,1\} \times L \to L$ are defined by

$$\alpha \,\&\, \beta = \begin{cases} \alpha & \text{if } \beta = 1 \\ 0 & \text{if } \beta = 0 \end{cases} \qquad \alpha \searrow \beta = \begin{cases} \beta & \text{if } \alpha = 1 \\ 1 & \text{if } \alpha = 0 \end{cases} .$$

Let us consider a set A, a value $\tau \in L$ and the L-fuzzy relation R defined by $R(a,a) = \tau$, for all $a \in A$, and 0 otherwise. Then, given $X \in L^A$, we have that

$$\begin{aligned} \varepsilon_R(X)(b) &= \bigwedge_{a \in A} R(a,b) \nearrow X(a) \\ &= R(b,b) \nearrow X(b) = \begin{cases} 1 & \text{if } \tau \leq X(b) \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

In other words, the associated L-fuzzy relational erosion ε_R coincides with a thresholding (with threshold τ).

Another advantage of using the L-fuzzy relational structure of Definition 5 is that it simplifies the definition of morphological operators on abstract algebraic structures.

Example 5. *Let us consider the complete residuated lattice $([0,1], \leq, \&_G, \to_G, 0, 1)$ with the Gödel connectives, and the set $A = \{0,1,2,3,4,5,6,7\}$. On such a set we* *define the following* $[0,1]$*-fuzzy relation:*

$a \diagdown b$	0	1	2	3	4	5	6	7
0	1	0.5	0	0	0	0	0	0.5
1	1	1	1	0.5	0	0	0	0.5
2	0	0.5	1	0.5	0	0	0	0
3	0	0.5	1	1	1	0.5	0	0
4	0	0	0	0.5	1	0.5	0	0
5	0	0	0	0.5	1	1	1	0.5
6	0	0	0	0	0	0.5	1	0.5
7	1	0.5	0	0	0	0.5	1	1

Note that, fixed $a \in A$*,* $R(a,b)$ *can be considered like a structuring element centred in* a *(in the sense of [17]). Specifically, if* a *is odd, then* $R(a,b)$ *is the fuzzy set given by*

$$R(a,b)(x) = \begin{cases} 1 & \textit{if } x = a \textit{ or } x = a-1 \textit{ or } x = a+1 \\ 0.5 & \textit{if } x = a-2 \textit{ or } x = a+2 \\ 0 & \textit{otherwise.} \end{cases}$$

Whereas, if a *is even, then* $R(a,b)$ *is the fuzzy set given by*

$$R(a,b)(x) = \begin{cases} 1 & \textit{if } x = a \\ 0.5 & \textit{if } x = a-1 \textit{ or } x = a+1 \\ 0 & \textit{otherwise.} \end{cases}$$

So the relational table above somehow represents the consideration of two different kinds of structuring elements, one for odd numbers and another for even numbers. Let us consider now the fuzzy set on A *given by* $X = \{0/_{0.5}, 1/_{0.8}, 2/_{0.4}, 3/_{0.8}, 7/_{0.8}\}$*. Then, the respective erosion and dilation applied to* X *are:*

$$\varepsilon_R(X) = \{0/_{0.5}, 1/_{0.4}, 2/_{0.4}\} \textit{ and}$$
$$\delta_R(X) = \{0/_{0.5}, 1/_{0.8}, 2/_{0.5}, 3/_{0.8}, 4/_{0.5}, 5/_{0.5}, 6/_{0.5}, 7/_{0.8}\}.$$

4. Properties of *L*-fuzzy relational mathematical morphology

This section aims at showing that the basic properties of the standard mathematical morphology also hold in the framework of *L*-fuzzy relational mathematical morphology based on adjoint triples. Firstly, we show that L-fuzzy relational erosions and dilations are algebraic erosions and dilations as well; then, we present a result concerning transformation invariance (which extends the well known translation invariance of the standard MM and fuzzy MM); and finally, a result concerning the duality between L-fuzzy relational fuzzy erosions and dilations. For the sake of presentation, hereinafter we will assume the following terminology related to the underlying structure given by Definition 5:

- A and B denote two arbitrary sets,
- L_1 and L_2 denote two complete lattices,
- P denotes an arbitrary poset,
- R denotes a P-fuzzy relation on $A \times B$ and,
- $(\&, \searrow, \nearrow)$ denotes an adjoint triple among P, L_2 and L_1.

4.1. Basic properties

The theorem given below shows that every pair of L-fuzzy relational erosion and dilation, with respect to the same relation, forms an adjunction.

Theorem 2. *The pair $(\varepsilon_R, \delta_R)$, as per Definition 5, forms an adjunction.*

Proof. Let us show that for all $X \in L_1^A$ and $Y \in L_2^B$ we have $Y \le \varepsilon_R(X)$ if and only if $\delta_R(Y) \le X$. So let $X \in L_1^A$ and $Y \in L_2^B$ such that $Y \le \varepsilon_R(X)$. Then,

$$
\begin{aligned}
Y \le \varepsilon_R(X) &\iff Y(b) \le \varepsilon_R(X)(b) \quad \text{for all } b \in B \\
&\iff Y(b) \le \bigwedge_{a \in A} R(a,b) \nearrow X(a) \text{ for all } b \in B \\
&\iff Y(b) \le R(a,b) \nearrow X(a) \text{ for all } b \in B \text{ and } a \in A \\
&\iff R(a,b) \,\&\, Y(b) \le X(a) \text{ for all } b \in B \text{ and } a \in A \\
&\iff \bigvee_{b \in B} R(a,b) \,\&\, Y(b) \le X(a) \text{ for all } a \in A \\
&\iff \delta_R(Y)(a) \le X(a) \text{ for all } a \in A \\
&\iff \delta_R(Y) \le X.
\end{aligned}
$$

□

As a consequence of the previous theorem, we have that L-fuzzy relational erosions and dilations commute with infimum and supremum, respectively. In other words, ε_R and δ_R are algebraic erosions and dilations, respectively.

