On the definition of suitable orderings to generate adjunctions over an unstructured codomain

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Abstract

Given a mapping $f\colon A \to B$ from a (pre-)ordered set A into an unstructured set B, we study the problem of defining a suitable (pre-)ordering relation on B such that there exists a mapping $g\colon B \to A$ such that the pair of mappings (f, g) forms an adjunction between (pre-)ordered sets. The necessary and sufficient conditions obtained are then expressed in terms of closure operators and closure systems.

Key words: Adjunction, Isotone Galois connection, preorder relation, partial order relation

1. Introduction

Adjunctions were introduced in 1958 by Kan [26]; although defined in a categorical context, or perhaps precisely because of this, an impressive amount of examples of adjunction can be found in a several research areas, ranging from the most theoretical to the most applied. In the realm of ordered structures, Ore [39] had introduced in 1944 the so-called Galois connections as a pair of antitone mappings, generalizing Birkhoff's theory of polarities to work with complete lattices; it turns out that, when instantiating an adjunction to categories of ordered sets, it can be seen that both constructions are fairly similar and, to some extent, are interdefinable: an adjunction between A and B is a Galois connection in which the order relation on B is reversed (this leads to the use of the term *isotone Galois connection* to refer to an adjunction between ordered structures).

The importance of Galois connections/adjunctions quickly increased to the extent that, for instance, the interest of category theorists moved from universal

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mapping properties and natural transformations to adjointness.

In recent years there has been a notable increase in the number of publications concerning Galois connections, both isotone and antitone. On the one hand, one can find lots of papers on theoretical developments or theoretical applications [9, 11, 28] On the other hand, of course, there exist as well a lot of applications to computer science, see [36] for a first survey on applications, although more specific references on certain topics can be found, for instance, to programming [38], data analysis [36], logic [14, 25]. One research topic which has benefitted recently from the use of the theory of Galois connections is that of approximate reasoning using rough sets [24, 40, 15].

Last but not least, it is worth noting that many of these works use Galois connection for dealing with Formal Concept Analysis (FCA), either theoretically or applicatively, for instance Antoni et al. [1] develop a general framework for fuzzy FCA, Butka et al. [6] prove the equivalence of two previously existing frameworks, Díaz and Medina [12] use Galois connections as building blocks for solving the multi-adjoint relation equations, Medina [33] develops new generalized frameworks for FCA, Dubois and Prade [13] introduce a relationship between FCA and possibility theory, Bĕlohlávek and Konečný [5] stress on the "duality" between isotone and antitone Galois connections in showing a case of mutual reducibility of the concept lattices generated by using each type of connection, etcetera.

It is not surprising to see so many works dealing with both Galois connections and FCA, since the derivation operators used to define the concepts form a (antitone) Galois connection. Valverde and Peláez have studied the extension of conceptualization modes in [42], and provided a general approach to the discipline.

The ability to build or define a Galois connection between two ordered structures is a matter of major importance, and not only for FCA. For instance, [10] establishes a Galois connection between valued constraint languages and sets of weighted polymorphisms in order to develop an algebraic theory of complexity for valued constraint languages.

A number of results can be found in the literature concerning sufficient or necessary conditions for a Galois connection between ordered structures to exist. The main results of this paper are related to the existence and construction of the adjoint pair to a given mapping f, but *in a more general framework*.

Our initial setting is to consider a mapping $f \colon A \to B$ from a partially ordered (resp. preordered) set A into an unstructured set B, and then characterize those situations in which the set B can be partially ordered (resp. preordered) and an isotone mapping $g \colon B \to A$ can be built such that the pair (f, g) is an adjunction. (Please note that, *for brevity's sake*, hereafter we will use exclusively the term adjunction instead of isotone Galois connection).

There is a tight relation between adjunctions and closure and kernel operators, in that every adjunction (f, g) leads to a closure operator $g \circ f$ and a kernel operator $f \circ g$. After obtaining the necessary and sufficient conditions to define a preorder on B, it makes sense to express those conditions in terms of the corresponding closure and/or kernel operators in a preordered setting.

Concerning potential applications of the present work, let us recall that the Galois connections used in FCA are given between the Boole algebras of the powersets of objects and the powerset of attributes. There exist several generalizations in FCA which weaken the structure on which a Galois connection is defined: for instance, in fuzzy FCA, the underlying structure used of the powerset of fuzzy sets is that of a residuated lattice. In [21], a general approach called *pattern structures* was proposed, which allows for extending FCA techniques to arbitrary partially ordered data descriptions. Using pattern structures, one can compute taxonomies, ontologies, implications, implication bases, association rules, concept-based (or JSM-) hypotheses in the same way it is done with standard concept lattices [30]. In this generalization, instead of associating each object with the set of attributes it satisfies, a pattern is given, which can be either a graph, or a sequence or an interval, and the semantics of these patterns can be different in each case. For instance, [20] represents scenarios of conflict between human agents, and [32] use gene expression data. These sets of patterns are provided with a partial ordering relation such as "*being a subgraph of*" or "*being a subchain of*".

The proposed formalism is not confined to applications in FCA. Rather, it is potentially useful in any domain in which the theory of partial orders can be applied, since the knowledge of the existence of a suitable ordering (whenever it exists), enables the full machinery of Galois connections to be used within the theory (whatever it might be). For instance, there are many works which suggest the use of the theory of order in the field of chemistry; one can even find special issues solely dedicated to the use of partial orders in this discipline. Specifically, in relation to FCA, [2] applies FCA to the classification of ancient objects (namely, ancient egyptian bronze artifacts); more recently, [27] advocates the use of partial orderings and the theory of formal concept analysis in environmental sciences, particularly, for studying a certain class of pesticides. Another interesting application field is linguistics: in [37], the study of grammatical inference in Lambek languages (both simple and mixed) is done in terms of Galois connections (the author used the term residuation); in [31], the author argues the use of iterated Galois connections in relation with an algebraic approach to the structure of sentences in a natural language. A third application field can be found in bioinformatics: [41] uses properties of Galois connections in order to identify sets of genes from microarray data sets; [18] applies Logic Information Systems (a framework based on logic-based FCA) to develop an

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application to help biologist extract information from raw genome data; [17] apply the framework of abstract interpretation (heavily based on the notion of Galois connection) to the formalization of further abstractions commonly used in systems biology as type systems, the authors analyze several different semantics, finding difficulties in one which they could not relate with a simple Galois connection.

The structure of the paper is as follows: in Section 2, given $f \colon A \to B$ we focus on the case in which domain A has a poset structure, after introducing the preliminary definitions, the necessary and sufficient conditions for the existence of a unique ordering on B and a mapping g such that (f, g) is an adjunction; then, in Section 3 we reproduce the study done in the previous section, although the main ideas underlying the results are the same, the absence of antisymmetry makes the proof of the results much more involved. Later, in Section 5 we state the necessary and sufficient conditions obtained in the previous section in terms of closure operators and closure systems. Finally, in Section 6, we draw some conclusions and discuss future work.

It is worth to remark that, although all the results will be stated in terms of the existence and construction of adjunctions on the right, all of them can be straightforwardly modified for the existence and construction of adjunctions on the left.

2. Building adjunctions between partially ordered sets

2.1. Preliminaries

We assume basic knowledge of the properties and constructions related to a partially ordered set. For the sake of self-completion, we include below the formal definitions of the main concepts to be used in this section.

Definition 1. *Given a partially ordered set $\langle A, \leq_A \rangle$, $X \subseteq A$, and $a \in A$.*

- *An element M is said to be the* maximum *of X, denoted $\max X$, if $M \in X$ and $x \leq M$ for all $x \in X$.*

- *An element m is said to be the* minimum *of X, denoted $\min X$, if $m \in X$ and $m \leq x$ for all $x \in X$.*

- *The* downward closure *$a^{\downarrow}$ of a is defined as $a^{\downarrow} = \{x \in A \mid x \leq_A a\}$.*

- *The* upward closure *$a^{\uparrow}$ of a is defined as $a^{\uparrow} = \{x \in A \mid a \leq_A x\}$.*

In addition, given a poset $\langle B, \leq_B \rangle$, a mapping $f \colon A \to B$ is said to be

- isotone *if $a_1 \leq_A a_2$ implies $f(a_1) \leq_B f(a_2)$, for all $a_1, a_2 \in A$.*

- antitone *if $a_1 \leq_A a_2$ implies $f(a_2) \leq_B f(a_1)$, for all $a_1, a_2 \in A$.*

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In the particular case in which $A = B$,

- *f is* inflationary *(also called extensive) if $a \leq_A f(a)$ for all $a \in A$.*

- *f is* deflationary *(also called contractive) if $f(a) \leq_A a$ for all $a \in A$.*

As we are including the necessary definitions for the development of the construction of adjunctions between posets, we state below the definition of adjunction we will be working with.

