

## ARTICLE

### On Homogeneous $L$ -bonds and Heterogeneous $L$ -bonds

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In this paper we deal with suitable generalizations of the notion of bond between contexts, as part of the research area of Formal Concept Analysis. We study different generalizations of the notion of bond within the  $L$ -fuzzy setting. Specifically, given a formal context there are three prototypical pairs of concept-forming operators, and this immediately leads to three possible versions of the notion of bond (so-called *homogeneous bond* wrt certain pair of concept-forming operators). The first results show a close correspondence between a homogeneous bond between two contexts and certain special types of mappings between the sets of extents (or intents) of the corresponding concept lattices. Later, we introduce the so-called *heterogeneous bonds* (considering simultaneously two types of concept-forming operators) and generalize the previous relationship to mappings between the sets of extents (or intents) of the corresponding concept lattices.

#### 1. Introduction

Formal Concept Analysis (FCA) has become a very active research topic, both theoretical and practical; its wide applicability justifies the need of a deeper knowledge of its underlying mechanisms, and one important way to obtain this extra knowledge turns out to be via generalization.

Since the seminal paper (Burusco and Fuentes-González 1994), several fuzzy variants of generalized FCA have been introduced and developed both from the theoretical and the practical side. The consideration of the adjointness property in residuated lattices as the main building blocks of fuzzy concept lattices was an important milestone simultaneously developed by (Pollandt 1997, Belohlavek 1998).

More recently, a number of new generalizations have been introduced, either based on fuzzy set theory (Alcalde *et al.* 2010, 2011), or the multi-adjoint framework (Medina *et al.* 2009, Medina and Ojeda-Aciego 2010, 2013) or heterogeneous approaches (Butka *et al.* 2012, Medina and Ojeda-Aciego 2012, Díaz *et al.* 2014).

FCA has been extended as well by considering alternative paradigms, for instance one can find generalizations of the framework and scope of FCA based on from possibility theory (Dubois and Prade 2012) or rough set theory (Wu and Liu 2009, Lei and Luo 2009, Lai and Zhang 2009, Medina 2012, Kang *et al.* 2013).

Concerning applications of techniques of generalized formal concept analysis, one can see papers ranging from ontology merging (Chen *et al.* 2011) and resolution of fuzzy or multi-adjoint relational equations (Alcalde *et al.* 2012, Díaz and Medina

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2013), to applications to the Semantic Web by using the notion of concept similarity or rough sets (Formica 2012), and from noise control in document classification (Li and Tsai 2011) to ontology-based sentiment analysis (Kontopoulos *et al.* 2013), or the study of fuzzy databases, in areas such as functional dependencies (Mora *et al.* 2012), or even linguistics (Falk and Gardent 2014).

All the generalizations stated above focused on the development of a general framework of FCA including extra features (fuzzy, possibilistic, rough, etc.) and some of its possible applications. However, not much have been published on the suitable general version of certain specific notions, such as the bonds between formal contexts.

One of the motivations for introducing the notion of bond was to provide a tool for studying mappings between formal contexts, somehow mimicking the behavior of Galois connections between their corresponding concept lattices. In this paper we deal with generalizations of the notion of bond for which, to the best of our knowledge, only one general version has been introduced, see (Křídlo *et al.* 2012), wrt the standard concept-forming operators used in (Belohlavek 1998).

The notions of bonds, scale measures and infomorphisms were studied by Krötzsch *et al.* (2005) aiming at a thorough study of the theory of morphisms in FCA; in areas related to ontology research, just infomorphisms are used, whereas more general approaches, namely more general heterogeneous bonds, could be utilized. Křídlo *et al.* (2013) use bonds to include background knowledge into data; the heterogeneous bonds described in this paper enable us to give an alternative semantics the background knowledge. Another application of bonds can be seen in (Meschke 2010) where bonds are used to approximate concepts, allowing to focus on just a sub context without losing implicational knowledge and, hence, reducing the size of a concept lattice.

We study generalizations of the notion of bond within the  $L$ -fuzzy setting. Specifically, given a formal context there are three prototypical pairs of concept-forming operators, and this immediately leads to three possible versions of the notion of bond (so-called *homogeneous bond* wrt certain pair of concept-forming operators). The first results show a close correspondence between a homogeneous bond between two contexts and certain special types of mappings between the sets of extents (or intents) of the corresponding concept lattices. Later, we introduce the so-called *heterogeneous bonds* (considering simultaneously two types of concept-forming operators) and generalize the previous relationship to mappings between the sets of extents (or intents) of the corresponding concept lattices.

## 2. Preliminaries

### 2.1 Residuated Lattices, Fuzzy Sets, and Fuzzy Relations

We use complete residuated lattices as basic structures of truth-degrees. A complete residuated lattice is a structure  $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$  such that

- (i)  $\langle L, \wedge, \vee, 0, 1 \rangle$  is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;
- (ii)  $\langle L, \otimes, 1 \rangle$  is a commutative monoid, i.e.  $\otimes$  is a binary operation which is commutative, associative, and  $a \otimes 1 = a$  for each  $a \in L$ ;
- (iii)  $\otimes$  and  $\rightarrow$  satisfy adjointness, i.e.  $a \otimes b \leq c$  iff  $a \leq b \rightarrow c$ .

Recall that the partial order of  $\mathbf{L}$  is denoted by  $\leq$ , elements 0 and 1 denote the least and greatest elements and, note that throughout this work,  $\mathbf{L}$  denotes an arbitrary complete residuated lattice whose multiplicative unit is also its greatest

element in the spirit of Goguen (1967).

Elements  $a$  of  $L$  are called truth degrees. Operations  $\otimes$  (multiplication) and  $\rightarrow$  (residuum) play the role of (truth functions of) “fuzzy conjunction” and “fuzzy implication”. Furthermore, we define the complement of  $a \in L$  as

$$\neg a = a \rightarrow 0 \quad (1)$$

An  $\mathbf{L}$ -set (or  $\mathbf{L}$ -fuzzy set)  $A$  in a universe set  $X$  is a mapping assigning to each  $x \in X$  some truth degree  $A(x) \in L$ . The set of all  $\mathbf{L}$ -sets in a universe  $X$  is denoted  $\mathbf{L}^X$ .

The operations with  $\mathbf{L}$ -sets are defined componentwise. For instance, the intersection of  $\mathbf{L}$ -sets  $A, B \in \mathbf{L}^X$  is an  $\mathbf{L}$ -set  $A \cap B$  in  $X$  such that  $(A \cap B)(x) = A(x) \wedge B(x)$  for each  $x \in X$ , etc. An  $\mathbf{L}$ -set  $A \in \mathbf{L}^X$  is also denoted  $\{A(x)/x \mid x \in X\}$ . If for all  $y \in X$  distinct from  $x_1, x_2, \dots, x_n$  we have  $A(y) = 0$ , we also write

$$\{A(x_1)/x_1, A(x_2)/x_2, \dots, A(x_n)/x_n\}. \quad (2)$$

Furthermore, in (2) we write just  $x$  instead of  $1/x$ .

An  $\mathbf{L}$ -set  $A \in \mathbf{L}^X$  is called crisp if  $A(x) \in \{0, 1\}$  for each  $x \in X$ . Crisp  $\mathbf{L}$ -sets can be identified with ordinary sets. For a crisp  $A$ , we also write  $x \in A$  for  $A(x) = 1$  and  $x \notin A$  for  $A(x) = 0$ . An  $\mathbf{L}$ -set  $A \in \mathbf{L}^X$  is called empty (denoted by  $\emptyset$ ) if  $A(x) = 0$  for each  $x \in X$ . For  $a \in L$  and  $A \in \mathbf{L}^X$ , the  $\mathbf{L}$ -sets  $a \otimes A$ ,  $a \rightarrow A$ ,  $A \rightarrow a$ , and  $\neg A$  in  $X$  are defined by

$$(a \otimes A)(x) = a \otimes A(x), \quad (3)$$

$$(a \rightarrow A)(x) = a \rightarrow A(x), \quad (4)$$

$$(A \rightarrow a)(x) = A(x) \rightarrow a, \quad (5)$$

$$\neg A(x) = A(x) \rightarrow 0. \quad (6)$$

For  $A \in \mathbf{L}^X$  the  $\mathbf{L}$ -sets  $a \otimes A$ ,  $a \rightarrow A$ ,  $A \rightarrow a$  are called  $a$ -multiplication,  $a$ -shift, and  $a$ -complement, respectively.

Binary  $\mathbf{L}$ -relations (binary  $\mathbf{L}$ -fuzzy relations) between  $X$  and  $Y$  can be thought of as  $\mathbf{L}$ -sets in the universe  $X \times Y$ . That is, a binary  $\mathbf{L}$ -relation  $I \in \mathbf{L}^{X \times Y}$  between a set  $X$  and a set  $Y$  is a mapping assigning to each  $x \in X$  and each  $y \in Y$  a truth degree  $I(x, y) \in L$  (a degree to which  $x$  and  $y$  are related by  $I$ ). By  $I^T$  we denote the transpose of  $I$ ; i.e.  $I^T \in \mathbf{L}^{Y \times X}$  with  $I^T(y, x) = I(x, y)$  for all  $x \in X, y \in Y$ .