Corollary 1. *L-fuzzy relational dilations and L-fuzzy relational erosions are algebraic dilations and algebraic erosions, respectively.*

Proof. Direct consequence of Theorem 2. □

As a consequence of the previous result, we can conclude that L-fuzzy relational erosion and dilation operators are monotonic.

Corollary 2. *Let $X_1, X_2 \in L_1^A$ and $Y_1, Y_2 \in L_2^B$ such that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. Then,*

$$\varepsilon_R(X_1) \subseteq \varepsilon_R(X_2) \quad \text{and} \quad \delta_R(Y_1) \subseteq \delta_R(Y_2).$$

Note that the L-fuzzy relation R used to define dilations and erosions in the L-fuzzy relational MM (Definition 5) can be considered as a parameter. So we can speak about monotonicity-antitonicity of ε_R and δ_R with respect to this parameter. Specifically we have the following result.

Proposition 1. *Let R_1 and R_2 be two P-fuzzy relations on $A \times B$ such that $R_1 \leq R_2$. Then, for all $X \in L_1^A$ and $Y \in L_2^B$:*

$$\varepsilon_{R_1}(X) \supseteq \varepsilon_{R_2}(X) \quad \text{and} \quad \delta_{R_1}(Y) \subseteq \delta_{R_2}(Y).$$

Proof. Straightforward from the monotonicity of $\nearrow$ and & (Lemma 1). □

Note that Proposition 1 is similar to the one obtained in L-fuzzy mathematical morphology (based on fuzzy sets as structuring elements) about monotonicity with respect to structuring elements.

The following result concerns the composition between two erosions and two dilations. Note that to perform such a composition, the sets A and B, and the lattices L_1 and L_2 considered in Definition 5 must coincide. The following result relates the composition of erosions (resp. dilations) with the composition between L-fuzzy relations.

Proposition 2. *Let R_1 and R_2 be two L-fuzzy relations on $A \times A$ and let $(\&, \searrow, \nearrow)$ be an adjoint triple between the same complete lattice L such that & is associative, then*

$$\varepsilon_{R_1}(\varepsilon_{R_2}(X)) = \varepsilon_{R_2 \circ R_1}(X)) \quad \textit{and} \quad \delta_{R_1}(\delta_{R_2}(Y)) = \delta_{R_1 \circ R_2}(Y))$$

where $R_1 \circ R_2$ is the L-fuzzy relation on $A \times A$ given by

$$R_1 \circ R_2(a,b) = \bigvee_{c \in A} R_1(a,c) \& R_2(c,b)$$

for all $(a,b) \in A \times A$.

Proof. Let us firstly prove the equality for the composition of dilations. Let $a \in A$, then

$$\delta_{R_1}(\delta_{R_2}(Y))(a) = \bigvee_{b \in A} R_1(a,b) \& (\delta_{R_2}(Y)(b)) = \bigvee_{b \in A} R_1(a,b) \& \big(\bigvee_{c \in A} R_2(b,c) \& Y(c) \big)$$
$$= \bigvee_{b \in A} (\bigvee_{c \in A} \big(R_1(a,b) \& \big(R_2(b,c) \& Y(c) \big) \big) = \bigvee_{c \in A} (\bigvee_{b \in A} \big(R_1(a,b) \& R_2(b,c) \big) \& Y(c)$$
$$= \bigvee_{c \in A} R_1 \circ R_2(a,c) \& Y(c) = \delta_{R_1 \circ R_2}(Y)(a).$$

The result for erosions arises from Theorem 2 and the fact that if $(\varepsilon_1, \delta_1)$ and $(\varepsilon_2, \delta_2)$ are adjunctions, then $(\varepsilon_2 \circ \varepsilon_1, \delta_1 \circ \delta_2)$ is an adjunction as well (see [22, Theorem 2.7]). □

Note that the previous result is related to one in the original mathematical morphology [35] based on translations of sets (as structuring elements). Such a result states that the composition of erosions (resp. dilations) coincides with the erosion associated to the dilation of structuring elements. In our approach, the composition of L-fuzzy relations can be considered as a dilation of "structuring elements" by considering an L-fuzzy relation R as an L-fuzzy set R^b for each $b \in A$ given by $R^b(a) = R(a,b)$. Then,

$$R_1 \circ R_2(a,b) = \delta_{R_1}(R_2^b)(a)$$

and, by a slightly abuse of notation by identifying $\delta_{R_1}(R_2^b)$ with $\delta_{R_1}(R_2)$, Proposition 2 can be reformulated as

$$\varepsilon_{R_1}(\varepsilon_{R_2}(X)) = \varepsilon_{\delta_{R_2}(R_1)}(X) \qquad \text{and} \qquad \delta_{R_1}(\delta_{R_2}(Y)) = \delta_{\delta_{R_1}(R_2)}(Y)$$

for all L-fuzzy sets $X, Y \in L^A$ and all L-fuzzy relations R_1 and R_2, which clearly resembles the classical result of mathematical morphology. As a consequence of Proposition 2 we can obtain a result about commutativity of erosions and dilations.