Definition 2. *Let $\mathcal{A} = \langle A, \leq_A \rangle$ and $\mathcal{B} = \langle B, \leq_B \rangle$ be posets, $f \colon A \to B$ and $g \colon B \to A$ be two mappings. The pair (f, g) is said to be an* adjunction between *$\mathcal{A}$ and $\mathcal{B}$, denoted by $(f, g) \colon \mathcal{A} \leftrightharpoons \mathcal{B}$, whenever for all $a \in A$ and $b \in B$ we have that*

$$f(a) \leq_B b \qquad \text{if and only if} \qquad a \leq_A g(b)$$

The mapping f is called left adjoint *and g is called* right adjoint.

The following theorem states equivalent definitions of adjunction between posets that can be found in the literature [22].

Theorem 1. *Let $\mathcal{A} = \langle A, \leq_A \rangle, \mathcal{B} = \langle B, \leq_B \rangle$ be two posets, $f \colon A \to B$ and $g \colon B \to A$ be two mappings. The following statements are equivalent:*

1. *$(f, g) \colon \mathcal{A} \leftrightharpoons \mathcal{B}$.*
2. *f and g are isotone, $g \circ f$ is inflationary, and $f \circ g$ is deflationary.*
3. *$f(a)^\uparrow = g^{-1}(a^\uparrow)$ for all $a \in A$.*
4. *$g(b)^\downarrow = f^{-1}(b^\downarrow)$ for all $b \in B$.*
5. *f is isotone and $g(b) = \max f^{-1}(b^\downarrow)$ for all $b \in B$.*
6. *g is isotone and $f(a) = \min g^{-1}(a^\uparrow)$ for all $a \in A$.*

We introduce the technical lemma below which shows that, in some cases, it is possible to get rid of the downward closure (as used in item 5 of the previous theorem).

Lemma 1. *Let $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$ be posets, $f \colon A \to B$ an isotone mapping and let $b \in f(A)$. If $\max f^{-1}(b^\downarrow)$ exists, then $\max f^{-1}(b)$ exists and $\max f^{-1}(b^\downarrow) = \max f^{-1}(b)$.*

Proof. Let us write $m = \max f^{-1}(b^\downarrow)$. We will prove that $a \leq_A m$, for all $a \in f^{-1}(b)$, and $m \in f^{-1}(b)$, so that we have $m = \max f^{-1}(b)$.

Consider $a \in f^{-1}(b)$, then $f(a) = b \in b^\downarrow$ and $a \in f^{-1}(b^\downarrow)$, hence $a \leq_A m$.

Now, isotonicity of f shows that $f(a) = b \leq_B f(m)$. For the other inequality, simply consider that $m = \max f^{-1}(b^\downarrow)$ implies $m \in f^{-1}(b^\downarrow)$, which means $f(m) \leq_B b$. Therefore, $f(m) = b$ because of antisymmetry of $\leq_B$. □

2.2. The construction

In general, given a poset $\langle A, \leq_A \rangle$ together with an equivalence relation $\sim$ on A, it is customary to consider the set $A_\sim = A/\sim$, the quotient set of A wrt $\sim$, and the natural projection $\pi \colon A \to A_\sim$. As usual, the equivalence class of an element $a \in A$ is denoted $[a]_\sim$ and, then, $\pi(a) = [a]_\sim$.

With the aim of finding conditions for building a right adjoint to a mapping f from a poset $(A, \leq_A)$ to an unstructured set B, we will naturally consider the canonical decomposition of $f \colon A \to B$ through $A_{\equiv_f}$, the quotient set of A wrt the **kernel relation** $\equiv_f$ defined as $a \equiv_f b$ if and only if $f(a) = f(b)$:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ f\ \ } & B \\
{\scriptstyle \pi}\downarrow & & \uparrow{\scriptstyle i} \\
A_{\equiv_f} & \dashrightarrow[\varphi] & f(A)
\end{array}
$$

The following lemma provides sufficient conditions for the natural projection being the left component of an adjunction.

Lemma 2. *Let $\langle A, \leq_A \rangle$ be a poset and $\sim$ an equivalence relation on A. Suppose that the following conditions hold*

1. *There exists $\max([a]_\sim)$, for all $a \in A$.*
2. *If $a_1 \leq_A a_2$ then $\max([a_1]_\sim) \leq_A \max([a_2]_\sim)$, for all $a_1, a_2 \in A$.*

Then, the relation $\leq_{A_\sim}$ defined by $[a_1]_\sim \leq_{A_\sim} [a_2]_\sim$ if and only if $a_1 \leq_A \max([a_2]_\sim)$ is an ordering in $A_\sim$ and, moreover, the pair $(\pi, \max)$ is an adjunction between $\langle A, \leq_A \rangle$ and $\langle A_\sim, \leq_{A_\sim} \rangle$.

Proof. To begin with, the relation $\leq_{A_\sim}$ is well defined since, by the first hypothesis, $\max([a]_\sim)$ exists for all $a \in A$.

Reflexivity Obvious, since $[a]_\sim \leq_{A_\sim} [a]_\sim$ if and only if $a \leq_A \max([a]_\sim)$, and the latter holds for all $a \in A$.

Transitivity Assume $[a_1]_\sim \leq_{A_\sim} [a_2]_\sim$ and $[a_2]_\sim \leq_{A_\sim} [a_3]_\sim$.

From $[a_1]_\sim \leq_{A_\sim} [a_2]_\sim$, by definition, we have $a_1 \leq_A \max([a_2]_\sim)$. Now, from $[a_2]_\sim \leq_{A_\sim} [a_3]_\sim$ we obtain, by definition of the ordering and the second hypothesis that $\max([a_2]_\sim) \leq_A \max([a_3]_\sim)$. As a result, we obtain $[a_1] \leq_{A_\sim} \max([a_3]_\sim)$, that is, $[a_1]_\sim \leq_{A_\sim} [a_3]_\sim$.

Antisymmetry Consider two elements $a_1, a_2 \in A$ such that $[a_1]_\sim \leq_{A_\sim} [a_2]_\sim$ and $[a_2]_\sim \leq_{A_\sim} [a_1]_\sim$.

By hypothesis, we have that $a_1 \leq_A \max([a_2]_\sim)$ then $\max([a_1]_\sim) \leq_A \max([a_2]_\sim)$, and $a_2 \leq_A \max([a_1]_\sim)$ then $\max([a_2]_\sim) \leq_A \max([a_1])_\sim$. Since $\leq_A$ is antisymmetric, then $\max([a_1]_\sim) = \max([a_2]_\sim)$; now, we have

that the intersection of the two classes $[a_1]_\sim$ and $[a_2]_\sim$ is non-empty, therefore $[a_1]_\sim = [a_2]_\sim$.

Once again by the first hypothesis, max can be seen as a mapping $A_\sim \to A$. Whence, the adjunction follows by the definition of π and the ordering:

$$\pi(a_1) \leq_{A_\sim} [a_2]_\sim \quad \text{if and only if} \quad [a_1]_\sim \leq_{A_\sim} [a_2]_\sim$$
$$\text{if and only if} \quad a_1 \leq_A \max([a_2]_\sim)$$

$\square$

The previous lemma gave sufficient conditions for π being a left adjoint; the following result states that the conditions are also necessary, and that the ordering relation and the right adjoint are uniquely defined.

Lemma 3. *Let $\langle A, \leq_A \rangle$ be a poset and $\sim$ an equivalence relation on A. Let $A_\sim = A/\sim$ be the quotient set of A wrt $\sim$, and $\pi \colon A \to A_\sim$ the natural projection. If there exists an ordering relation $\leq_{A_\sim}$ in $A_\sim$ and $g \colon A_\sim \to A$ such that $(\pi, g) \colon \langle A, \leq_A \rangle \leftrightharpoons \langle A_\sim, \leq_{A_\sim} \rangle$ then,*

1. *$g([a]_\sim) = \max([a]_\sim)$ for all $a \in A$.*
2. *$[a_1]_\sim \leq_{A_\sim} [a_2]_\sim$ if and only if $a_1 \leq_A \max([a_2]_\sim)$ for all $a_1, a_2 \in A$.*
3. *If $a_1 \leq_A a_2$ then $\max([a_1]_\sim) \leq_A \max([a_2]_\sim)$ for all $a_1, a_2 \in A$.*

Proof.