Various composition operators for binary  $\mathbf{L}$ -relations were extensively studied by Kohout and Bandler (1985); we will use the following three composition operators, defined for relations  $A \in \mathbf{L}^{X \times F}$  and  $B \in \mathbf{L}^{F \times Y}$ :

$$(A \circ B)(x, y) = \bigvee_{f \in F} A(x, f) \otimes B(f, y), \quad (7)$$

$$(A \triangleleft B)(x, y) = \bigwedge_{f \in F} A(x, f) \rightarrow B(f, y), \quad (8)$$

$$(A \triangleright B)(x, y) = \bigwedge_{f \in F} B(f, y) \rightarrow A(x, f). \quad (9)$$

All of them have natural verbal descriptions. For instance,  $(A \circ B)(x, y)$  is the truth degree of the proposition “there is factor  $f$  such that  $f$  applies to object  $x$  and attribute  $y$  is a manifestation of  $f$ ”;  $(A \triangleleft B)(x, y)$  is the truth degree of “for

every factor  $f$ , if  $f$  applies to object  $x$  then attribute  $y$  is a manifestation of  $f$ ". Note also that for  $L = \{0, 1\}$ ,  $A \circ B$  coincides with the well-known composition of binary relations.

We will occasionally use some of the following properties concerning the associativity of several composition operators, see (Belohlavek 2002).

**Theorem 2.1:** *The operators above have the following properties concerning composition.*

- *Associativity:*

$$R \circ (S \circ T) = (R \circ S) \circ T, \quad (10)$$

$$R \triangleleft (S \triangleright T) = (R \triangleleft S) \triangleright T, \quad (11)$$

$$R \triangleleft (S \triangleleft T) = (R \circ S) \triangleleft T, \quad (12)$$

$$R \triangleright (S \circ T) = (R \triangleright S) \triangleright T. \quad (13)$$

- *Distributivity:*

$$\left(\bigcup_i R_i\right) \circ S = \bigcup_i (R_i \circ S), \quad \text{and} \quad R \circ \left(\bigcup_i S_i\right) = \bigcup_i (R \circ S_i), \quad (14)$$

$$\left(\bigcap_i R_i\right) \triangleright S = \bigcap_i (R_i \triangleright S), \quad \text{and} \quad R \triangleright \left(\bigcup_i S_i\right) = \bigcap_i (R \triangleright S_i), \quad (15)$$

$$\left(\bigcup_i R_i\right) \triangleleft S = \bigcap_i (R_i \triangleleft S), \quad \text{and} \quad R \triangleleft \left(\bigcap_i S_i\right) = \bigcap_i (R \triangleleft S_i). \quad (16)$$

## 2.2 Formal fuzzy concept analysis

An  $\mathbf{L}$ -context is a triplet  $\langle X, Y, I \rangle$  where  $X$  and  $Y$  are (ordinary nonempty) sets and  $I \in \mathbf{L}^{X \times Y}$  is an  $\mathbf{L}$ -relation between  $X$  and  $Y$ . Elements of  $X$  are called objects, elements of  $Y$  are called attributes,  $I$  is called an incidence relation.  $I(x, y) = a$  is read: "The object  $x$  has the attribute  $y$  to degree  $a$ ."

Consider the following pairs of operators induced by an  $\mathbf{L}$ -context  $\langle X, Y, I \rangle$ . First, the pair  $\langle \uparrow, \downarrow \rangle$  of operators  $\uparrow : \mathbf{L}^X \rightarrow \mathbf{L}^Y$  and  $\downarrow : \mathbf{L}^Y \rightarrow \mathbf{L}^X$  is defined, for all  $A \in \mathbf{L}^X$  and  $B \in \mathbf{L}^Y$ , by

$$A^\uparrow(y) = \bigwedge_{x \in X} A(x) \rightarrow I(x, y), \quad B^\downarrow(x) = \bigwedge_{y \in Y} B(y) \rightarrow I(x, y). \quad (17)$$

Second, the pair  $\langle \wedge, \vee \rangle$  of operators  $\wedge : \mathbf{L}^X \rightarrow \mathbf{L}^Y$  and  $\vee : \mathbf{L}^Y \rightarrow \mathbf{L}^X$  is defined by

$$A^\wedge(y) = \bigvee_{x \in X} A(x) \otimes I(x, y), \quad B^\vee(x) = \bigwedge_{y \in Y} I(x, y) \rightarrow B(y), \quad (18)$$

Third, the pair  $\langle \hat{\wedge}, \hat{\vee} \rangle$  of operators  $\hat{\wedge} : \mathbf{L}^X \rightarrow \mathbf{L}^Y$  and  $\hat{\vee} : \mathbf{L}^Y \rightarrow \mathbf{L}^X$  is defined by

$$A^{\hat{\wedge}}(y) = \bigwedge_{x \in X} I(x, y) \rightarrow A(x), \quad B^{\hat{\vee}}(x) = \bigvee_{y \in Y} B(y) \otimes I(x, y), \quad (19)$$

The three previous pairs are those more commonly used in the literature related to residuated lattice-based generalizations of FCA. In this respect, it is worth to

note that there exists a fourth pair of concept-forming operators not considered in the present work which can be viewed as a double dualization on the first pair.

**Remark 2.2** Notice that the three different pairs of concept-forming operators can be interpreted as instances of the composition operators between relations. Applying the isomorphisms  $\mathbf{L}^{1 \times X} \cong \mathbf{L}^X$  and  $\mathbf{L}^{Y \times 1} \cong \mathbf{L}^Y$  whenever necessary, one could write them, alternatively, as follows:

$$\begin{array}{lll} A^\uparrow = A \triangleleft I & A^\wedge = A \circ I & A^\wedge = A \triangleright I \\ B^\downarrow = I \triangleright B & B^\cup = I \triangleleft B & B^\vee = I \circ B \end{array}$$

Furthermore, denote the corresponding sets of fixed points by  $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ ,  $\mathcal{B}^{\cup\cup}(X, Y, I)$ , and  $\mathcal{B}^{\wedge\vee}(X, Y, I)$ , i.e.

$$\begin{aligned} \mathcal{B}^{\uparrow\downarrow}(X, Y, I) &= \{\langle A, B \rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^\uparrow = B, B^\downarrow = A\}, \\ \mathcal{B}^{\cup\cup}(X, Y, I) &= \{\langle A, B \rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^\wedge = B, B^\cup = A\}, \\ \mathcal{B}^{\wedge\vee}(X, Y, I) &= \{\langle A, B \rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^\wedge = B, B^\vee = A\}. \end{aligned}$$

The sets of fixpoints are complete lattices (Pollandt 1997, Belohlavek 1999, Georgescu and Popescu 2004), called the standard (resp. object-oriented, and property-oriented)  $\mathbf{L}$ -concept lattices associated with  $I$ , and their elements are called formal concepts.

For a concept lattice  $\mathcal{B}^{\Delta\vee}(X, Y, I)$ , where  $\mathcal{B}^{\Delta\vee}$  is either of  $\mathcal{B}^{\uparrow\downarrow}$ ,  $\mathcal{B}^{\cup\cup}$ , or  $\mathcal{B}^{\wedge\vee}$ , denote the corresponding sets of extents and intents by  $\text{Ext}^{\Delta\vee}(X, Y, I)$  and  $\text{Int}^{\Delta\vee}(X, Y, I)$ . That is,

$$\begin{aligned} \text{Ext}^{\Delta\vee}(X, Y, I) &= \{A \in \mathbf{L}^X \mid \langle A, B \rangle \in \mathcal{B}^{\Delta\vee}(X, Y, I) \text{ for some } B\}, \\ \text{Int}^{\Delta\vee}(X, Y, I) &= \{B \in \mathbf{L}^Y \mid \langle A, B \rangle \in \mathcal{B}^{\Delta\vee}(X, Y, I) \text{ for some } A\}, \end{aligned}$$

The operators induced by an  $\mathbf{L}$ -context and their sets of fixpoints have extensively been studied, see e.g. (Pollandt 1997, Belohlavek 1999, 2001, 2004, Georgescu and Popescu 2004).

We will need the following result by Belohlavek and Konecny (2012b).

**Theorem 2.3:** Consider  $\mathbf{L}$ -contexts  $\langle X, Y, I \rangle$ ,  $\langle X, F, A \rangle$ , and  $\langle F, Y, B \rangle$ .