Corollary 3. *Let R_1 and R_2 be two L-fuzzy relations on $A \times A$ and let $(\&, \searrow, \nearrow)$ be an adjoint triple between the same complete lattice L and such that & is associative. If $R_1 \circ R_2(a,b) = R_2 \circ R_1(a,b)$ for all $(a,b) \in A \times A$ then, $\varepsilon_{R_1}(\varepsilon_{R_2}(X)) = \varepsilon_{R_2}(\varepsilon_{R_1}(X))$ and $\delta_{R_1}(\delta_{R_2}(X)) = \delta_{R_2}(\delta_{R_1}(X))$ for all $X \in L^A$.*

4.2. Generalized transformation invariance

Translation invariance is an important property of the standard morphological operators in which erosions and dilations commute with translations. As we are considering L-fuzzy relations and, in general we do not require extra structure in A and B, translations are not directly applicable. Instead, we can consider interchangeability with more general transformations between universes A and B. From a mathematical point of view, transformations are usually referred to mappings of an affine space into itself, such as translations, reflections, rotations, etc. We keep the terminology, but due to the generality of our approach, in our context transformations are simply mappings with the same domain and codomain.

Given two surjective transformations $T_A\colon A \to A$ and $T_B\colon B \to B$, *Zadeh's extension principle* allows us to extend them to the L-fuzzy powersets of A and B, respectively. Specifically, $T_A\colon A \to A$ can be extended to every $X \in L^A$ as the L-fuzzy set given by the following membership function

$$\begin{array}{rcl} T_A(X)\colon A & \longrightarrow & L \\ a & \mapsto & \displaystyle\bigvee_{x \in T_A^{-1}(a)} X(x) \end{array}$$

where $T_A^{-1}(a) = \{x \in A \mid T_A(x) = a\}$. Note that when T_A is bijective, its extension on $X \in L^A$ given by Zadeh's extension principle has the following easy form: $T_A(X)(a) = X(T_A^{-1}(a))$.

Zadeh's extension principle is the most common way to extend transformations of a set into its L-fuzzy powerset [47]. Specifically, given an affine space A, the translation $T_{\vec{v}}(x) = x + \vec{v}$ by a vector $\vec{v}$ in A is extended to any L-fuzzy set $X \in L^A$, by means of Zadeh's extension principle, by

$$(X + \vec{v})(a) = T_{\vec{v}}(X)(a) = \bigvee_{x \in T_{\vec{v}}^{-1}(a)} X(x) = \bigvee_{x = a - \vec{v}} X(x) = X(a - \vec{v}).$$

The above extension is the one used by all the approaches in fuzzy mathematical morphology based on translations of structuring elements [5, 8, 12, 17, 20, 21].

The next definition introduces the notion of a pair of transformations that are, in some sense, 'compatible' with an L-fuzzy relation R.

Definition 6. *Let $R\colon A\times B\to L$ be an L-fuzzy relation. Two transformations $T_A\colon A\to A$ and $T_B\colon B\to B$ are said to be R-compatible if $R(a,b)=R(T_A(a),T_B(b))$, for all $a\in A$ and $b\in B$.*

The following proposition shows that R-compatibility for two bijective transformations is a sufficient condition for invariance.

Proposition 3. *Let $\varepsilon_R\colon L_1^A\to L_2^B$ be an L-fuzzy relational erosion and $\delta_R\colon L_2^B\to L_1^A$ be an L-fuzzy relational dilation. If both transformations $T_A\colon A\to A$ and $T_B\colon B\to B$ are R-compatible and bijective, then*

$$\varepsilon_R\circ T_A=T_B\circ\varepsilon_R \qquad \textit{and} \qquad T_A\circ\delta_R=\delta_R\circ T_B.$$

Proof. For any $X\in L_1^A$ and $b\in B$, we have:

$$\begin{aligned}
T_B(\varepsilon_R(X))(b) &= \varepsilon_R(X)(T_B^{-1}(b))\\
&= \bigwedge_{a\in A} R(a,T_B^{-1}(b)) \nearrow X(a)\\
&\overset{(\star)}{=} \bigwedge_{a\in A} R(T_A^{-1}(a),T_B^{-1}(b)) \nearrow X(T_A^{-1}(a))\\
&\overset{(\triangle)}{=} \bigwedge_{a\in A} R(a,b) \nearrow X(T_A^{-1}(a))\\
&= \varepsilon_R(T_A(X))(b)
\end{aligned}$$

where $(\star)$ follows by bijectivity of T_A, and $(\triangle)$ follows by the equality given by R-compatibility.

The equality $T_A\circ\delta_R=\delta_R\circ T_B$ can be proved similarly. □

The next example shows that bijectivity of T_A cannot be omitted in order to obtain the invariance of erosions.