1. By Theorem 1, we have $g([a]_\sim) = \max \pi^{-1}([a]_\sim^\downarrow)$. Now, Lemma 1 leads to $\max \pi^{-1}([a]_\sim^\downarrow) = \max \pi^{-1}([a]_\sim) = \max([a]_\sim)$.

   Note that there is a slight abuse of notation, in that $[a]_\sim$ is sometimes considered as a single element, i.e. one equivalence class of the quotient set, and sometimes as the set of elements of the equivalence class. The context helps to clarify which meaning is intended in each case.

2. By the adjointness of (π, g), definition of π, and the previous item we have the following chain of equivalences

$$[a_1]_\sim \leq_{A_\sim} [a_2]_\sim \quad \text{if and only if} \quad \pi(a_1) \leq_{A_\sim} [a_2]_\sim$$
$$\text{if and only if} \quad a_1 \leq_A g([a_2]_\sim)$$
$$\text{if and only if} \quad a_1 \leq_A \max([a_2]_\sim)$$

3. Finally, since π and g are isotone maps, $a_1 \leq_A a_2$ implies $[a_1]_\sim \leq_{A_\sim} [a_2]_\sim$, and $g([a_1]_\sim) \leq_A g([a_2]_\sim)$, therefore $\max([a_1]_\sim) \leq_A \max([a_2]_\sim)$ by the first item above. $\square$

Continuing with the analysis of the decomposition, we naturally arrive to the following result.

Lemma 4. *Consider a poset $\langle A, \leq_A \rangle$ and a bijective mapping $\varphi \colon A \to B$, then there exists a unique ordering relation in B, which is defined as $b_1 \leq_B b_2$ if and only if $\varphi^{-1}(b_1) \leq_A \varphi^{-1}(b_2)$, such that $(\varphi, \varphi^{-1}) \colon \langle A, \leq_A \rangle \leftrightharpoons \langle B, \leq_B \rangle$.*

Proof. Straightforward. □

As a consequence of the previous results, we have established necessary and sufficient conditions ensuring the existence and uniqueness of right adjoint for any *surjective* mapping f from a poset A to an unstructured set B.

Theorem 2. *Given a poset $\langle A, \leq_A \rangle$ and a surjective mapping $f \colon A \to B$, let $\equiv_f$ be the kernel relation. Then, there exists an ordering $\leq_B$ in B and a mapping $g \colon B \to A$ such that $(f, g) \colon \langle A, \leq_A \rangle \leftrightharpoons \langle B, \leq_B \rangle$ if and only if*

1. *There exists $\max([a]_{\equiv_f})$ for all $a \in A$.*
2. *For all $a_1, a_2 \in A$, $a_1 \leq_A a_2$ implies $\max([a_1]_{\equiv_f}) \leq_A \max([a_2]_{\equiv_f})$.*

Proof. Assume that there exists an adjunction $(f, g) \colon A \leftrightharpoons B$ and let us prove items 1 and 2.

Given $a \in A$, item 1 holds because of the following chain of equalities, where the first equality follows from Theorem 1, the second one follows from Lemma 1, and the third because of the definition of $[a]_{\equiv_f}$:

$$g(f(a)) = \max f^{-1}(f(a)^{\downarrow}) = \max f^{-1}(f(a)) = \max([a]_{\equiv_f}) \qquad (1)$$

Now, item 2 is straightforward, because if $a_1 \leq_A a_2$ then, by isotonicity, $f(a_1) \leq_B f(a_2)$ and $g(f(a_1)) \leq_A g(f(a_2))$. Therefore, by Equation (1) above, $\max([a_1]_{\equiv_f}) \leq_A \max([a_2]_{\equiv_f})$.

Conversely, given $\langle A, \leq_A \rangle$ and $f \colon A \to B$ and items 1 and 2, let us prove that f is the left adjoint of a mapping $g \colon B \to A$. To begin with, consider the canonical decomposition of f through the quotient set A_f of A wrt $\equiv_f$, see below, where $\pi \colon A \to A_f$ is the natural projection, $\pi(a) = [a]_{\equiv_f}$, and $\varphi([a]_{\equiv_f}) = f(a)$.

Firstly, by Lemma 2, using conditions 1 and 2, and the fact that $[a]_{\equiv_f} = \pi(a)$, we obtain that $(\pi, \max) \colon \langle A, \leq_A \rangle \leftrightharpoons \langle A_{\equiv_f}, \leq_{\equiv_f} \rangle$.

Moreover, since the mapping $\varphi \colon A_{\equiv_f} \to B$ is bijective, we can apply Lemma 4 in order to induce an ordering $\leq_B$ on B such that we have another adjunction, the pair $(\varphi, \varphi^{-1}) \colon \langle A_{\equiv_f}, \leq_{\equiv_f} \rangle \leftrightharpoons \langle B, \leq_B \rangle$.

Finally, the composition $g = \max \circ \varphi^{-1} \colon B \to A$ is such that (f, g) is an adjunction. □

The third part of this section is devoted to considering the case in which f is not surjective. In this case, in general, there are several possible orderings

on B which allows us to define the right adjoint. The crux of the construction is related to the definition of an order-embedding of the image into the codomain set.

More generally, the idea is to extend an ordering defined just on a subset of a set to the whole set.

Proposition 1. *Given a subset $X \subseteq B$, and a fixed element $m \in X$, any preordering $\leq_X$ in X can be extended to a preordering $\leq_m$ on B, defined as the reflexive and transitive closure of the relation $\leq_X \cup \{(m, y) \mid y \notin X\}$.*

Note that the relation above can be described, for all $x, y \in B$, as $x \leq_m y$ if and only if some of the following holds:

(a) $x, y \in X$ and $x \leq_X y$

(b) $x \in X, y \notin X$ and $x \leq_X m$

(c) $x, y \notin X$ and $x = y$

It is not difficult to check that if the initial relation $\leq_X$ is an ordering relation, then $\leq_m$ is an ordering as well. Formally, we have

Lemma 5. *Given a subset $X \subseteq B$, and a fixed element $m \in X$, then $\leq_X$ is an ordering in X if and only if $\leq_m$ is an ordering on B.*

Proof. Just some routine computations are needed to check that $\leq_m$ is anti-symmetric using the properties of $\leq_X$.

Conversely, if $\leq_m$ is an ordering, then $\leq_X$ is an ordering as well, since it is a restriction of $\leq_m$. $\square$

Lemma 6. *Let X be a subset of B, consider a fixed element $m \in X$, and an ordering $\leq_X$ in X. Define the mapping $j_m \colon \langle B, \leq_m \rangle \to \langle X, \leq_X \rangle$ as*

$$j_m(x) = \begin{cases} x & \text{if } x \in X \\ m & \text{if } x \notin X \end{cases}$$

Then, $(i, j_m) \colon \langle X, \leq_X \rangle \leftrightharpoons \langle B, \leq_m \rangle$ where i denotes the inclusion $X \hookrightarrow B$.

Proof. It follows easily by routine computation. $\square$

Theorem 3. *Given a poset $\langle A, \leq_A \rangle$ and a mapping $f \colon A \to B$, let $\equiv_f$ be the kernel relation. Then, there exists an ordering $\leq_B$ in B and a mapping $g \colon B \to A$ such that $(f, g) \colon \langle A, \leq_A \rangle \leftrightharpoons \langle B, \leq_B \rangle$ if and only if*

1. *There exists $\max([a]_{\equiv_f})$ for all $a \in A$.*
2. *$a_1 \leq_A a_2$ implies $\max([a_1]_{\equiv_f}) \leq_A \max([a_2]_{\equiv_f})$, for all $a_1, a_2 \in A$.*

Proof. Assume that there exists an adjunction $(f,g)\colon \langle A, \leq_A \rangle \leftrightharpoons \langle B, \leq_B \rangle$. The proof of items 1 and 2 is exactly the same of that in Theorem 2, in which assumption of f being surjective was not used.

Conversely, given $\langle A, \leq_A \rangle$ and $f\colon A \to B$ satisfying items 1 and 2, let us prove that f is the left adjoint of a mapping $g\colon B \to A$. By Theorem 2, there exists an ordering $\leq_{f(A)}$ on $f(A)$ and a mapping $g'\colon f(A) \to B$ such that $(f,g')\colon \langle A, \leq_A \rangle \leftrightharpoons \langle f(A), \leq_{f(A)} \rangle$.

Now, considering an arbitrary element $m \in f(A)$, the ordering $\leq_{f(A)}$ induces an ordering $\leq_m$ on B, as stated in Lemma 5, and a mapping $j_m\colon B \to f(A)$ such that $(i, j_m)\colon \langle f(A), \leq_{f(A)} \rangle \leftrightharpoons \langle B, \leq_B \rangle$.