- (a)  $\text{Int}^{\cup\cup}(X, Y, I) \subseteq \text{Int}^{\cup\cup}(F, Y, B)$  if and only if there exists  $A' \in \mathbf{L}^{X \times F}$  such that  $I = A' \circ B$ ,
- (b)  $\text{Ext}^{\wedge\vee}(X, Y, I) \subseteq \text{Ext}^{\wedge\vee}(X, F, A)$  if and only if there exists  $B' \in \mathbf{L}^{F \times Y}$  such that  $I = A \circ B'$ ,
- (c)  $\text{Int}^{\uparrow\downarrow}(X, Y, I) \subseteq \text{Int}^{\uparrow\downarrow}(F, Y, B)$  if and only if there exists  $A' \in \mathbf{L}^{X \times F}$  such that  $I = A' \triangleleft B$ ,
- (d)  $\text{Ext}^{\uparrow\downarrow}(X, Y, I) \subseteq \text{Ext}^{\uparrow\downarrow}(X, F, A)$  if and only if there exists  $B' \in \mathbf{L}^{F \times Y}$  such that  $I = A \triangleright B'$ ,
- (e)  $\text{Ext}^{\uparrow\downarrow}(X, Y, I) \subseteq \text{Ext}^{\cup\cup}(X, F, A)$  if and only if there exists  $B' \in \mathbf{L}^{F \times Y}$  such that  $I = A \triangleleft B'$ ,
- (f)  $\text{Int}^{\uparrow\downarrow}(X, Y, I) \subseteq \text{Int}^{\wedge\vee}(F, Y, B)$  if and only if there exists  $A' \in \mathbf{L}^{X \times F}$  such that  $I = A' \triangleright B$ .

In addition, we also have

- (g)  $\text{Ext}^{\cup\cup}(X, Y, A \circ B) \subseteq \text{Ext}^{\cup\cup}(X, F, A)$ .
- (h)  $\text{Int}^{\wedge\vee}(X, Y, A \circ B) \subseteq \text{Int}^{\wedge\vee}(F, Y, B)$ .

We will also utilize following lemma by Belohlavek and Konecny (2011).

**Lemma 2.4:** *Let  $I, J \in L^{X \times Y}$ . We have  $B^{U_I} = B^{U_J}$  for each  $B \in L^Y$  iff  $I = J$ .*

### 2.3 Morphisms of closure and interior systems

A system of  $\mathbf{L}$ -sets  $V \subseteq \mathbf{L}^X$  is called an  $\mathbf{L}$ -interior system if

- $V$  is closed under  $\otimes$ -multiplication, i.e. for every  $a \in L$  and  $C \in V$  we have  $a \otimes C \in V$ ;
- $V$  is closed under union, i.e.  $\bigcup_{j \in J} C_j \in V$  whenever  $C_j \in V$  for all  $j \in J$ .

$V \subseteq \mathbf{L}^X$  is called an  $\mathbf{L}$ -closure system if

- $V$  is closed under left  $\rightarrow$ -multiplication (or  $\rightarrow$ -shift), i.e. for every  $a \in L$  and  $C \in V$  we have  $a \rightarrow C \in V$ ;
- $V$  is closed under intersection, i.e.  $\bigcap_{j \in J} C_j \in V$  whenever  $C_j \in V$  for all  $j \in J$ .

One can find examples of  $\mathbf{L}$ -closure and  $\mathbf{L}$ -interior systems in the framework of formal fuzzy concept analysis as follows: for an  $\mathbf{L}$ -context  $\langle X, Y, I \rangle$ , the sets  $\text{Ext}^{\uparrow\downarrow}(X, Y, I)$ ,  $\text{Ext}^{\cup}(X, Y, I)$ ,  $\text{Int}^{\wedge\vee}(X, Y, I)$ , and  $\text{Int}^{\uparrow\downarrow}(X, Y, I)$  are  $\mathbf{L}$ -closure systems, while  $\text{Ext}^{\wedge\vee}(X, Y, I)$  and  $\text{Int}^{\cup}(X, Y, I)$  are  $\mathbf{L}$ -interior systems, see (Belohlavek and Konecny 2011, 2012b, Konecny 2012).

#### Definition 2.5:

- (a) A mapping  $h : V \rightarrow W$  from an  $\mathbf{L}$ -interior system  $V \subseteq \mathbf{L}^X$  into an  $\mathbf{L}$ -interior system  $W \subseteq \mathbf{L}^Y$  is called an *i-morphism* if it is a  $\otimes$ - and  $\bigvee$ -morphism, i.e.
- $h(a \otimes C) = a \otimes h(C)$  for each  $a \in L$  and  $C \in V$ ;
  - $h(\bigvee_{k \in K} C_k) = \bigvee_{k \in K} h(C_k)$  for every collection of  $C_k \in V$  ( $k \in K$ ).
- An i-morphism  $h : V \rightarrow W$  is said to be an *extendable i-morphism* if  $h$  can be extended to an i-morphism of  $\mathbf{L}^X$  to  $\mathbf{L}^Y$ , i.e. if there exists an i-morphism  $h' : \mathbf{L}^X \rightarrow \mathbf{L}^Y$  such that for every  $C \in V$  we have  $h'(C) = h(C)$ .
- (b) A mapping  $h : V \rightarrow W$  from an  $\mathbf{L}$ -closure system  $V \subseteq \mathbf{L}^X$  into an  $\mathbf{L}$ -closure system  $W \subseteq \mathbf{L}^Y$  is called a *c-morphism* if it is a  $\rightarrow$ - and  $\bigwedge$ -morphism and it preserves  $a$ -complements, i.e. if
- $h(a \rightarrow C) = a \rightarrow h(C)$  for each  $a \in L$  and  $C \in V$ ;
  - $h(\bigwedge_{k \in K} C_k) = \bigwedge_{k \in K} h(C_k)$  for every collection of  $C_k \in V$  ( $k \in K$ );
  - if  $C$  is an  $a$ -complement then  $h(C)$  is an  $a$ -complement.

A c-morphism  $h : V \rightarrow W$  is called an *extendable c-morphism* if  $h$  can be extended to a c-morphism of  $\mathbf{L}^X$  to  $\mathbf{L}^Y$ , i.e. if there exists a c-morphism  $h' : \mathbf{L}^X \rightarrow \mathbf{L}^Y$  such that for every  $C \in V$  we have  $h'(C) = h(C)$ .

- (c) A mapping  $h : V \rightarrow W$  from an  $\mathbf{L}$ -interior system  $V \subseteq \mathbf{L}^X$  into an  $\mathbf{L}$ -closure system  $W \subseteq \mathbf{L}^Y$  is called an *a-morphism* if
- $h(a \otimes C) = a \rightarrow h(C)$  for each  $a \in L$  and  $C \in V$ ;
  - $h(\bigvee_{k \in K} C_k) = \bigwedge_{k \in K} h(C_k)$  for every collection of  $C_k \in V$ .
- An a-morphism  $h : V \rightarrow W$  is called an *extendable a-morphism* if  $h$  can be extended to an a-morphism of  $\mathbf{L}^X$  to  $\mathbf{L}^Y$ , i.e. if there exists an a-morphism  $h' : \mathbf{L}^X \rightarrow \mathbf{L}^Y$  such that for every  $C \in V$  we have  $h'(C) = h(C)$ .

In this paper we will consider only extendable morphisms, for which the following results will be used hereafter, see (Belohlavek and Konecny 2011, 2012b, Konecny 2012).

**Lemma 2.6:** *For  $V \subseteq \mathbf{L}^X$ ,*

- (a) if  $h : V \rightarrow \mathbf{L}^Y$  is an *i*-morphism then there exists an  $\mathbf{L}$ -relation  $R \in \mathbf{L}^{X \times Y}$  such that  $h(C) = C \circ R$  for every  $C \in V$ .
- (b) if  $h : V \rightarrow \mathbf{L}^Y$  is a *c*-morphism then there exists an  $\mathbf{L}$ -relation  $R \in \mathbf{L}^{X \times Y}$  such that  $h(C) = C \triangleright R$  for every  $C \in V$ .
- (c) if  $h : V \rightarrow \mathbf{L}^Y$  is an *a*-morphism then there exists an  $\mathbf{L}$ -relation  $R \in \mathbf{L}^{X \times Y}$  such that  $h(C) = C \triangleleft R$  for every  $C \in V$ .

**Lemma 2.7:** Let  $R \in \mathbf{L}^{Y \times X}$ ,

- (a) the mapping  $h_R : \mathbf{L}^X \rightarrow \mathbf{L}^Y$  defined by  $h_R(C) = R \circ C$  and the mapping  $g_R : \mathbf{L}^Y \rightarrow \mathbf{L}^X$  defined by  $g_R(C) = C \circ R$  are *i*-morphisms.
- (b) the mapping  $h_R : \mathbf{L}^X \rightarrow \mathbf{L}^Y$  defined by  $h_R(C) = R \triangleleft C$  and the mapping  $h_g : \mathbf{L}^Y \rightarrow \mathbf{L}^X$  defined by  $g_R(C) = C \triangleright R$  are *c*-morphisms.
- (c) the mapping  $g_R : \mathbf{L}^X \rightarrow \mathbf{L}^Y$  defined by  $h_R(C) = R \triangleright C$  and the mapping  $g_R : \mathbf{L}^Y \rightarrow \mathbf{L}^X$  defined by  $g_R(C) = C \triangleleft R$  are *a*-morphisms.