Example 6. *Assume $P = L_1 = L_2 = \{0,1\}$, and the crisp connectives of classical logic for conjunction and implication & and $\nearrow$. Let us consider two sets, $A = \{a_1, a_2\}$ and $B = \{b\}$, the transformations $T_A\colon A \to A$ and $T_B\colon B \to B$ defined by $T_A(a_1) = T_A(a_2) = a_1$ and $T_B(b) = b$ and, the relation $R\colon A \times B \to \{0,1\}$ defined by $R(a_1,b) = R(a_2,b) = 1$ (i.e., the constant relation 1).*

Consider $X = A$, and let us check that we do not have the invariance of ε_R with respect to T_A and T_B. On the one hand, we have for each $b \in B$ that

$$\begin{aligned} T_B(\varepsilon_R(X))(b) &= \varepsilon_R(X)(T_B^{-1}(b)) = \varepsilon_R(X)(b) \\ &= \bigwedge_{a\in A} R(a,b) \nearrow X(a) \\ &= \big(R(a_1,b) \nearrow X(a_1)\big) \wedge \big(R(a_2,b) \nearrow X(a_2)\big) \\ &= (1 \nearrow 1) \wedge (1 \nearrow 1) = 1. \end{aligned}$$

On the other hand, taking into account that a_2 is not in the image of T_A, we have that $T_A(X)(a_2) = 0$, since $T_A^{-1}(a_2) = \varnothing$. Hence,

$$\begin{aligned} \varepsilon_R(T_A(X))(b) &= \bigwedge_{a\in A} R(a,b) \nearrow T_A(X)(a) \\ &= R(a_1,b) \nearrow T_A(X)(a_1) \wedge R(a_2,b) \nearrow T_A(X)(a_2) \\ &= (1 \nearrow 1) \wedge (1 \nearrow 0) = 0 \end{aligned}$$

for all $b \in B$. Therefore the equality in Proposition 3 does not hold.

Example 7. *Let us consider again the erosion and dilation of Example 5. It is clear that we can introduce a group structure in A by considering the sum modulo 8, (formally, consider A to be the group $(\mathbb{Z}_8, +_8)$). Moreover, is it not difficult to check that $R(x,y) = R(x +_8 2, y +_8 2)$. Then, if we consider the transformation on A given by $T(x) = x +_8 2$, we have that $T \circ \varepsilon_R = \varepsilon_R \circ T$ and $T \circ \delta_R = \delta_R \circ T$, since T is bijective and R-compatible.*

Remark 1. *In Example 4 we showed that thresholding is a special case of L-fuzzy relational erosion. Moreover, recall that the L-fuzzy relation used was defined by $R(a,a) = 1$, for all $a \in A$, and 0, otherwise. Hence, if we consider any bijective transformation $T : A \to A$, we have that $R(a,b) = R(T(a), T(b))$ straightforwardly. In other words, every bijective transformation in A commutes with any thresholding.*

Remark 2. *Let us assume that the universes A and B coincide and have the structure of an Abelian group and that the fuzzy relation R used to define* ε_R *and* δ_R *holds that* $R(a+c,b+c) = R(a,b)$ *for* $c \in A$*. Then, by Proposition 3,* ε_R *and* δ_R *are translation invariant by c; in other words* $\varepsilon_R(X+c) = \varepsilon_R(X)+c$ *and* $\delta_R(Y+c) = \delta_R(Y)+c$ *for all* $X \in L_1^A$ *and* $Y \in L_2^B$*.*

4.3. Duality

Another important property of erosions and dilations is related to duality, for which we need to assume that P, L_1, and L_2 coincide, i.e., $P = L_1 = L_2$, therefore we will be working with just one complete lattice L. Moreover, we have to assume the existence of an involutive negation in L, i.e., an antitonic operator $n\colon L \to L$ such that $n(0) = 1$, $n(1) = 0$ and $n(n(x)) = x$ for all $x \in L$. In this section, we assume such requirements together with the following construction associated with a given adjoint triple.

Definition 7. *Let* $(\&, \searrow, \nearrow)$ *be an adjoint triple defined in a lattice L with an involutive negation n. The n-adjoint triple* $(\&^n, \searrow^n, \nearrow^n)$ *of* $(\&, \searrow, \nearrow)$ *is given by the following operators:*

- $x \nearrow^n y = n(x \,\&\, n(y))$ *for all* $x, y \in L$
- $x \,\&^n y = n(x \nearrow n(y))$ *for all* $x, y \in L$
- $x \searrow^n y = n(y) \searrow n(x)$ *for all* $x, y \in L$.

The following result justifies the use of the term 'adjoint triple' in the previous definition.

Lemma 2. *Let* $(\&, \searrow, \nearrow)$ *be an adjoint triple defined in a lattice L, then* $(\&^n, \searrow^n, \nearrow^n)$ *is an adjoint triple as well.*

Proof. We need to show that $(\&^n, \searrow^n, \nearrow^n)$ satisfies the chain of equivalences in Equation (3). Given $x, y, z \in L$, we have

$$\begin{aligned} x \le y \searrow^n z &\iff x \le n(z) \searrow n(y) \\ &\iff x \,\&\, n(z) \le n(y) \\ &\overset{(*)}{\iff} y \le n(x \,\&\, n(z)) \\ &\iff y \le x \nearrow^n z \end{aligned} \tag{4}$$

where step $(*)$ holds because n is both antitone and involutive (this is also used in the derivation below).