The composition $g = g' \circ j_m\colon B \to A$ is such that (f,g) is an adjunction. $\square$

Pictorially, we have the following commutative diagram of adjoint pairs, in which the mapping g' above is the composition $\max \circ \varphi^{-1}$

$$g = \max \circ \varphi^{-1} \circ j_m$$

We finish this section with two counterexamples showing that the conditions in the theorem cannot be removed.

Example 1. *Let $A = \{a, b, c\}$ and $B = \{d, e\}$ be two sets and $f\colon A \to B$ defined as $f(a) = d$ and $f(b) = f(c) = e$.*

Condition 1 cannot be removed: *Consider the situation depicted below, in which we have a poset $\langle A, \leq \rangle$ with $a \leq b, a \leq c$ and b, c are not related, together with a mapping f to an unstructured set B. Then $[b]_{\equiv_f} = \{b, c\}$ and there does not exist $\max([b]_{\equiv_f})$.*
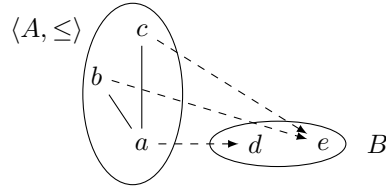
The right adjoint cannot exist because $\max f^{-1}(e^{\downarrow})$ does not exist for any ordering in B.

Condition 2 cannot be removed: *Consider $\langle A, \leq \rangle$, where $b \leq a \leq c$.*
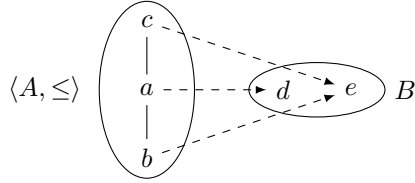
In this case, Condition 1 holds, since there exist both $\max[a]_{\equiv_f} = a$ and $\max[b]_{\equiv_f} = c$, but Condition 2 clearly does not. Again, the right adjoint does not exist because f will never be isotone in any possible ordering defined in B.

3. Building adjunctions between preordered sets

In this section we extend the previous results to the framework of preordered sets. The idea underlying the construction is similar to that above, but the absence of antisymmetry makes the low level computations much more involved than in the partially ordered case.

3.1. Preliminaries

The definitions of downward (resp. upward) closure of an element in a pre-ordered set, and those of isotone, antitone, inflationary and deflationary mapping between preordered sets are exactly the same as those given for posets.

The notion of maximum or minimum element of a subset of a preordered set is defined as usual. Note, however, that due to the absence of antisymmetry, these elements need not be unique. This is an important difference which justifies the introduction of special terminology in this context.

Definition 3. *Given a preordered set $\langle A, \lesssim_A \rangle$ and a subset $X \subseteq A$, an element $a \in A$ is said to be a p-maximum (resp., p-minimum) of X if $a \in X$ and $x \lesssim_A a$ (resp., $a \lesssim_A x$) for all $x \in X$. The set of p-maxima (resp., p-minima) of X will be denoted as p-max(X) (resp., p-min(X)).*

Notice that p-max(X) (resp., p-min(X)) need not be a singleton. In the event that, say $a, b \in$ p-max(X), then the two relations $a \lesssim_A b$ and $b \lesssim_A a$ hold. As this situation will repeat several times, we introduce the equivalence relation $\approx_A$ in any preordered set $\langle A, \lesssim_A \rangle$, defined as follows for $a_1, a_2 \in A$:

$$a_1 \approx_A a_2 \quad \text{if and only if} \quad a_1 \lesssim_A a_2 \text{ and } a_2 \lesssim_A a_1 \qquad (2)$$

In this section we will assume a mapping $f \colon A \to B$ such that the original set is preordered. In order to study the existence of adjoints in this framework, we will need to use the previously defined relation $\approx_A$ and we will keep using the kernel relation $\equiv_f$.

The two relations above are used together in the definition of the *p-kernel* relation defined below:

Definition 4. *Let $\langle A, \lesssim_A \rangle$ be a preordered set, and $f\colon A \to B$ a mapping. The p-kernel relation $\cong_A$ on A is the equivalence relation obtained as the transitive closure of the union of the relations $\approx_A$ and $\equiv_f$.*

It is well-known that the transitive closure in the definition above can be described as follows: given $a_1, a_2 \in A$, we have that $a_1 \cong_A a_2$ if and only if there exists a finite chain $\{x_i\}_{i \in \{1,\ldots,n\}} \subseteq A$ such that $x_1 = a_1$, $x_n = a_2$ and, for all $i \in \{1, \ldots, n-1\}$, either $x_i \equiv_f x_{i+1}$ or $x_i \approx_A x_{i+1}$.

The following theorem states different equivalent characterizations of the notion of adjunction between preordered sets that will be used in the main construction of the right adjoint. As expected, the general structure of the definitions is preserved, but those concerning the actual definition of the adjoints have to be modified by using the notions of p-maximum and p-minimum.

Theorem 4 ([22]). *Let $\mathcal{A} = \langle A, \lesssim_A \rangle, \mathcal{B} = \langle B, \lesssim_B \rangle$ be two preordered sets, and $f\colon \mathcal{A} \to \mathcal{B}$ and $g\colon \mathcal{B} \to \mathcal{A}$ be two mappings. The following statements are equivalent:*

1. *$(f, g) : \mathcal{A} \leftrightharpoons \mathcal{B}$.*
2. *f and g are isotone, and $g \circ f$ is inflationary, $f \circ g$ is deflationary.*
3. *$f(a)^\uparrow = g^{-1}(a^\uparrow)$ for all $a \in A$.*
4. *$g(b)^\downarrow = f^{-1}(b^\downarrow)$ for all $b \in B$.*
5. *f is isotone and $g(b) \in \text{p-max}\, f^{-1}(b^\downarrow)$ for all $b \in B$.*
6. *g is isotone and $f(a) \in \text{p-min}\, g^{-1}(a^\uparrow)$ for all $a \in A$.*

Once again, the absence of antisymmetry leads to slight modifications of some well-known properties of adjunctions, as stated in the result below and its corollary.

Theorem 5. *Let $\mathcal{A} = \langle A, \lesssim_A \rangle, \mathcal{B} = \langle B, \lesssim_B \rangle$ be two preordered sets, and $f\colon A \to B$ and $g\colon B \to A$ be two mappings. If $(f, g)\colon \mathcal{A} \leftrightharpoons \mathcal{B}$ then, $(f \circ g \circ f)(a) \approx_B f(a)$ for all $a \in A$, and $(g \circ f \circ g)(b) \approx_A g(b)$ for all $b \in B$.*

Corollary 1. *Let $\mathcal{A} = \langle A, \lesssim_A \rangle, \mathcal{B} = \langle B, \lesssim_B \rangle$ be two preordered sets, and $f\colon A \to B$ and $g\colon B \to A$ be two mappings. If $(f, g)\colon \mathcal{A} \leftrightharpoons \mathcal{B}$ then, $(g \circ f \circ g \circ f)(a) \approx_A (g \circ f)(a)$ for all $a \in A$, and $(f \circ g \circ f \circ g)(b) \approx_B (f \circ g)(b)$ for all $b \in B$.*

The following definition recalls the notion of Hoare ordering between subsets of a preordered set, and then introduces an alternative statement in the subsequent lemma.

Definition 5. *Let $\langle A, \lesssim_A \rangle$ be a preordered set, and consider $X, Y \subseteq A$.*

- *We will denote by $\sqsubseteq_H$ the Hoare relation, $X \sqsubseteq_H Y$ if and only if, for all $x \in X$, there exists $y \in Y$ such that $x \lesssim_A y$.*

- *We define $X \sqsubseteq Y$ if and only if there exist $x \in X$ and $y \in Y$ such as $x \lesssim_A y$.*

Lemma 7. *Let $\langle A, \lesssim_A \rangle$ be a preordered set, and consider $X, Y \subseteq A$ such that p-min$(X) \neq \varnothing$ and p-min$(Y) \neq \varnothing$. The following statements are equivalent:*

1. *p-min$(X) \sqsubseteq_H$ p-min(Y).*
2. *p-min$(X) \sqsubseteq$ p-min(Y).*
3. *For all $x \in$ p-min(X) and for all $y \in$ p-min(Y), $x \lesssim_A y$.*

Proof. The implications $1 \Rightarrow 2$ and $3 \Rightarrow 1$ are straightforward. Let us prove, $2 \Rightarrow 3$. For this, consider any $x \in$ p-min(X) and $y \in$ p-min(Y). Using the hypothesis and $x \in$ p-min(X), we have that, there exists $y_1 \in$ p-min(Y) such that $x \lesssim_A y_1$. Since $y_1 \in$ p-min(Y), we have that $y_1 \lesssim_A y$ for all $y \in Y$. Therefore, $x \lesssim_A y$ for all $x \in$ p-min(X) and $y \in$ p-min(Y). $\square$

We finish this section of preliminaries introducing the notation UB(X) to mean the set of upper bounds of the subset X of a preordered set, together with the operator φ which will be frequently used in the construction of the right adjoint g.