The previous lemmas together with Remark 2.2 allow for establishing a link between {i,c,a}-morphisms with formal fuzzy concept analysis in that, for instance,  $h_R(C)$  in (a) coincides with  $C^\vee$  just using  $R$  as incidence relation (hence we will denote the corresponding concept-forming operator as  $\vee_R$ ). Similarly, we will use  $\downarrow_R$ ,  $\cup_R$  and so on.

### 3. Homogeneous L-bonds

This section introduces some new notions studied in this work. To begin with, we introduce the notion of *homogeneous L-bond* as a convenient generalization of bond. Firstly, it will be convenient to recall the classical notion of bond.

A bond between two contexts  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  and  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  is a relation  $\beta \subseteq X_1 \times Y_2$  such that

- (B1) For all  $x \in X_1$ , the set  $\beta(x)$  is an intent of  $\langle X_2, Y_2, I_2 \rangle$ ;
- (B2) For all  $y \in Y_2$ , the set  $\beta^{-1}(y)$  is an extent of  $\langle X_1, Y_1, I_1 \rangle$ ;

where  $\beta(x) \subseteq Y_2, \beta^{-1}(y) \subseteq X_1$  s.t.  $(\beta(x))(y) = \beta(x, y) = \beta^{-1}(y)(x)$ .

Note that, in the classical case, these conditions are equivalent to

- (B1') Each extent of  $\langle X_1, Y_2, \beta \rangle$  is an extent of  $\langle X_1, Y_1, I_1 \rangle$ .
- (B2') Each intent of  $\langle X_1, Y_2, \beta \rangle$  is an intent of  $\langle X_2, Y_2, I_2 \rangle$ .

These conditions lead us to the following generalization to the  $\mathbf{L}$ -fuzzy case.

**Definition 3.1:** A homogeneous bond wrt  $\langle \Delta, \nabla \rangle$  between two  $\mathbf{L}$ -contexts  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  and  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  is an  $\mathbf{L}$ -relation  $\beta \in L^{X_1 \times Y_2}$  s.t.

$$\text{Ext}^{\Delta \nabla}(X_1, Y_2, \beta) \subseteq \text{Ext}^{\Delta \nabla}(X_1, Y_1, I_1) \text{ and } \text{Int}^{\Delta \nabla}(X_1, Y_2, \beta) \subseteq \text{Int}^{\Delta \nabla}(X_2, Y_2, I_2).$$

Now, we can explain the use of the term *homogeneous* in that the same pair of concept-forming operators is used in both inclusions in the definition above. Later, in Section 4 we will consider *heterogeneous* bonds in which the concept-forming operators appear mixed in the inclusions above.

In this section we study homogeneous bonds with respect to  $\langle \cap, \cup \rangle$  and homogeneous bonds with respect to  $\langle \wedge, \vee \rangle$ .

#### Remark 3.2

- (a) Note that homogeneous bonds with respect to  $\langle \uparrow, \downarrow \rangle$  were studied in (Křídlo *et al.* 2012). In Section 4.2 we will provide a comparison of our results with

those in the previous reference. See also Remark 3.15.

- (b) One can observe that homogeneous bonds wrt  $\langle \wedge, \vee \rangle$  from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  are transposes of homogeneous bonds wrt  $\langle \cap, \cup \rangle$  from  $\langle Y_2, X_2, I_2^T \rangle$  to  $\langle Y_1, X_1, I_1^T \rangle$ .

Homogeneous bonds can be put in relation to that of c-morphism.

**Theorem 3.3:**

- (a) The homogeneous bonds from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  wrt  $\langle \cap, \cup \rangle$  are in one-to-one correspondence with the c-morphisms from  $\text{Ext}^{\cap\cup}(X_2, Y_2, I_2)$  to  $\text{Ext}^{\cap\cup}(X_1, Y_1, I_1)$ .
- (b) The homogeneous bonds from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  wrt  $\langle \wedge, \vee \rangle$  are in one-to-one correspondence with the c-morphisms from  $\text{Int}^{\wedge\vee}(X_1, Y_1, I_1)$  to  $\text{Int}^{\wedge\vee}(X_2, Y_2, I_2)$ .

**Proof:**

- (a) We show procedures to obtain the c-morphism from a homogeneous bond and vice versa.

“ $\Rightarrow$ ”: Let  $\beta$  be a homogeneous bond from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  wrt  $\langle \cap, \cup \rangle$ . By Definition 3.1 we have  $\text{Int}^{\cap\cup}(X_1, Y_2, \beta) \subseteq \text{Int}^{\cap\cup}(X_2, Y_2, I_2)$ ; thus by Theorem 2.3 there exists  $R \in L^{X_1 \times X_2}$  such that  $\beta = R \circ I_2$ . Now, by Lemma 2.7, the induced operator of this type  $\cup_R: L^{X_2} \rightarrow L^{X_1}$ , such that  $C^{\cup_R} = R \triangleleft C$ , is a c-morphism.

It only remains to check that when  $C$  is an extent of  $\mathbb{K}_2$ , its image  $R \triangleleft C$  is an extent of  $\mathbb{K}_1$ . Assume  $C \in \text{Ext}^{\cap\cup}(X_2, Y_2, I_2)$ , then we have that  $C = D^{\cup_{I_2}} = I_2 \triangleleft D$  for some  $D \in L^{Y_2}$ ; now using this expression in  $R \triangleleft C$  we have

$$R \triangleleft C = R \triangleleft (I_2 \triangleleft D) = (R \circ I_2) \triangleleft D = \beta \triangleleft D = D^{\cup_\beta}$$

and, as a result, we obtain that  $R \triangleleft C$  is in  $\text{Ext}^{\cap\cup}(X_1, Y_2, \beta)$  and, therefore, as  $\beta$  is a homogeneous bond, it is also an extent of  $\mathbb{K}_1$ .

Now, let us show that the previous construction, given a homogeneous bond  $\beta$  from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  wrt  $\langle \cap, \cup \rangle$ , produces a unique c-morphism  $f_\beta: \text{Ext}^{\cap\cup}(X_2, Y_2, I_2) \rightarrow \text{Ext}^{\cap\cup}(X_1, Y_1, I_1)$ .

It is enough to check that the construction does not depend on the relation used to factorize  $\beta$ , i.e. for any  $R, S \in L^{X_1 \times X_2}$  satisfying  $\beta = R \circ I_2 = S \circ I_2$  we have that the equality

$$C^{\cup_R} = C^{\cup_S} \tag{20}$$

holds for all  $C \in \text{Ext}^{\cap\cup}(X_2, Y_2, I_2)$ . Now, by definition,  $\text{Ext}^{\cap\cup}(X_2, Y_2, I_2) = \{D^{\cup_{I_2}} \mid D \in L^{Y_2}\}$  the equality (20) is equivalent to

$$D^{\cup_{I_2} \cup_R} = D^{\cup_{I_2} \cup_S} \quad \text{for all } D \in L^{Y_2}. \tag{21}$$

but we have that

$$\begin{aligned} D^{\cup_{I_2} \cup_R} &= R \triangleleft (I_2 \triangleleft D) = (R \circ I_2) \triangleleft D = \beta \triangleleft D \\ D^{\cup_{I_2} \cup_S} &= S \triangleleft (I_2 \triangleleft D) = (S \circ I_2) \triangleleft D = \beta \triangleleft D \end{aligned}$$

Thus, equality (21) holds true, and both relations  $R$  and  $S$  induce the same c-morphism  $f_\beta: \text{Ext}^{\cap\cup}(X_2, Y_2, I_2) \rightarrow \text{Ext}^{\cap\cup}(X_1, Y_1, I_1)$ .



“ $\Leftarrow$ ”: For a  $c$ -morphism  $f: \text{Ext}^{\cup}(X_2, Y_2, I_2) \rightarrow \text{Ext}^{\cup}(X_1, Y_1, I_1)$ , by Lemma 2.6, there is an  $L$ -relation  $S \in L^{X_2 \times X_1}$  s.t.  $f(C) = C^{\cup S} = S^T \triangleleft C = C \triangleright S$  for each  $C \in \text{Ext}^{\cup}(X_2, Y_2, I_2)$ .