Now, using (4), we prove the equivalence with respect to the conjunction. That is, for all $x, y, z \in L$ we have

$$\begin{aligned} x \le y \searrow^n z &\overset{(4)}{\iff} x \,\&\, n(z) \le n(y) \\ &\iff n(z) \le x \nearrow n(y) \\ &\overset{(*)}{\iff} n(x \nearrow n(y)) \le z \\ &\iff x \,\&^n y \le z. \end{aligned}$$

□

Given an involutive negation n, one can consider the corresponding n-complement of an L-fuzzy set X as $X^c(x) = n(X(x))$. Let us show how erosion and dilation of a complement can be expressed with the help of operations from the n-adjoint triple.

Given an adjoint triple $(\&, \searrow, \nearrow)$ defined in L, an involutive negation n on L, and an L-fuzzy relation R, let us write ε_R and δ_R to denote the L-fuzzy relational dilation and erosion given by Definition 5. In addition, let us write ${\varepsilon_{\overline{R}}}^d$ and ${\delta_{\overline{R}}}^d$ to refer to the erosion and dilation, again defined as in Definition 5 by using the n-adjoint triple $(\&^n, \searrow^n, \nearrow^n)$ of $(\&, \searrow, \nearrow)$ and w.r.t. the converse of R, defined by $\overline{R}(x,y) = R(y,x)$.

Proposition 4. *Given the pairs $(\varepsilon_R, \delta_R)$ and $({\varepsilon_{\overline{R}}}^d, {\delta_{\overline{R}}}^d)$ introduced above, the following equalities hold for all $X \in L^A$*

$$(\varepsilon_R(X^c))^c = {\delta_{\overline{R}}}^d(X) \qquad\qquad (\delta_R(X^c))^c = {\varepsilon_{\overline{R}}}^d(X).$$

Proof. We prove here just the first equality since the second one is obtained similarly. Given $X \in L^A$ and $b \in B$

$$\begin{aligned}(\varepsilon(X_R^c))^c(b) &= n\big(\bigwedge_{a\in A} R(a,b) \nearrow n(X(a))\big)\\ &= \bigvee_{a\in A} n\big(R(a,b) \nearrow n(X(a))\big)\\ &= \bigvee_{a\in A} R(a,b)\,\&^n X(a)\\ &= \bigvee_{a\in A} \overline{R}(b,a)\,\&^n X(a) = \delta_{\overline{R}}^{\,d}(X)(b).\end{aligned}$$

□

It is not true in general that δ_R and ε_R are dual with respect to each other; i.e. by using the same L-fuzzy relation. Note that the proposition above proves that the dual of an L-fuzzy relational dilation is an L-fuzzy relational erosion and vice versa, but by considering the versions based on the converse relation $\overline{R}$.

Remark 3. *The result in Proposition 4 can be seen as a general version of the well-known result about duality of fuzzy MM on additive abelian groups. Such an approach is a subclass of the L-fuzzy relational MM where fuzzy relations R_S are defined from fuzzy sets S (called structuring elements) as $R_S(a,b) = S(a-b)$. It is not hard to prove that in such a context, the converse relation of R_S can be identified with the symmetric of S with respect to the origin.*

Example 8. *Let us continue with Example 5, consider the negation $n(x) = 1-x$ and the n-adjoint triple of $(\&_G, \to_G)$, that is the operators given by the expressions:*

$$x \nearrow_G^n y = 1-\min(x,1-y) = \max(1-x,y)$$

$$x \&_G^n y = \begin{cases} y & \text{if } x > 1-y \\ 0 & \text{otherwise} \end{cases}$$

$$x \searrow_G^n y = \begin{cases} 1 & \text{if } y > x \\ 1-x & \text{otherwise} \end{cases}$$

Given $X = \{0/_{0.5}, 1/_{0.8}, 2/_{0.4}, 3/_{0.8}, 7/_{0.8}\}$*, let us consider the complement of* X *with respect to n:*

$$Y = X^c = \{0/_{0.5}, 1/_{0.2}, 2/_{0.6}, 3/_{0.2}, 4/_{1}, 5/_{1}, 6/_{1}, 7/_{0.2}\}$$

and the relation $\overline{R}$ *defined by* $\overline{R}(x,y) = R(y,x)$*. Then, the dilation of* Y *with respect to the n-adjoint operators above and* $\overline{R}$ *coincides with the complementary of* $\varepsilon_R(X)$ *computed in Example 5 above. That is:*

$$\begin{aligned}\delta_{\overline{R}}^{\,d}(Y) &= \varepsilon_R(X)^c = \{0/_{0.5}, 1/_{0.8}, 2/_{0.4}\}^c \\ &= \{0/_{0.5}, 1/_{0.2}, 2/_{0.6}, 3/_{1}, 4/_{1}, 5/_{1}, 6/_{1}, 7/_{1}\}.\end{aligned}$$

5. Representation Theorems

The goal of this section is to prove that the operators of algebraic morphology can be represented in terms of the L-fuzzy relational ones. Specifically, our aim is to prove the theorem that *every algebraic erosion (resp. dilation) is an L-fuzzy relational erosion (resp. dilation).* Obviously, the crux of the proof is the definition of the L-fuzzy relation which leads to the relational construction.