Definition 6. *Let $\langle A, \lesssim_A \rangle$ be a preordered set and let X, S be subsets of A. The set of upper bounds of X is defined as follows*

$$UB(X) = \{b \in A \mid x \leq b \text{ for all } x \in X\}$$

The mapping $\varphi_S \colon A \to 2^A$ is defined as

$$\varphi_S(a) = \text{p-min}(\text{UB}([a]_{\cong_A}) \cap S)$$

where $[a]_{\cong_A}$ denotes the equivalence class of a wrt the p-kernel relation $\cong_A$.

3.2. The construction

Given a mapping $f \colon \mathcal{A} \to B$ from a preordered set $\mathcal{A} = \langle A, \lesssim_A \rangle$ to an unstructured set B, our first goal is to find sufficient conditions to define a suitable preordering on B such that a right adjoint exists, in the style of Lemma 2. Notice that there is much more than a mere adaptation of the result for posets.

Lemma 8. *Let $\mathcal{A} = \langle A, \lesssim_A \rangle$ be a preordered set and $f \colon \mathcal{A} \to B$ a surjective mapping. Consider $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$ such that the following conditions hold:*

- *$\varphi_S(a) \neq \varnothing$, for all $a \in A$.*

- *If $a_1 \lesssim_A a_2$, then $\varphi_S(a_1) \sqsubseteq \varphi_S(a_2)$.*

Then, there exists a preorder $\lesssim_B$ in B and a map g such that $(f, g) \colon \mathcal{A} \leftrightharpoons \mathcal{B}$.

Proof. The definition of the preorder $\lesssim_B$ in B, given $b_1, b_2 \in B$, is as follows:

$b_1 \lesssim_B b_2$ if and only if $\hspace{5cm}$ (3)

$\hspace{2cm}$ there is $a_1 \in f^{-1}(b_1)$ and $a_2 \in f^{-1}(b_2)$ with $\varphi_s(a_1) \sqsubseteq \varphi_s(a_2)$

Let us prove that it is a preordering:

Reflexivity: By the first hypothesis, we have that $\varphi_s(a) \neq \varnothing$. Now, trivially, $\varphi_s(a) \sqsubseteq \varphi_s(a)$ holds for any $a \in f^{-1}(b)$. Therefore, $b \lesssim_B b$ for any $b \in B$.

Transitivity: Assume $b_1 \lesssim_B b_2$ and $b_2 \lesssim_B b_3$.

From $b_1 \lesssim_B b_2$, there exist $a_i \in f^{-1}(b_i)$, and $c_i \in \varphi_s(a_i)$ for $i \in \{1, 2\}$ such that $c_1 \lesssim_A c_2$.

From $b_2 \lesssim_B b_3$, there exist $a'_j \in f^{-1}(b_j)$, and $c'_j \in \varphi_s(a'_j)$ for $j \in \{2, 3\}$ such that $c'_2 \lesssim_A c'_3$.

As $a_2, a'_2 \in f^{-1}(b_2)$, we have that $[a_2]_{\cong_A} = [a'_2]_{\cong_A}$, which implies that $c_2 \approx_A c'_2$. Therefore, $c_1 \lesssim_A c_2 \approx_A c'_2 \lesssim_A c'_3$ and, as a result, $b_1 \lesssim_B b_3$.

In order to define $g \colon B \to A$, notice that using the axiom of choice, as f is onto, for all $b \in B$ we can choose $x_b \in A$ with $f(x_b) = b$. By hypothesis, $\varphi_s(x_b) \neq \varnothing$ for all $b \in B$ and, therefore, there exists a choice function. Any of these functions can be used to define g, in such a manner that $g(b) \in \varphi_s(x_b)$.

To finish the proof, we have just to check that $(f, g) \colon \langle A, \lesssim_A \rangle \leftrightarrows \langle B, \lesssim_B \rangle$.

Assume $f(a) \lesssim_B b$, then there exist $a_1 \in f^{-1}(f(a))$, $a_2 \in f^{-1}(b)$, $c_1 \in \varphi_s(a_1)$ and $c_2 \in \varphi_s(a_2)$ with $c_1 \lesssim_A c_2$; as $[a_1]_{\cong_A} = [a]_{\cong_A}$, and $c_1 \in \mathrm{UB}[a_1]_{\cong_A}$, we also have $a \lesssim_A c_1$. Let $x \in f^{-1}(b)$ such that $g(b) \in \varphi_s(x)$. Then $[a_2]_{\cong_A} = [x]_{\cong_A}$, and $\varphi_s(a_2) = \varphi_s(x)$. Thus, $c_2 \approx_A g(b)$ and, as $a \lesssim_A c_1 \lesssim_A c_2$, then $a \lesssim_A g(b)$.

Assuming now that $a \lesssim_A g(b)$, let us prove $f(a) \lesssim_B b$. Let x_b be the element in $f^{-1}(b)$ used in the definition of $g(b)$, that is, $g(b) \in \varphi_s(x_b)$. We will prove the inequality in three steps:

- Firstly, we prove that $g(b) \in \varphi_S(g(b))$: From $g(b) \in \varphi_S(x_b)$, particularly $g(b) \in S$, and using the hypothesis on S, we have that there exists $c \in A$ such that $g(b) \in \text{p-max}[c]_{\cong_A}$ and $g(b) \in [c]_{\cong_A} = [g(b)]_{\cong_A}$. Thus, $g(b) \in \text{p-max}[g(b)]_{\cong_A} \subseteq \mathrm{UB}[g(b)]_{\cong_A}$. Therefore, $g(b) \in \mathrm{UB}[g(b)]_{\cong_A} \cap S$ and, from the definition of upper bound, we have $g(b) \in \text{p-min}(\mathrm{UB}[g(b)]_{\cong_A} \cap S) = \varphi_S(g(b))$.

- Now, we prove $\varphi_s(a) \sqsubseteq \varphi_s(x_b)$. From $a \lesssim_A g(b)$ and the second hypothesis we have $\varphi_s(a) \sqsubseteq \varphi_s(g(b))$. By Lemma 7 and $g(b) \in \varphi_S(g(b))$, we have that $z \lesssim_A g(b)$ for all $z \in \varphi_s(a)$. Since $g(b) \in \varphi_S(x_b)$, by the definition of $g(b)$, we obtain $\varphi_s(a) \sqsubseteq \varphi_s(x_b)$.

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- Finally, since $\varphi_S(a) \sqsubseteq \varphi_S(x_b)$ for $a \in f^{-1}(f(a))$ and $x_b \in f^{-1}(b)$, by the definition of the preorder given in (3), we have that $f(a) \lesssim_B b$. $\qquad\square$

The following lemma will be used in the proof of the main theorem in this section, namely Theorem 6. It corresponds to the adaptation to preorders of Lemma 5 and the final part of the proof of Theorem 3.

Lemma 9. *Consider $\langle A, \lesssim_A \rangle$ a preordered set, B a set, and $f\colon A \to B$. Then, there exist both a preorder $\lesssim_B$ and an adjunction $(f,g)\colon \langle A, \lesssim_A \rangle \leftrightharpoons \langle B, \lesssim_B \rangle$ if and only if there exist a preorder $\lesssim_{f(A)}$ and an adjunction $(f,g')\colon \langle A, \lesssim_A \rangle \leftrightharpoons \langle f(A), \lesssim_{f(A)} \rangle$.*

Proof. The direct implication is trivial, by considering $\lesssim_{f(A)}$ and g' as the restrictions to $f(A)$ of $\lesssim_B$ and g, respectively.

Conversely, consider the adjunction $(f,g')\colon \langle A, \lesssim_A \rangle \leftrightharpoons \langle f(A), \lesssim_{f(A)} \rangle$, fix $m \in f(A)$, and choose $\lesssim_B$ to be its associated preorder, as introduced in Proposition 1. It is just a matter of straightforward computation to check that we have an adjunction $(f,g)\colon \langle A, \lesssim_A \rangle \leftrightharpoons \langle B, \lesssim_B \rangle$ where g is the extension of g' defined as follows:

$$g(x) = \begin{cases} g'(x) & \text{if } x \in f(A) \\ g'(m) & \text{if } x \notin f(A) \end{cases}$$

$\qquad\square$

The corresponding version of Theorem 3 is a twofold extension of the statement of Lemma 8 in that, firstly, the mapping f need not be onto and, secondly, it gives a necessary and sufficient condition for the existence of adjunction.