By considering  $\beta = S^T \circ I_2$ , and using Theorem 2.3(a) one obtains that  $\text{Int}^{\cup}(X_1, Y_2, \beta) \subseteq \text{Int}^{\cup}(X_2, Y_2, I_2)$ . For the inclusion between the extents, it is sufficient to show that  $\text{Ext}^{\cup}(X_1, Y_2, \beta) \subseteq \text{Im}(f)$ : assume  $C \in \text{Ext}^{\cup}(X_1, Y_2, \beta)$ , then there exists a  $D$  such that  $C = \beta \triangleleft D$ . Unfolding the definition of  $\beta$  and applying some relational equalities we obtain the following:

$$C = \beta \triangleleft D = (S^T \circ I_2) \triangleleft D = S^T \triangleleft (I_2 \triangleleft D) = (I_2 \triangleleft D)^T \triangleright S = f((I_2 \triangleleft D)^T)$$

As, by assumption,  $\text{Im}(f) \subseteq \text{Ext}^{\cup}(X_1, Y_1, I_1)$ , we have that  $\beta$  is a homogeneous bond from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  w.r.t  $\langle \cap, \cup \rangle$ .

Let us prove now that this construction produces a unique homogeneous bond  $\beta_f$  from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  for a given  $c$ -morphism  $f: \text{Ext}^{\cup}(X_2, Y_2, I_2) \rightarrow \text{Ext}^{\cup}(X_1, Y_1, I_1)$  wrt  $\langle \cap, \cup \rangle$ . It is enough to show

$$R^T \circ I_2 = S^T \circ I_2. \quad (22)$$

for all relations  $R, S \in L^{X_2 \times X_1}$  satisfying

$$f(C) = C \triangleright R = C \triangleright S \quad \text{for all } C \in \text{Ext}^{\cup}(X_2, Y_2, I_2) \quad (23)$$

Since  $\text{Ext}^{\cup}(X_2, Y_2, I_2) = \{D^{\cup I_2} \mid D \in L^{Y_2}\}$  the condition (23) is equivalent to

$$f(D^{\cup I_2}) = D^{\cup I_2} \triangleright R = D^{\cup I_2} \triangleright S \quad \text{for all } D \in L^{Y_2}. \quad (24)$$

We have  $D^{\cup I_2} \triangleright R = (I_2 \triangleleft D) \triangleright R = R^T \triangleleft (I_2 \triangleleft D) = (R^T \circ I_2) \triangleleft D$  and similarly  $D^{\cup I_2} \triangleright S = (S^T \circ I_2) \triangleleft D$ , hence the condition (23) is equivalent to

$$\begin{aligned} f(D^{\cup I_2}) &= (R^T \circ I_2) \triangleleft D = (S^T \circ I_2) \triangleleft D \\ &= D^{\downarrow R^T \circ I_2} = D^{\downarrow S^T \circ I_2} \quad \text{for all } D \in L^{Y_2}. \end{aligned}$$

By Lemma 2.4 we have  $R^T \circ I_2 = S^T \circ I_2$  and (22) is satisfied and  $\beta_f$  is well-defined.

Finally, the one-to-one correspondence stated by the theorem will be completely proved if  $\beta_{f_\beta} = \beta$  and  $f_{\beta_f} = f$ . For this, it is worth to recall both directions of the correspondence in purely relational terms:

- Given  $\beta$ , if  $\beta = R \circ I_2$ , then  $f_\beta(C) = R \triangleleft C = C \triangleright R^T$
- Given  $f$ , if  $f(C) = C \triangleright S$ , then  $\beta_f = S^T \circ I_2$

Assume that  $\beta = R \circ I_2$ , then  $\beta_{f_\beta} = S^T \circ I_2$  for some relation  $S$  which is a right factor of  $f_\beta$  wrt  $\triangleright$ , by definition of  $f_\beta$  it is possible to consider  $S^T = R$ . As a result, we obtain  $\beta_{f_\beta} = \beta$ .

Now, assume  $f$  can be written as  $f(C) = C \triangleright S$ , then  $\beta_f = S^T \circ I_2$  which, in its turn, implies that  $f_{\beta_f} = S^T \triangleleft C = C \triangleright S = f$ .

(b) Follows from (a) and Remark 3.2(b).

□

The previous remark and theorem show that the homogeneous bonds wrt  $\langle \cap, \cup \rangle$  are different from homogeneous bonds wrt  $\langle \wedge, \vee \rangle$

**Theorem 3.4:** *The system of all homogeneous bonds wrt  $\langle \cap, \cup \rangle$  (resp. wrt  $\langle \wedge, \vee \rangle$ ) from  $\mathbb{K}_1$  to  $\mathbb{K}_2$  is an  $\mathbf{L}$ -interior system.*

**Proof:** We prove the result only for  $\langle \cap, \cup \rangle$ ; the other part then follows from Remark 3.2(b).

Consider a family  $\{\beta_j \in L^{X_1 \times X_2} \mid j \in J\}$  of homogeneous bonds from  $\mathbb{K}_1$  to  $\mathbb{K}_2$  and let us show that  $\beta = \bigcup_j \beta_j$  is a homogeneous bond; i.e. that  $A^{\cap\beta} \in \text{Int}^{\cup}(X_2, Y_2, I_2)$  and  $B^{\cup\beta} \in \text{Ext}^{\cup}(X_1, Y_1, I_1)$ .

$$A^{\cap\beta} = A \circ \beta = A \circ \left( \bigcup_j \beta_j \right) = \bigcup_j (A \circ \beta_j) = \bigcup_j A^{\cap\beta_j}$$

Thus we have that  $A^{\cap\beta} = \bigcup_{j \in J} A^{\cap\beta_j}$  proving that  $A^{\cap\beta} \in \text{Int}^{\cup}(X_2, Y_2, I_2)$  since  $\text{Int}^{\cup}(X_2, Y_2, I_2)$  is an  $\mathbf{L}$ -interior system.

Similarly we have

$$B^{\cup\beta} = \beta \triangleleft B = \left( \bigcup_j \beta_j \right) \triangleleft B = \bigcap_j (\beta_j \triangleleft B) = \bigcap_j B^{\cup\beta_j}$$

Thus we have that  $B^{\cup\beta} = \bigcap_{j \in J} B^{\cup\beta_j}$  proving that  $B^{\cup\beta} \in \text{Ext}^{\cup}(X_2, Y_2, I_2)$  since  $\text{Ext}^{\cup}(X_2, Y_2, I_2)$  is an  $\mathbf{L}$ -closure system.

Second, we show that if  $\beta$  is a homogeneous bond from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  then  $a \otimes \beta$  is a homogeneous bond as well. For every  $A \in L^{X_1}$  we have

$$A^{\cap a \otimes \beta} = A \circ (\text{Id}_a \circ \beta) = (A \circ \text{Id}_a) \circ \beta = (a \otimes A)^{\cap\beta},$$

where  $\text{Id}_a$  the identity relation on  $X_1$  multiplied by  $a \in L$ ; i.e.  $\text{Id}_a = a \otimes \text{Id}$ . Thus  $A^{\cap a \otimes \beta} = (a \otimes A)^{\cap\beta} \in \text{Int}^{\cup}(X_2, Y_2, I_2)$ .

For every  $B \in L^{Y_2}$  we have

$$B^{\cup a \otimes \beta} = (\text{Id}_a \circ \beta) \triangleleft B = \text{Id}_a \triangleleft (\beta \triangleleft B) = a \rightarrow B^{\cup\beta}.$$

Thus  $B^{\cup a \otimes \beta} = a \rightarrow B^{\cup\beta}$ , proving that  $B^{\cup a \otimes \beta} \in \text{Ext}^{\cup}(X_2, Y_2, I_2)$  since  $\text{Ext}^{\cup}(X_2, Y_2, I_2)$  is an  $\mathbf{L}$ -closure system.

The system of all homogeneous bonds is closed under union and multiplication, whence it is an  $\mathbf{L}$ -interior system. □

### 3.1 Strong homogeneous bonds

In this section, we will consider homogeneous bonds wrt both pairs of isotone concept-forming operators simultaneously; the antitone pair  $\langle \uparrow, \downarrow \rangle$  will be considered in Section 4.2. Formally, we introduce the notion of *strong homogeneous bond* as follows:

**Definition 3.5:** A strong homogeneous bond from  $\mathbf{L}$ -context  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbf{L}$ -context  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  is an  $\mathbf{L}$ -relation  $\beta \in L^{X_1 \times Y_2}$  s.t.  $\beta$  is a homogeneous bond wrt both  $\langle \cap, \cup \rangle$  and  $\langle \wedge, \vee \rangle$ .

The following shows that there exist homogeneous bonds which are not strong homogeneous bonds. as the following example shows.