Let us consider the set $\mathscr{E}$ of algebraic erosions from L_1 to L_2. The set $\mathscr{E}$ can be viewed as a complete lattice when considering the following ordering:

$$\varepsilon_1 \leq \varepsilon_2 \quad \text{if and only if} \quad \varepsilon_2(x) \leq \varepsilon_1(x) \text{ for all } x \in L_1. \tag{5}$$

Now we define the following operators:

$$
\begin{aligned}
\&\colon \mathscr{E} \times L_2 &\to L_1 \\
(\varepsilon, y) &\mapsto \bigwedge\{x \in L_1 \mid y \leq \varepsilon(x)\} \\[2ex]
\nearrow\colon \mathscr{E} \times L_1 &\to L_2 \\
(\varepsilon, x) &\mapsto \varepsilon(x) \\[2ex]
\searrow\colon L_2 \times L_1 &\to \mathscr{E} \\
(y, x) &\mapsto (y \searrow x)(z) = \begin{cases} 1 & \text{if } z = 1 \\ y & \text{if } 1 > z \geq x \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
\tag{6}
$$

Note that, given an erosion $\varepsilon\colon L_1 \to L_2$ and $y \in L_2$, the conjunction $\varepsilon \,\&\, y$ coincides with $\delta_\varepsilon(y)$, where δ_ε is the residuated dilation associated with ε in the corresponding adjunction (see Theorem 1); the implication $\nearrow$ is just the application of an erosion to an argument; and, finally, $\searrow$ is a parameterized cut of level x with value y which behaves like an erosion, as shown in the following result.

Lemma 3. *For all $x \in L_1$ and $y \in L_2$, the mapping $\varepsilon_{y,x}\colon L_1 \to L_2$, determined by $y \searrow x$, is an erosion.*

Proof. Let us show that $y \searrow x$ commutes with infimum. Consider a set $Z \subseteq L_1$. Then:

$$
(y \searrow x)(\bigwedge Z) = \begin{cases} 1 & \text{if } \bigwedge Z = 1 \\ y & \text{if } 1 > \bigwedge Z \geq x \\ 0 & \text{otherwise} \end{cases} .
$$

Reasoning by cases, if $\bigwedge Z = 1$ then, the result holds straightforwardly. If $1 > \bigwedge Z \geq x$, then we have that $z \geq x$ for all $z \in Z$ and, in particular, $(y \searrow x)(z) \geq y$ for all $z \in Z$. Moreover, since $\bigwedge Z \neq 1$, there exists $z_0 \in Z$ such that $(y \searrow x)(z) = y$. Hence

$$
(y \searrow x)(\bigwedge Z) = y = \bigwedge_{z \in Z} (y \searrow x)(z).
$$

Finally, if it is not the case that $\bigwedge Z = 1$ or $1 > \bigwedge Z \geq x$, then there exists $z_0 \in Z$ such that $z_0 \geq x$ does not hold and, as a result, $(y \searrow x)(z_0) = 0$. Therefore,

$$(y \searrow x)(\bigwedge Z) = 0 = \bigwedge_{z \in Z} (y \searrow x)(z).$$

□

Now, we will prove that the three operators constructed in (6) actually determine an adjoint triple.

Lemma 4. $(\&, \searrow, \nearrow)$ *is an adjoint triple on* $\mathscr{E}, L_2, L_1$.

Proof. The chain of equivalences in Equation (3) holds by recalling that, given an erosion ε, the operator $\delta_\varepsilon(y) = \bigwedge\{x \in L_1 \mid y \leq \varepsilon(x)\}$ is the only dilation such that $(\varepsilon, \delta_\varepsilon)$ forms an adjunction. Thus, for every $x \in L_1, y \in L_2$ and $\varepsilon \in \mathscr{E}$, we have

$$\begin{aligned} \varepsilon \,\&\, y \leq x &\iff \delta_\varepsilon(y) \leq x \\ &\iff y \leq \varepsilon(x) \\ &\iff y \leq \varepsilon \nearrow x. \end{aligned}$$

It is only left to prove $\varepsilon \leq y \searrow x$ if and only if $y \leq \varepsilon \nearrow x$. Assume $\varepsilon \leq y \searrow x$, then by the ordering between erosions given in (5) we have $(y \searrow x)(z) \leq \varepsilon(z)$ for all z. For $x = 1$ we have $(y \searrow x)(x) = 1 = \varepsilon(x)$. For $z = x \neq 1$ we get $y = (y \searrow x)(x) \leq \varepsilon(x)$. From both cases we obtain that $y \leq \varepsilon \nearrow x$.

Conversely, assuming $y \leq \varepsilon \nearrow x$, i.e. $y \leq \varepsilon(x)$, by (5) it is sufficient to prove that $(y \searrow x)(z) \leq \varepsilon(z)$ for all $z \in L_1$. Note that it is enough to consider $1 > z \geq x$, since otherwise the inequality is trivial; by definition of $\searrow$, the hypothesis $y \leq \varepsilon(x)$, and monotonicity of ε we have that

$$(y \searrow x)(z) = y \leq \varepsilon(x) \leq \varepsilon(z).$$

□

Now, we introduce a family of ϕ-mappings which will be used in the proof of the main representation result. They are defined as follows: given a set A, a lattice L, and elements $a_0 \in A$ and $x \in L$, the mapping $\phi_{a_0,x}\colon A \to L$ is defined by

$$\phi_{a_0,x}(a) = \begin{cases} x & \text{if } a = a_0 \\ 1 & \text{otherwise} \end{cases}. \tag{7}$$

Note that the mappings in (7) represent somehow complements of singletons in A.

It is remarkable that the previous definition can be seen as an instantiation of a more general mapping $\phi\colon A \times L \to L^A$ by considering $\phi(a_o, x) = \phi_{a_0,x}$. It is straightforward to check the following result.