Theorem 6. *Given any preordered set $\mathcal{A} = \langle A, \lesssim_A \rangle$ and a mapping $f\colon \mathcal{A} \to B$, there exists a preorder $\mathcal{B} = \langle B, \lesssim_B \rangle$ and $g\colon B \to A$ such that $(f,g)\colon \mathcal{A} \leftrightharpoons \mathcal{B}$ if and only if there exists a subset S of A such that the following conditions hold:*

1. $S \subseteq \displaystyle\bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
2. $\varphi_S(a) \neq \varnothing$, *for all $a \in A$.*
3. *If $a_1 \lesssim_A a_2$, then $\varphi_S(a_1) \sqsubseteq \varphi_S(a_2)$, for $a_1, a_2 \in A$.*

Proof. Assume the existence of the preordering in B and the mapping g such that $(f,g)\colon \mathcal{A} \leftrightharpoons \mathcal{B}$, and let us prove the three properties in the statement.

For property 1, define $S = g(f(A))$, consider $g(f(a)) \in S$, and let us show that $g(f(a)) \in \text{p-max}[g(f(a))]_{\cong_A}$. Consider $x \in [g(f(a))]_{\cong_A}$, by a straightforward induction argument we obtain $f(x) \approx_B f(g(f(a)))$; now, using $f(g(f(a))) \approx_B f(a)$ we have $f(x) \approx_B f(a)$. Since $f(x) \lesssim_B f(a)$, by using the adjunction, we obtain $x \lesssim_A g(f(a))$, hence $g(f(a)) \in \text{p-max}[g(f(a))]_{\cong_A}$.

For property 2, we will check that $g(f(a)) \in \varphi_S(a)$. To begin with, by definition $g(f(a)) \in S$; then, we will prove that $g(f(a)) \in \text{UB}[a]_{\cong_A}$. That

is, $x \lesssim_A g(f(a))$ for all $x \in [a]_{\cong_A}$. Specifically, we are going to prove, by induction on the length n of any chain $\{a_i\}_{i \in \{0,\dots,n\}} \subseteq A$ such that $a_i \approx_A a_{i+1}$ or $f(a_i) = f(a_{i+1})$ for all $i \in \{0,\dots,n-1\}$, that condition $a_n \lesssim_A g(f(a_0))$ holds:

- For $n = 0$, we have $a_0 \lesssim_A g(f(a_0))$ by properties of adjunction.

- As induction hypothesis, assume the result is true for any chain of length k. Let $\{a_i\}_{i \in \{0,\dots,k+1\}} \subseteq A$ be a chain such that, for all $0 \le i \lesssim k$, either $a_i \approx_A a_{i+1}$ or $f(a_i) = f(a_{i+1})$. By induction hypothesis, $a_k \lesssim_A g(f(a_0))$ holds. There are two possibilities:

  - If $a_k \approx_A a_{k+1}$, trivially $a_{k+1} \lesssim_A g(f(a_0))$.
  - If $f(a_k) = f(a_{k+1})$, using the hypothesis of induction and the properties of adjunction twice we firstly obtain $f(a_{k+1}) = f(a_k) \lesssim_A f(g(f(a_0)))$ and, then, $a_{k+1} \lesssim_A g(f(g(f(a_0)))) \approx_A g(f(a_0))$.

We have just proved that $g(f(a)) \in \mathrm{UB}[a]_{\cong_A} \cap S$, the remaining part is to prove that it is a p-minimum element. Consider $x \in \mathrm{UB}[a]_{\cong_A} \cap S$; then $z \lesssim_A x$ for all $z \in [a]_{\cong_A}$ and, by definition of S, $x = g(f(a_1))$. Particularly, for $z = a$ we have that, $a \lesssim_A g(f(a_1))$, by properties of adjunction, $g(f(a)) \lesssim_A g(f(g(f(a_1)))) \approx_A g(f(a_1)) = x$, i.e. $g(f(a)) \lesssim_A x$.

For Property 3, assume $a_1 \lesssim_A a_2$, by adjunction, f and g are isotone maps, then $g(f(a_1)) \lesssim_A g(f(a_2))$. From this, we directly obtain $\varphi_s(a_1) \sqsubseteq \varphi_s(a_2)$ since we have just proved above that $g(f(a)) \in \varphi_s(a)$, for all $a \in A$.

Conversely, if we suppose the conditions 1, 2, and 3, then by Lemma 8 and Lemma 9, there exists a preorder $\mathcal{B} = \langle B, \lesssim_B \rangle$ and a mapping g such that $(f,g) : \mathcal{A} \leftrightharpoons \mathcal{B}$. $\qquad\square$

4. On the uniqueness of right adjoints and the induced ordered structure in the codomains

The unicity of the right adjoint between posets is well-known. Specifically, given two posets $\mathcal{A} = \langle A, \le_A \rangle$ and $\mathcal{B} = \langle B, \le_B \rangle$ and a mapping $f \colon A \to B$, if there exists $g \colon B \to A$ such that the pair (f,g) is an adjunction, then it is unique.

This behavior was analyzed in Section 2, where the uniqueness property was extended, in the case of surjective mappings, not only to the right adjoint, but also to the ordering relation in the codomain: namely, there exists just one partial ordering on the codomain B such that a right adjoint exists. That is, given a surjective mapping f from a poset $\mathcal{A}$ to an unstructured set B, we introduced necessary and sufficient conditions to ensure the existence of an

ordering $\leq_B$ in B and a mapping $g: B \to A$ such that (f, g) is an adjunction. Moreover, both $\leq_B$ and g are uniquely determined by $\leq_A$ and f.

Contrariwise to the partially ordered case, given two preordered sets $\mathcal{A} = \langle A, \lesssim_A \rangle$ and $\mathcal{B} = \langle B, \lesssim_B \rangle$ and a mapping $f: A \to B$, the unicity of the mapping $g: B \to A$ satisfying $(f, g): \mathcal{A} \leftrightharpoons \mathcal{B}$, when it exists, cannot be guaranteed. However, it is well known that if g_1 and g_2 are right adjoints, then $g_1(b) \approx_A g_2(b)$ for all $b \in B$, and one usually says that the right adjoint is *essentially unique*. This scenario is much more similar to what occurs in category theory: if one functor F has two right adjoints G_1 and G_2, then G_1 and G_2 are naturally isomorphic.

However, and this is the interesting part, the unicity of the ordering cannot be extended in general in the preordered case when the codomain is unstructured. Hereafter we introduce a couple examples supporting this statement, all of them based on the same mapping $f: A \to B$.

Example 2. *Let $A = \{a, b, c, d\}$, $B = \{o, p, q\}$ be two sets and $f: A \to B$ defined as $f(a) = f(c) = p$, $f(b) = o$ and $f(d) = q$. Consider $\langle A, \leq_A \rangle$ ordered by $a \leq_A b \leq_A c \leq_A d$. We have $[a]_{\cong_A} = [c]_{\cong_A} = \{a, c\}$, $[b]_{\cong_A} = \{b\}$ and $[d]_{\cong_A} = \{d\}$ and $\bigcup_{x \in A} \text{p-max}[x]_{\cong_A} = \{b, c, d\}$.*
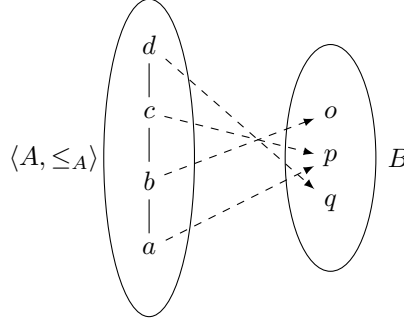
Notice that f is surjective, and does not fulfill the conditions in Theorem 3, specifically the second one. Thus, there does not exist any partial ordering relation in B for which some $g: B \to A$ would be a right adjoint to f. Notice, however, that if we relax the requirement to be an adjunction between preordered sets, then there exists a preordering (actually more than one) which generates a right adjoint to f. Some examples are worked out below to illustrate the previous situation.