**Example 3.6** Consider  $L$  a finite chain  $0 < a < b < 1$  with  $\otimes$  defined as follows:

$$x \otimes y = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for each  $x, y \in L$ . One can easily see that  $x \otimes \bigvee_j y_j = \bigvee_j (x \otimes y_j)$  and thus an adjoint operation  $\rightarrow$  exists such that  $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$  is a complete residuated lattice. Namely,  $\rightarrow$  is given as follows for all  $x, y \in L$ :

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x = 1, \\ b & \text{otherwise,} \end{cases}$$

Consider the sets  $X_1 = X_2 = \{x\}$ ,  $Y_1 = Y_2 = \{y\}$ , and the relations  $I_1 = \{^a/\langle x, y \rangle\}$  and  $I_2 = \{^b/\langle x, y \rangle\}$ . One can check that, we have  $\text{Ext}^{\cup}(\{x\}, \{y\}, I_1) = \text{Ext}^{\cup}(\{x\}, \{y\}, I_2) = \{\{^b/x\}, \{x\}\}$  and, trivially,  $\text{Int}^{\cup}(\{x\}, \{y\}, I_2) = \text{Int}^{\cup}(\{x\}, \{y\}, I_2)$ . Thus  $I_2$  is a homogeneous bond between  $I_1$  and  $I_2$  wrt  $\langle \wedge, \vee \rangle$ .

On the other hand,  $I_2$  is not a homogeneous bond between  $I_1$  and  $I_2$  wrt  $\langle \wedge, \vee \rangle$  since  $\text{Ext}^{\wedge}(\{x\}, \{y\}, I_1) = \{\emptyset, \{^a/x\}\} \not\supseteq \{\emptyset, \{^b/x\}\} = \text{Ext}^{\wedge}(\{x\}, \{y\}, I_2)$ .

The following lemma introduces alternative characterizations of the notion of strong homogeneous bond.

**Lemma 3.7:** *The following statements are equivalent:*

- (1)  $\beta$  is a strong homogeneous bond from  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$ .
- (2)  $\beta$  satisfies both  $\text{Ext}^{\wedge}(X_1, Y_2, \beta) \subseteq \text{Ext}^{\wedge}(X_1, Y_1, I_1)$  and  $\text{Int}^{\cup}(X_1, Y_2, \beta) \subseteq \text{Int}^{\cup}(X_2, Y_2, I_2)$ .
- (3)  $\beta$  satisfies both  $\{y\}^{\vee\beta} \in \text{Ext}^{\wedge}(X_1, Y_1, I_1)$  and  $\{x\}^{\cap\beta} \in \text{Int}^{\cup}(X_2, Y_2, I_2)$  for each  $x \in X_1, y \in Y_2$ .
- (4)  $\beta = S_e \circ I_2 = I_1 \circ S_i$  for some  $S_e \in L^{X_1 \times X_2}$  and  $S_i \in L^{Y_1 \times Y_2}$ .

**Proof:** (1)  $\Rightarrow$  (2): Directly from definition of strong homogeneous bond.

(2)  $\Rightarrow$  (3): Trivial, since  $\{y\}^{\vee\beta} \in \text{Ext}^{\wedge}(X_1, Y_2, \beta)$  and  $\{x\}^{\cap\beta} \in \text{Int}^{\cup}(X_1, Y_2, \beta)$ .

(3)  $\Rightarrow$  (4): Each  $\mathbf{L}$ -set  $A$  in  $\mathbf{L}^{X_1}$  can be written in the following form  $\bigcup_{x \in X_1} A(x) \otimes \{x\}$ . Then we have:

$$\begin{aligned} A^{\cap\beta}(y) &= \bigvee_{x' \in X_1} \left( \bigcup_{x \in X_1} A(x) \otimes \{x\} \right) (x') \otimes \beta(x', y) \\ &= \bigvee_{x' \in X_1} \left( \bigvee_{x \in X_1} A(x) \otimes \{x\} (x') \right) \otimes \beta(x', y) \\ &= \bigvee_{x \in X_1} \bigvee_{x' \in X_1} A(x) \otimes \{x\} (x') \otimes \beta(x', y) \\ &= \bigvee_{x \in X_1} A(x) \otimes \bigvee_{x' \in X} \{x\} (x') \otimes \beta(x', y) \\ &= \bigvee_{x \in X_1} A(x) \otimes \{x\}^{\cap\beta}(y) \\ &= \left( \bigcup_{x \in X_1} A(x) \otimes \{x\}^{\cap\beta} \right) (y). \end{aligned}$$

As a result we obtain  $A^{\wedge\beta} \in \text{Int}^{\cup}(X_2, Y_2, I_2)$  since  $\{x\}^{\wedge\beta} \in \text{Int}^{\cup}(X_2, Y_2, I_2)$  for each  $x \in X_1$  and  $\text{Int}^{\cup}(X_2, Y_2, I_2)$  is an  $\mathbf{L}$ -interior system. Because each intent in  $\text{Int}^{\cup}(X_1, Y_2, \beta)$  has the form  $A^{\wedge\beta}$ , we get  $\text{Int}^{\cup}(X_1, Y_2, \beta) \subseteq \text{Int}^{\cup}(X_2, Y_2, I_2)$ . The existence of  $S_e$  now follows from Theorem 2.3. The existence of  $S_i$  can be proved similarly.

(4)  $\Rightarrow$  (1): By Theorem 2.3 items (a),(b),(g),(h). □

**Remark 3.8** It is worth noting that, although conditions (B1')-(B2') are equivalent to (B1)-(B2) for the concept-forming operators  $\langle \uparrow, \downarrow \rangle$ , they are no longer equivalent for other concept-forming operators, i.e.  $\langle \cap, \cup \rangle$  and  $\langle \wedge, \vee \rangle$ , instead the conditions (B1')-(B2') are weaker. Definition 3.5 corresponds to conditions (B1)-(B2) as Lemma 3.7 (3) shows.

Strong homogeneous bonds can be related to the i-morphisms.

**Theorem 3.9:** *The strong homogeneous bonds from  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  are in one-to-one correspondence with*

- (a) *i-morphisms from  $\text{Int}^{\cup}(X_1, Y_1, I_1)$  to  $\text{Int}^{\cup}(X_2, Y_2, I_2)$ .*
- (b) *i-morphisms from  $\text{Ext}^{\wedge\vee}(X_2, Y_2, I_2)$  to  $\text{Ext}^{\wedge\vee}(X_1, Y_1, I_1)$ .*

**Proof:** We prove only (a), the proof of (b) is dual. We show procedures to obtain the i-morphism from  $\text{Int}^{\cup}(X_1, Y_1, I_1)$  to  $\text{Int}^{\cup}(X_2, Y_2, I_2)$  from a strong homogeneous bond and vice versa.

“ $\Rightarrow$ ”: Let  $\beta$  be a strong homogeneous bond from  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$ . By Lemma 3.7 there is  $S_i \in L^{Y_1 \times Y_2}$  such that  $\beta = I_1 \circ S_i$ . The induced operator  $\cap_{S_i}$  is an i-morphism from  $\text{Int}^{\cup}(X_1, Y_1, I_1)$  to  $\text{Int}^{\cup}(X_2, Y_2, I_2)$  by Lemma 2.7(a).

“ $\Leftarrow$ ”: For i-morphism  $f$  from  $\text{Int}^{\cup}(X_1, Y_1, I_1)$  to  $\text{Int}^{\cup}(X_2, Y_2, I_2)$  there is an  $\mathbf{L}$ -relation  $S_i$  s.t.  $f(B) = B^{\cap_{S_i}}$  for each  $B \in \text{Int}^{\cup}(X_1, Y_1, I_1)$  by Lemma 2.6(a). Denote  $\beta = I_1 \circ S_i$ . Each  $C \in \text{Int}^{\cup}(X_1, Y_2, \beta)$  is equal to  $A^{\wedge\beta}$  for some  $A \in \mathbf{L}^{X_1}$  and  $A^{\wedge\beta} = A \circ \beta = A \circ (I_1 \circ S_i) = (A \circ I_1) \circ S_i = (A \circ I_1)^{\cap_{S_i}} = f(A \circ I_1) \in \text{Im}(f)$ . Thus, we have  $\text{Int}^{\cup}(X_1, Y_2, \beta) \subseteq \text{Im}(f) \subseteq \text{Int}^{\cup}(X_2, Y_2, I_2)$ ; furthermore we have  $\text{Ext}^{\wedge\vee}(X_1, Y_2, \beta) \subseteq \text{Ext}^{\wedge\vee}(X_1, Y_1, I_1)$  by Theorem 2.3(b). Hence  $\beta$  is strong homogeneous bond by Lemma 3.7.

The fact that the two mappings between bonds and i-morphisms are mutually inverse can be checked as in the proof of Theorem 3.3. □

**Theorem 3.10:** *The system of all strong homogeneous bonds is an  $\mathbf{L}$ -interior system.*

**Proof:** Using Lemma 3.7 (2), it is an intersection of the  $\mathbf{L}$ -interior systems from Theorem 3.4. □

### 3.2 Direct $\circ$ -products

In the previous section we have studied the properties of homogeneous bonds, in particular its one-to-one correspondence with c-morphisms and i-morphisms. In this section, somehow paraphrasing Ganter (2007), we introduce the parameterized<sup>1</sup> direct product of contexts in order to elegantly describe the different families of generalized bonds between two given contexts.