Lemma 5. *Let us consider the lattice $(L^A, \leq)$ where $\leq$ is the ordering induced by L. Then, the operator ϕ is an erosion with respect to the second component.*

We have now all the tools needed to state and prove the representation theorem by following a similar strategy than in [22, Proposition 2.10].

Theorem 3. *Let L_1 and L_2 be complete lattices, let $\mathscr{E}$ be the set of algebraic erosions from L_1 to L_2 and let A and B be ordinary sets. Then, every algebraic erosion $\varepsilon\colon L_1^A \to L_2^B$, can be represented in the form of an L-fuzzy relational erosion ε_R under the adjoint triple $(\&, \searrow, \nearrow)$ defined in* (6) *and the $\mathscr{E}$-fuzzy relation R*

$$R(a,b)(x) = \varepsilon(\phi_{a,x})(b) \quad \textit{for all } x \in L_1,$$

where $\phi_{a,x}$ denotes a ϕ-mapping given in Equation (7) *from A to L_1.*

Proof. Let us begin by proving that the $\mathscr{E}$-fuzzy relation R provided in the statement is well defined. It is not difficult to check that $R(a,b)(x) \in L_2$. In order to verify that $R(a,b)$ is indeed an erosion from L_1 to L_2, we prove that $R(a,b)$ preserves infima. Firstly, we apply (7) and use Lemma 5 so that $\phi_{a,\bigwedge_{i\in I} x_i} = \bigwedge_{i\in I} \phi_{a,x_i}$. Then, we easily obtain $\varepsilon(\phi_{a,\bigwedge_{i\in I} x_i})(b) = \varepsilon(\bigwedge_{i\in I} \phi_{a,x_i})(b) = \bigwedge_{i\in I} \varepsilon(\phi_{a,x_i})(b)$.

Let us prove that $\varepsilon_R = \varepsilon$. Firstly, we verify the equality $\varepsilon_R = \varepsilon$ for ϕ-mappings. That is, let us consider arbitrary elements $a_0 \in A$ and $x \in L_1$, and let us prove that

$\varepsilon_R(\phi_{a_0,x})(b) = \varepsilon(\phi_{a_0,x})(b)$ for all $b \in B$. Indeed

$$\begin{aligned}
\varepsilon_R(\phi_{a_0,x})(b) &= \bigwedge_{a\in A} R(a,b) \nearrow \phi_{a_0,x}(a) = \\
&\overset{(\star)}{=} \bigwedge_{a\in A} \varepsilon(\phi_{a,\phi_{a_0,x}(a)})(b) = \\
&= \varepsilon(\phi_{a_0,x})(b) \wedge \bigwedge_{a\in A, a\neq a_0} \varepsilon(\phi_{a,\phi_{a_0,x}(a)})(b) = \\
&= \varepsilon(\phi_{a_0,x})(b) \wedge \bigwedge_{a\in A, a\neq a_0} \varepsilon(\phi_{a,1})(b) = \\
&\overset{(\bullet)}{=} \varepsilon(\phi_{a_0,x})(b).
\end{aligned}$$

where $(\star)$ follows from the definition of ϕ-mapping, and $(\bullet)$ from monotonicity of ε.

Secondly, we prove the desired equality $\varepsilon_R = \varepsilon$ for a general element of $X \in L_1^A$. It is easy to see that $X \in L_1^A$ can be represented as a meet of ϕ-mappings so that $X = \bigwedge_{a\in A} \phi_{a,X(a)}$. Thus, we have, for all $b \in B$

$$\begin{aligned}
\varepsilon_R(X)(b) &= \varepsilon_R\Big(\bigwedge_{a\in A} \phi_{a,X(a)}\Big)(b) \\
&= \bigwedge_{a\in A} \varepsilon_R(\phi_{a,X(a)})(b) \\
&= \bigwedge_{a\in A} \varepsilon(\phi_{a,X(a)})(b) \\
&= \varepsilon\Big(\bigwedge_{a\in A} \phi_{a,X(a)}\Big)(b) \\
&= \varepsilon(X)(b).
\end{aligned}$$

□

As a consequence of Theorems 2 and 3, we obtain the following result.

Corollary 4. *Let L_1 and L_2 be complete lattices, A and B ordinary sets. Let $\varepsilon\colon L_1^A \to L_2^B$ and $\delta\colon L_2^B \to L_1^A$ be an algebraic erosion and an algebraic dilation, respectively, such that (ε,δ) forms an adjunction. Let moreover $(\&, \searrow, \nearrow)$ be the adjoint triple defined in* (6)*. Then, there exists an $\mathscr{E}$-fuzzy relation R such that $\varepsilon_R = \varepsilon$ and $\delta_R = \delta$.*

Proof. From Theorem 3 there exists an $\mathscr{E}$-fuzzy relation R such that $\varepsilon_R = \varepsilon$. From Theorem 2 and the unicity given in Theorem 1 we have finally that $\delta_R = \delta$. □

6. Related work and discussion

This section emphasizes similarities and differences of this contribution with respect to some related approaches concerning fuzzy MM [5, 8, 17], fuzzy relational MM [9, 10, 37] and representation theorems in the context of algebraic MM [22, 26].