Example 3. *Consider $\mathcal{B} = \langle B, \lesssim_B \rangle$ preordered with $o \approx_B p$, $o \lesssim_B q$ and $p \lesssim_B q$, and the mapping $g_1: B \to A$ defined as $g_1(o) = g_1(p) = c$ and $g_1(q) = d$.*
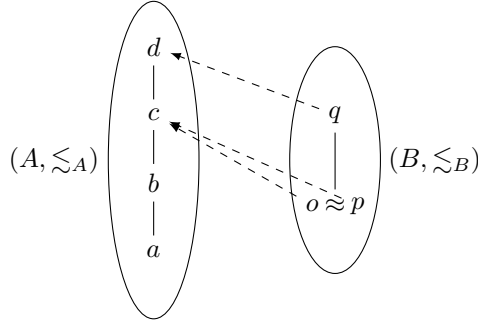
Firstly, we have that $S = g_1 f(A) = \{c, d\}$ is a subset of $\bigcup_{x \in A} \text{p-max}[x]_{\cong_A}$.
For the two other conditions in Theorem 6, it is not difficult to check that $\text{p-min}(\text{UB}[x]_{\cong_A} \cap S) \neq \varnothing$ for all $x \in A$. Specifically, we have

$$\text{p-min}(\text{UB}[a]_{\cong_A} \cap S) = \text{p-min}(\text{UB}[b]_{\cong_A} \cap S) = \text{p-min}(\text{UB}[c]_{\cong_A} \cap S) = \{c, d\}$$
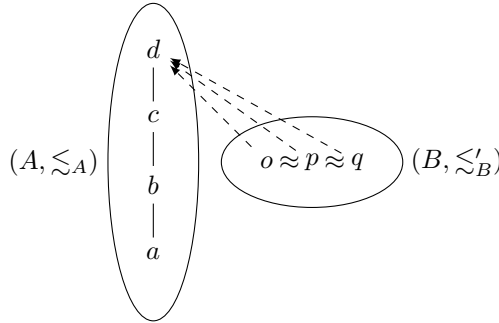
and

$$\text{p-min}(\text{UB}[d]_{\cong_A} \cap S) = \{d\}$$

Finally, with the previous computation, it is straightforward to check that if $a_1 \lesssim_A a_2$ then $\text{p-min}(\text{UB}[a_1]_{\cong_A} \cap S) \sqsubseteq \text{p-min}(\text{UB}[a_2]_{\cong_A} \cap S)$.
As a result, the pair (f, g_1) is an adjunction between $\mathcal{A}$ and $\mathcal{B}$. □

Example 4. *Now, consider $\mathcal{B}' = \langle B, \lesssim'_B \rangle$ preordered by $o \approx'_B p$ and $p \approx'_B q$, and the mapping $g_2 \colon B \to A$ defined as $g'_2(o) = g_2(p) = g_2(q) = d$.*

Again we will check the conditions in Theorem 6.
In this case, $S = g_2 f(A) = \{d\}$ which is a subset of $\bigcup_{x \in A} \text{p-max}[x]_{\cong_A} = \{b, c, d\}$. The second condition holds since $\text{p-min}(\text{UB}[a]_{\cong_A} \cap S) = \text{p-min}(\text{UB}[b]_{\cong_A} \cap S) = \text{p-min}(\text{UB}[c]_{\cong_A} \cap S) = \text{p-min}(\text{UB}[d]_{\cong_A} \cap S) = \{d\}$. As all the previous sets coincide, the third condition follows trivially.
As a result, the pair (f, g_2) is an adjunction between the preorders $\mathcal{A}$ and $\mathcal{B}'$.
□

5. Closure operators on preorders

There is a tight relation between adjunctions and closure and kernel operators, in that every adjunction (f, g) leads to a closure operator $g \circ f$ and a kernel $f \circ g$. In this section, we state the necessary and sufficient conditions obtained in the previous section in terms of closure operators and closure systems. To begin with, we have to recall their formal definitions:

Definition 7. *Let A be a poset.*

- *A mapping $f \colon A \to A$ is a* closure operator *if it is inflationary, isotone and idempotent (that is if $f \circ f = f$).*

- *A subset $S \subseteq A$ is a* closure system *if there exist a minimum element in $a^\uparrow \cap S$ for all $a \in A$.*

Closure operators and closure systems are different approaches to the same phenomenon. We will focus now on the development of the well-known link between these two notions on a partially ordered set, but in the more general framework of preordered sets.

To begin with, both notions have to be adapted to the lack of antisymmetry. This involves the use of the equivalence relation $\approx$ introduced in the previous sections.

Definition 8. *Let $\mathcal{A} = \langle A, \lesssim_A \rangle$ be a preordered set.*

1. *A mapping $c \colon A \to A$ is said to be a $\approx_A$-closure operator if c is inflationary, isotone and $\approx_A$-idempotent, i.e. $(c \circ c)(a) \approx_A c(a)$, for all $a \in A$.*
2. *A subset $S \subseteq A$ is a $\approx_A$-closure system if the set p-min$(a^\uparrow \cap S)$ is non-empty for all $a \in A$.*

The notion of $\approx_A$-closed set can be found in [16], whereas the previous version of $\approx_A$-closure system is, to the best of our knowledge, a novel notion.

Remark 1. *It is not difficult to see that given a $\approx_A$-closure operator $c \colon A \to A$, the set $S_c = \{x \in A \mid c(a) = a\}$ is a $\approx_A$-closure system; conversely, given a $\approx_A$-closure system S, any mapping $c \colon A \to A$ satisfying $c(a) \in$ p-min$(a^\uparrow \cap S)$ for all $a \in A$ is a $\approx_A$-closure operator; we will say that c is associated to S.*

As usual, it is convenient to introduce the notion of compatibility with an equivalence relation.

Definition 9. *Let $\mathcal{A} = \langle A, \lesssim_A \rangle$ be a preordered set an equivalence relation $\sim$ on A.*

1. *A $\approx_A$-closure operator $c \colon A \to A$ is said to be compatible with $\sim$ if $a \sim b$ implies $c(a) \approx_A c(b)$ for all $a, b \in A$.*

2. *Similarly, a $\approx_A$-closure system S is said to be compatible with $\sim$ if $a \lesssim_A s$ implies $[a]_\sim \subseteq s^\downarrow$, for all $a \in A, s \in S$.*

The notion of compatibility in the previous definition is preserved when moving between operators and systems in the sense of Remark 1. This is formally stated in the following result:

Lemma 10. *Let $c\colon A \to A$ be a $\approx_A$-closure operator compatible with an equivalence relation $\sim$ on A, then the $\approx_A$-closure system $S_c = \{x \in A \mid c(a) = a\}$ is compatible with $\sim$.*

Conversely, let S be a $\approx_A$-closure system compatible with $\sim$, then any $\approx_A$-closure operator c associated to S is compatible with $\sim$ as well.

Proof. Consider a $\approx_A$-closure operator $c\colon A \to A$ compatible with $\sim$ and the closure system $S_c = \{a \in A \mid c(a) = a\}$. Let $a \in A$ and $s \in S_c$ such that $a \lesssim s$. By compatibility of c with $\sim$, for all $x \in [a]_\sim$, we have that $x \sim a$ and $c(x) \approx_A c(a)$, and, since c is a $\approx_A$-closure operator, we conclude that $x \lesssim_A c(x) \approx_A c(a) \lesssim_A c(s) = s$. Therefore, $[a]_\sim \subseteq s^\downarrow$ and S_c is compatible with $\sim$.

Conversely, let S be a $\approx_A$-closure system compatible with $\sim$ and let c be a $\approx_A$-closure operator satisfying $c(a) \in \text{p-min}(a^\uparrow \cap S)$ for all $a \in A$. Consider $a, b \in A$ such that $a \sim b$. Since $a \leq c(a)$ and $b \in [a]_\sim$, by compatibility of S with $\sim$, we have that $b \lesssim_A c(a)$ and, since c is a $\approx_A$-closure operator, $c(b) \lesssim_A c(c(a)) \approx_A c(a)$. Similarly, from $b \lesssim_A c(b)$ and $a \in [b]$, we obtain $c(a) \lesssim_A c(c(b)) \approx_A c(b)$. Therefore, $c(a) \approx_A c(b)$. $\qquad\square$

Lemma 11. *Let $\mathcal{A} = \langle A, \lesssim_A \rangle$ be a preordered set and a mapping $f\colon \mathcal{A} \to B$. A $\approx_A$-closure system is compatible with $\equiv_f$ if and only if it is compatible with $\cong_A$.*

Proof. Assuming that S is a $\approx_A$-closure system compatible with $\equiv_f$, and $a \in A$ and $s \in S$ such that $a \leq s$, by hypothesis, we have $[a]_{\equiv_f} \subseteq s^\downarrow$; we will have to show that $[a]_{\cong_A} \subseteq s^\downarrow$ as well.

Given $x \in [a]_{\cong_A}$ we will reason by induction on the length of the chain connecting a and x:

- The base case of length one is straightforward, both in the case of $x \approx_A a$ and in the case of $f(x) = f(a)$.