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<sup>1</sup>We introduce here the  $\circ$ -product, but we will later introduce the  $\triangleright$ -product and the  $\triangleleft$ -product as well.

**Definition 3.11:** Let  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle, \mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  be  $\mathbf{L}$ -contexts. The direct  $\circ$ -product of  $\mathbb{K}_1$  and  $\mathbb{K}_2$  is defined as the  $\mathbf{L}$ -context

$$\mathbb{K}_1 \boxtimes \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta \rangle$$

with  $\Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) = I_1(x_1, y_1) \otimes I_2(x_2, y_2)$ .

**Theorem 3.12:**

- (a) The  $\langle \wedge, \vee \rangle$ -intents of  $\mathbb{K}_1 \boxtimes \mathbb{K}_2$  are strong homogeneous bonds from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ .
- (b) The  $\langle \wedge, \vee \rangle$ -extents of  $\mathbb{K}_1 \boxtimes \mathbb{K}_2$  are strong homogeneous bonds from  $\mathbb{K}_2$  to  $\mathbb{K}_1$ .

**Proof:** We prove only (a); the (b)-part is dual. We have

$$\begin{aligned} \phi^{\wedge\Delta}(x_1, y_2) &= \bigvee_{\langle x_2, y_1 \rangle \in X_2 \times Y_1} \phi(x_2, y_1) \otimes \Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) \\ &= \bigvee_{\langle x_2, y_1 \rangle \in X_2 \times Y_1} \phi(x_2, y_1) \otimes I_1(x_1, y_1) \otimes I_2(x_2, y_2) \\ &= \bigvee_{y_1 \in Y_1} \bigvee_{x_2 \in X_2} \phi(x_2, y_1) \otimes I_1(x_1, y_1) \otimes I_2(x_2, y_2) \\ &= \bigvee_{y_1 \in Y_1} I_1(x_1, y_1) \otimes \bigvee_{x_2 \in X_2} \phi(x_2, y_1) \otimes I_2(x_2, y_2) \\ &= \bigvee_{y_1 \in Y_1} I_1(x_1, y_1) \otimes (\phi^T \circ I_2)(y_1, y_2) \\ &= (I_1 \circ \phi^T \circ I_2)(x_1, y_2). \end{aligned}$$

Now, notice that  $(I_1 \circ \phi^T) \circ I_2 = I_1 \circ (\phi^T \circ I_2) = \beta$  is a strong homogeneous bond by Lemma 3.7.  $\square$

**Remark 3.13** It is worth mentioning that not every strong homogeneous bond is included in  $\text{Int}^{\wedge\Delta}(X_1 \times Y_2, X_2 \times Y_1, \Delta)$  since there are strong homogeneous bonds which are not of the form  $I_1 \circ \phi^T \circ I_2$ . For instance, using the same structure of truth degrees and  $I_1$  as in Example 3.6, obviously  $I_1$  is a strong homogeneous bond on  $\mathbb{K}_1$  (i.e. from  $\mathbb{K}_1$  to  $\mathbb{K}_1$ ), but  $\text{Int}^{\wedge\Delta}(X_1 \times Y_2, X_2 \times Y_1, \Delta)$  contains only an empty set.

**Corollary 3.14:** *The intents of  $\mathbb{K}_1 \boxtimes \mathbb{K}_2$  are exactly those strong homogeneous bonds from  $\mathbb{K}_1$  to  $\mathbb{K}_2$  which can be decomposed as  $I_1 \circ \phi^T \circ I_2$  for some  $\phi \in L^{X_2 \times Y_1}$ .*

**Proof:** The final line of the proof of Theorem 3.12 explains which strong homogeneous bonds are intents of  $\mathbb{K}_1 \boxtimes \mathbb{K}_2$ .  $\square$

**Remark 3.15** The relationship with the homogeneous bonds wrt  $\langle \uparrow, \downarrow \rangle$  introduced in (Křídlo *et al.* 2012) is the following: If the double negation law holds true in  $\mathbf{L}$  we have the equality  $\text{Ext}^{\uparrow\downarrow}(X, Y, I) = \text{Ext}^{\wedge\Delta}(X, Y, \neg I)$ . Thus, for a strong homo-

geneous bond  $\beta \in L^{X_1 \times Y_2} = S_e \circ I_2 = I_1 \circ S_i$  from  $\mathbb{K}_1$  to  $\mathbb{K}_2$  we have

$$\begin{aligned}
 (\neg\beta)(x_1, y_2) &= \neg(S_e \circ I_2)(x_1, y_2) \\
 &= \left( \bigvee_{x_2 \in X_2} (S_e(x_1, x_2) \otimes I_2(x_2, y_2)) \right) \rightarrow 0 \\
 &= \bigwedge_{x_2 \in X_2} (S_e(x_1, x_2) \otimes I_2(x_2, y_2)) \rightarrow 0 \\
 &= \bigwedge_{x_2 \in X_2} (S_e(x_1, x_2) \rightarrow (I_2(x_2, y_2) \rightarrow 0)) \\
 &= (S_e \triangleleft \neg I_2)(x_1, y_2).
 \end{aligned}$$

for each  $x_1 \in X_1, y_2 \in Y_2$ . Similarly, we can show that  $\neg\beta = \neg I_1 \triangleright S_i$ . Thus  $\neg\beta$  is a homogeneous bond wrt  $\langle \uparrow, \downarrow \rangle$  from  $\neg\mathbb{K}_1$  to  $\neg\mathbb{K}_2$ .

Some papers (Ganter 2007, Krídlo *et al.* 2012) have considered direct products in the crisp and the fuzzy settings, respectively, for the concept-forming operators  $\langle \uparrow, \downarrow \rangle$ . In (Krídlo *et al.* 2012) conditions are specified under which a homogeneous bond wrt  $\langle \uparrow, \downarrow \rangle$  is present in the concept lattice of the direct product. Corollary 3.14 and Remark 3.15 provide a simplification of these conditions.

A different direct product of contexts  $\mathbb{K}_1 \boxplus \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta \rangle$  was defined in (Krídlo *et al.* 2012), with the incidence relation given by

$$\begin{aligned}
 \Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) &= \neg I_1(x_1, y_1) \rightarrow I_2(x_2, y_2) \\
 &= (\neg I_2(x_2, y_2) \rightarrow I_1(x_1, y_1)).
 \end{aligned} \tag{25}$$

For the concept-forming operator  $\uparrow_\Delta$  we have

$$\begin{aligned}
 \phi^{\uparrow_\Delta}(x_1, y_2) &= \bigwedge_{\langle x_2, y_1 \rangle \in X_2 \times Y_1} \phi(x_2, y_1) \rightarrow (\neg I_1(x_1, y_1) \rightarrow I_2(x_2, y_2)) \\
 &= \bigwedge_{\langle x_2, y_1 \rangle \in X_2 \times Y_1} \neg I_1(x_1, y_1) \rightarrow (\phi(x_2, y_1) \rightarrow I_2(x_2, y_2)) \\
 &= \bigwedge_{x_2 \in X_2} \bigwedge_{y_1 \in Y_1} \neg I_1(x_1, y_1) \rightarrow (\phi(x_2, y_1) \rightarrow I_2(x_2, y_2)) \\
 &= \bigwedge_{y_1 \in Y_1} \neg I_1(x_1, y_1) \rightarrow \bigwedge_{x_2 \in X_2} (\phi(x_2, y_1) \rightarrow I_2(x_2, y_2)) \\
 &= \bigwedge_{y_1 \in Y_1} \neg I_1(x_1, y_1) \rightarrow (\phi^T \triangleleft I_2)(y_1, y_2) \\
 &= [\neg I_1 \triangleleft (\phi^T \triangleleft I_2)](x_1, y_2) \\
 &= [(\neg I_1 \circ \phi^T) \triangleleft I_2](x_1, y_2) \\
 &= [\neg(\neg I_1 \circ \phi^T \circ \neg I_2)](x_1, y_2).
 \end{aligned}$$

Whence a strong homogeneous bond wrt  $\langle \uparrow, \downarrow \rangle$  is an intent of the concept lattice of  $\mathbb{K}_1 \boxplus \mathbb{K}_2$  iff it is possible to write it as  $\neg(\neg I_1 \circ \phi^T \circ \neg I_2)$ , i.e. if its complement is an intent of  $\neg\mathbb{K}_1 \boxtimes \neg\mathbb{K}_2$ .