The standard approaches of fuzzy MM are based on the translation of structuring fuzzy sets by either focusing on duality [8] or adjointness [17]. Firstly, our approach considers fuzzy relations instead of translations of fuzzy sets, which increase the expressiveness of fuzzy MM and eliminate the restriction of assuming an affine space as universe for fuzzy erosions and dilations. Moreover, [5] shows that in fuzzy MM (based on residuated pairs) duality is incompatible with adjointness (i.e., isotone Galois connections) in most cases. In this paper, thanks to the use of adjoint triples, we show that adjointness and duality can coexist in our *fuzzy relational* framework.

There are three main approaches concerning morphological operators defined by means of relations. In [9] Bloch *et al.* provide an interesting overview of MM and introduce several definitions of dilations and erosions in different contexts, one of them given in terms of relations in the crisp framework (originally given in [4]). This definition turned out to be useful to determine conditions to ensure that a dilation (resp. erosion) coincides with a closing (resp. opening) and to establish relationships between mathematical morphology and other fields. Another definition of erosions and dilations based on crisp relations is given in [37]. Such operators are applied satisfactorily in graph theory and are used to establish bridges between rough sets and mathematical morphology.

Another interesting approach, apart from ours, that also includes *L*-fuzzy relations into the definition of fuzzy erosions and dilations is [10]. However, in contrast to our approach, [10] does not study properties of such morphological transformations but, instead, focuses on filters constructed from them, namely the openings and closings. Moreover, the consideration of adjoint triples in our definition instead of t-norms makes our approach more general; note that this consideration is crucial for the representation theorem obtained.

Finally, concerning representation theorems of algebraic MM, we point out [22] and [26]. The representation theorem of [22] allows to represent every L-fuzzy erosion (resp. L-fuzzy dilation) as an infimum (resp. supremum) of algebraic erosions (resp. dilations) in the underlying lattice L, whereas the one in [26] is based on *clodums* and *impulse response* functions. There are two main differences between Theorem 3 and those representations theorems. The first is that Theorem 3 includes the case where erosions (resp. dilations) are defined between different complete lattices, whereas [22, Proposition 2.10] and [26, Theorem 1] consider only erosions (resp. dilations) between the same complete lattice. The second difference concerns a set-theoretical interpretation, since Theorem 3 relates algebraic MM with L-fuzzy relational MM, which has a similar interpretation as fuzzy MM in terms of (fuzzy) set-theoretical operations. Note that [22, Proposition 2.10] is given entirely in the context of algebraic MM.

7. Conclusions and future work

Although this is neither the first approach to study morphological operators in terms of fuzzy relations [9, 10] nor the first one providing a representation theorem for algebraic erosions and dilations [22, 26], to the best of our knowledge, it is the first that provides a representation theorem for algebraic erosions and dilations in terms of L-fuzzy relational MM. In order to obtain such a theorem, we have included a novelty with respect to [9, 10] in the definition of L-fuzzy relational erosions and dilations: the inclusion of adjoint triples [15]. Adjoint triples play a crucial role in this approach, since from residuated pairs (as in Definition 3) it is not possible to prove Theorem 3; algebraic erosions and dilations can be defined between different complete lattices but residuated pairs used in [9, 10] require the same lattice as set of truth values. As a side effect, the use of adjoint triples provides two different extensions depending on the set in which we want to apply the erosion and the dilation. In addition, adjoint triples preserve the set-theoretical motivations of residuated pairs and other common operators used in fuzzy set theory [16].

Apart from the two possible definitions of L-fuzzy relational erosion and dilation based on adjoint triples and the representation theorem, we have provided other two noteworthy results. The first is related to the invariance of the morphological operators under arbitrary transformations. This result is motivated by the translation invariance of the original family of crisp morphological operators [28] and the *L*-fuzzy MM [9, 17]. However, the use of *L*-fuzzy relations allows the study of transformation invariance in a much general way than translation invariance; as already studied in [22] for algebraic MM.

The second significant contribution is related to duality of erosions and dilations. As in the case of translation invariance, although it has been studied deeply in the fuzzy framework [5], so far it has not been the case in the *L-fuzzy relational* framework. In the proof of this result, adjoint triples also play a crucial role, and as in the case of Theorem 3, it cannot be achieved in fuzzy MM with only the use of residuated pairs; Proposition 4 requires the use of an ad-hoc adjoint triple even in those cases where the given erosions and dilations are defined from residuated pairs.

This work provides the basis of a promising future research. In a series of examples, we have shown that classical thresholding is a particular case of *L*-fuzzy relational erosion; thus, other kinds of thresholding (for instance by taking into account values in the neighbourhood of the element) can be defined by using fuzzy relational morphological operators. This is potentially applicable to image processing, where thresholding is used to transforms grayscale images into black and white images, or 24 bits colour images into 8 bits colour images. On the other hand, algebraic erosions and dilations generalize many different operators (e.g. lattice F-transforms [39], concept-forming operators in formal concept analysis [19], etc) applicable in other areas far from image processing. In this way, the *L*-fuzzy relational structure provided for algebraic erosions and dilations can pave the way for new field of application of mathematical morphology beyond image processing; e.g. in data analysis for classification [40] or to deal with bipolar information [6].

Another interesting topic for further research is the classification (following [43]) of algebraic mathematical morphology operators according to the kind of properties of the *L*-fuzzy relation and adjoint triples that define them, for instance, mathematical morphology based on uninorms [18, 21] or translation invariant morphology operators [22]. Last but not least, it is also interesting to relate our approach to other fields different from mathematical morphology, like lattice F-transforms [39], fuzzy concept lattices [1] or graph theory [38].

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