- As induction hypothesis, assume the result is true for any chain of length k, and assume that there exists a finite chain $\{x_i\}_{i \in \{0,\ldots,k+1\}} \subseteq A$ such that $x_0 = a$, $x_{k+1} = x$ and, for all $i \in \{0,\ldots,k\}$, either $x_i \equiv_f x_{i+1}$ or $x_i \approx_A x_{i+1}$ and therefore, $x_k \lesssim_A s$. Now, we have that either $x_k \approx_A x_{k+1}$ (in this case we easily obtain $x_{k+1} \lesssim_A s$) or $f(x_k) = f(x_{k+1})$ (this case follows from the compatibility with $\equiv_f$).

The converse implication is trivial, since the relation $\equiv_f$ is contained in the relation $\cong_A$. $\qquad\square$

In the sections above, we have addressed the problem of defining a preordering on an unstructured set B, which is the codomain of a mapping $f\colon A \to B$, so that f is a left adjoint. If the answer was affirmative, then the composition of the two components of the adjunction leads to a $\approx_A$-closure operator which, moreover, is compatible with the kernel relation associated to f. As a result, the existence of a $\approx_A$-compatible system turns out to be a necessary condition. The following results states that this condition is also sufficient.

Theorem 7. *Let $\mathcal{A} = \langle A, \lesssim_A \rangle$ be a preordered set and a mapping $f\colon A \to B$. Then, there exists a preorder in B and a mapping $g\colon B \to A$ such that (f,g) forms an adjunction if and only if there exists a $\approx_A$-closure system S compatible with $\equiv_f$.*

Proof. The existence of the preordering and the adjunction implies the existence of a $\approx_A$-closure system (that associated to the composition $g \circ f$ as in Remark 1) which is obviously compatible with $\equiv_f$.

For the converse, let us assume the existence of the $\approx_A$-closure system S compatible with $\equiv_f$ and let us show the existence of right adjoint $g\colon B \to A$.

By Lemma 11, we have that the $\approx_A$-closure system S is compatible with $\cong_A$ and we can directly apply Theorem 6: we have just to show that the $\approx_A$-closure system S is such that the following three conditions hold:

- $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$.

  This holds because $x \in \text{p-max}[x]_{\cong_A}$, for all $x \in S$. In effect, from $x \in [x]_{\cong_A}$ and $x \in x^\downarrow$ one obtains $[x]_{\cong_A} \subseteq x^\downarrow$, that is, $y \lesssim_A x$, for all $y \in [x]_{\cong_A}$.

- $\varphi_S(a) = \text{p-min}(\text{UB}[a]_{\cong_A} \cap S) \neq \varnothing$, for all $a \in A$.

  Firstly, we will see that $\text{UB}[a]_{\cong_A} \cap S = a^\uparrow \cap S$. For this, as $\text{UB}[a]_{\cong_A} = \bigcap_{\{z \mid z \cong_A a\}} z^\uparrow$, it is sufficient to show that $a_1 \cong_A a_2$ implies $a_1{}^\uparrow \cap S = a_2{}^\uparrow \cap S$. Let $x \in a_1{}^\uparrow \cap S$, then $a_1 \lesssim_A x$ which implies $[a_1]_{\cong_A} \subseteq x^\downarrow$. In particular, $a_2 \lesssim_A x$, thus $x \in a_2{}^\uparrow \cap S$. Therefore, $a_1{}^\uparrow \cap S \subseteq a_2{}^\uparrow \cap S$. Similarly, $a_2{}^\uparrow \cap S \subseteq a_1{}^\uparrow \cap S$.

  Since S is a $\approx_A$-closure system, for all $a \in A$, the set $\varnothing \neq \text{p-min}(a^\uparrow \cap S) = \varphi_S(a)$.

- If $a_1 \lesssim_A a_2$, then $\varphi_S(a_1) \sqsubseteq \varphi_S(a_2)$.

  Suppose that $a_1 \lesssim_A a_2$ and let $x_i \in \varphi_S(a_i)$, for $i \in \{1,2\}$. As $x_2 \in \text{UB}[a_2]_{\cong_A}$, in particular, $x_2 \geq a_2 \geq a_1$ then, by compatibility wrt $\cong_A$ we obtain $[a_1]_{\cong_A} \subseteq x_2{}^\downarrow$. Therefore, $x_2 \in \text{UB}[a_1]_{\cong_A} \cap S$ which implies that $x_1 \lesssim_A x_2$. $\qquad\square$

We finish this section with another example showing special features of the results introduced so far.

In Examples 3 and 4, the domain of f, that is $\langle A, \leq_A \rangle$, was a linearly ordered set; this has been chosen by simplicity reasons but it is, by no means, required for the results to hold. In the following example, $\langle A, \lesssim_A \rangle$ is strictly a preorder (since antisymmetry does not hold), and even the poset obtained after making the quotient wrt the kernel relation is not a lattice, but a multilattice [7, 8, 34, 35].

Example 5. *Consider the mapping $f \colon (A, \lesssim_A) \to B$ depicted in the figure below. Notice that $(A, \lesssim_A)$ is a strict preorder (not a partial order).*
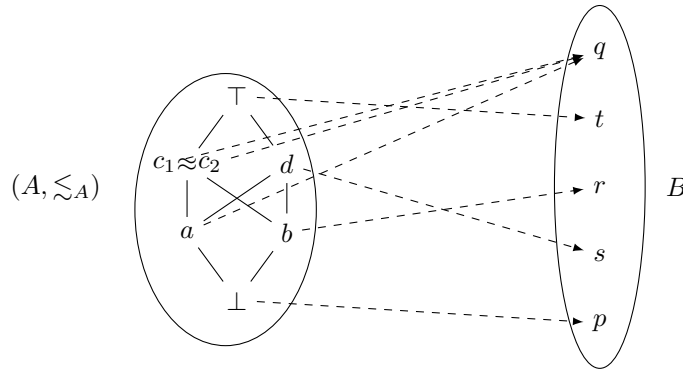
Several $\approx$-closure systems can be defined in $\langle A, \lesssim_A \rangle$ which, in general, lead to different right adjoints. For example, $S_1 = \{\top, c_1, c_2\}$ is a $\approx$-closure system that induces the preorder relation $\lesssim_B$ (as given in the proof of Lemma 8) depicted in the following picture.

$$
\begin{array}{c}
s \approx t \\
| \qquad (B, \leq_B) \\
p \approx q \approx r
\end{array}
$$

In this case, f admits eight different right adjoints such as:

- *g_1 with $g_1(s) = g_1(t) = \top$, $g_1(p) = g_1(q) = c_1$ and $g_1(r) = c_2$.*

- *g_2 with $g_2(s) = g_2(t) = \top$, $g_2(p) = c_1$ and $g_2(q) = g_2(r) = c_2$.*

The rest of the possible right adjoints are the rest of different assignments to the elements p, q, r in the subset $\{c_1, c_2\}$.

6. Conclusions

Given a mapping $f \colon A \to B$ from a (pre-)ordered set A into an unstructured set B, we have obtained necessary and sufficient conditions which allow us to

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define a suitable (pre-)ordering relation on B such that there exists a mapping $g \colon B \to A$ such that (f, g) forms an adjunction between (pre-)ordered sets.

It is worth underscoring that, although all the results have been stated in terms of the existence and construction of right adjoints, all of them can be straightforwardly modified for the existence and construction of left adjoints.

Whereas the study of the partially ordered case follows more or less the intuition of what should be expected (Theorem 3), the description of the conditions on the preordered case is much more involved (Theorem 6); only later, when we consider to use of $\approx$-closure systems, together with the convenient definition of compatibility with the kernel operator $\equiv_f$, can we rewrite the result in much more concise terms (Theorem 7). This result shows the convenience of considering closure systems in the study of Galois connections in more general carriers.

Since its introduction in [3], a number of papers have already been published on fuzzy Galois connections, see [4, 19, 23, 29] for some recent ones. As future work, we are planning to extend the results in this work to the fuzzy case, for instance to the framework of fuzzy posets and fuzzy preorders, including as well the corresponding study in terms of closure or kernel systems, and study the potential relationship to other approaches based on generalized structures, in the line of those given in [43, 44].

Concerning potential practical applications of the present work, we will explore the area of *pattern structures*, which allows for extending FCA techniques to arbitrary partially ordered data descriptions. The scenario in which this work could be applied is as follows: we start from a set of objects each one related to the set of patterns it satisfies, ignoring whether there exists some (pre-)ordering relation between patterns, but assuming that the semantics of the problem guarantees the existence of a Galois connection between them, the goal would be to obtain as much information as possible about the relation existing in the set of patterns.

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