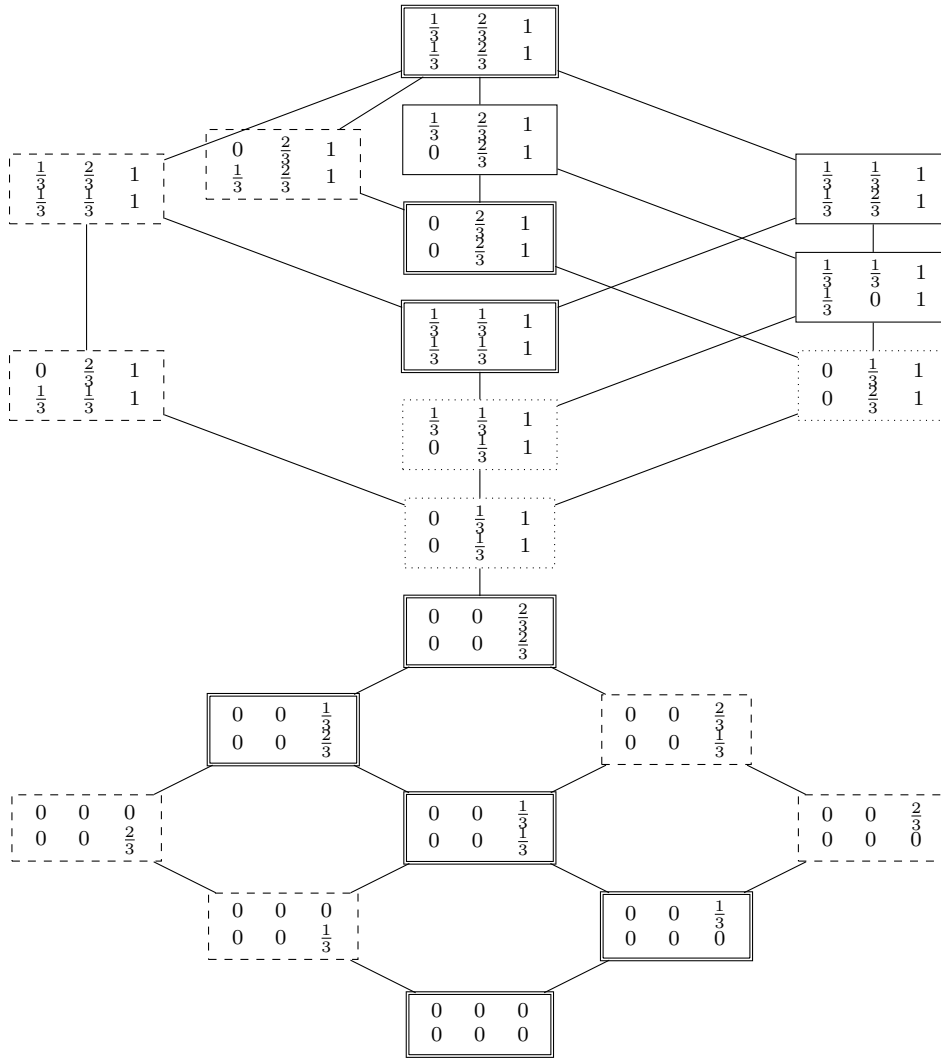


Figure 1. Lattice of all homogeneous bonds wrt isotone concept-forming operators on  $\mathbb{K}$  from Example 3.16. Homogeneous bonds wrt  $\langle n, u \rangle$  are drawn with dashed border; Homogeneous bonds wrt  $\langle \lambda, v \rangle$  are drawn with dotted border; strong homogeneous bonds are drawn with solid border; and intents of  $\mathbb{K} \boxtimes \mathbb{K}$  are drawn with double border.

**Example 3.16** Consider formal  $\mathbf{L}$ -context

$$\mathbb{K} = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 0 & 1 & 1 \\ \hline \end{array}.$$

Figure 1 depicts a lattice of all homogeneous bonds from  $\mathbb{K}$  to  $\mathbb{K}$  wrt  $\langle n, u \rangle$  and  $\langle \lambda, v \rangle$ .

#### 4. Heterogeneous L-bonds

This section introduces heterogeneous  $\mathbf{L}$ -bonds in the sense that conditions generalizing (B1') and (B2') relate different pairs of concept-forming operators. Particularly, we study so-called a-bonds and c-bonds defined as follows.

**Definition 4.1:**

- (a) An *a-bond* from  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  is an  $\mathbf{L}$ -relation

$\beta \in L^{X_1 \times Y_2}$  such that

$$\begin{aligned} \text{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta) &\subseteq \text{Ext}^{\cap\cup}(X_1, Y_1, I_1) \\ \text{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta) &\subseteq \text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2). \end{aligned} \tag{26}$$

- (b) A *c-bond* from two  $\mathbf{L}$ -contexts  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  is an  $\mathbf{L}$ -relation  $\beta \in L^{X_1 \times Y_2}$  s.t.

$$\begin{aligned} \text{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta) &\subseteq \text{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1) \\ \text{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta) &\subseteq \text{Int}^{\wedge\vee}(X_2, Y_2, I_2). \end{aligned} \tag{27}$$



**Remark 4.2**

- (a) The terms *a-bond* and *c-bond* have been chosen to match with the notions of a-morphism and c-morphism (Belohlavek and Konecny 2011, Konecny 2012, Belohlavek and Konecny 2012b). Later, in Theorem 4.4 we will show that a-bonds (resp. c-bonds) are in one-to-one correspondence with a-morphisms (resp. c-morphisms) on associated sets of intents.
- (b) Notice that both the sets of extents and intents in (26) and (27) are **L**-closure systems. From this point of view, the condition of subsethood is natural.
- (c) Notice also that a-bonds from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  are transposed versions of c-bonds from  $\langle Y_2, X_2, I_2^T \rangle$  to  $\langle Y_1, X_1, I_1^T \rangle$ .

The following theorem brings a characterization of a-bonds (resp. c-bonds) in terms of relational compositions.

**Theorem 4.3:**

- (a)  $\beta \in L^{X_1 \times Y_2}$  is an a-bond from  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  iff there exist **L**-relations  $S_i \in L^{Y_1 \times Y_2}$  and  $S_e \in L^{X_1 \times X_2}$ , such that

$$\beta = I_1 \triangleleft S_i = S_e \triangleleft I_2.$$

- (b)  $\beta \in L^{X_1 \times Y_2}$  is a c-bond from  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  iff there exist **L**-relations  $S_i \in L^{Y_1 \times Y_2}$  and  $S_e \in L^{X_1 \times X_2}$ , such that

$$\beta = I_1 \triangleright S_i = S_e \triangleright I_2.$$

**Proof:**

- (a) Follows from Definition 4.1 and Theorem 2.3, items (c) and (e).  
 (b) Follows from Definition 4.1 and Theorem 2.3, items (d) and (f).

□

**Theorem 4.4:**

- (a) The a-bonds from  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  are in one-to-one correspondence with
- a-morphisms from  $\text{Int}^{\cup}(X_1, Y_1, I_1)$  to  $\text{Int}^{\downarrow}(X_2, Y_2, I_2)$ ;
  - c-morphisms from  $\text{Ext}^{\downarrow}(X_2, Y_2, I_2)$  to  $\text{Ext}^{\cup}(X_1, Y_1, I_1)$ .
- (b) The c-bonds from  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  are in one-to-one correspondence with
- c-morphisms from  $\text{Int}^{\downarrow}(X_1, Y_1, I_1)$  to  $\text{Int}^{\vee}(X_2, Y_2, I_2)$ ;
  - a-morphisms from  $\text{Ext}^{\vee}(X_2, Y_2, I_2)$  to  $\text{Ext}^{\downarrow}(X_1, Y_1, I_1)$ .

**Proof:**

- (a) Let  $\beta$  be an a-bond from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ . By Theorem 4.3(a) we have  $\beta = I_1 \triangleleft S_i$ . By Lemma 2.7(c)  $f : \mathbf{L}^{X_2} \rightarrow \mathbf{L}^{X_1}$  defined by

$$f(B) = B \triangleleft S_i \quad (= B^{\uparrow s_i})$$

is an a-morphism. We need to show that it maps intents in  $\text{Int}^{\cup}(X_1, Y_1, I_1)$  to intents in  $\text{Int}^{\downarrow}(X_2, Y_2, I_2)$ .

For each  $\langle A, B \rangle \in \mathcal{B}^{\cup}(X_1, Y_1, I_1)$  we have  $B = A^{\wedge t_1}$ , which is equivalent to

$B = A \circ I_1$  by Remark 2.2. Then we have

$$\begin{aligned} f(B) &= B \triangleleft S_i = (A \circ I_1) \triangleleft S_i = A \triangleleft (I_1 \triangleleft S_i) = \\ &= A \triangleleft \beta = A^{\uparrow\beta} \in \text{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta) \subseteq \text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2). \end{aligned}$$

For the c-morphism, by Theorem 4.3(a) we have  $\beta = S_e \triangleleft I_2$ . By Lemma 2.7 (b)  $f : \mathbf{L}^{X_2} \rightarrow \mathbf{L}^{X_1}$  defined by

$$f(A) = S_e \triangleleft A \quad (= A^{U_{S_e}})$$

is a c-morphism. We need to show that it maps extents in  $\text{Ext}^{\uparrow\downarrow}(X_2, Y_2, I_2)$  to extents in  $\text{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)$ . For each  $\langle A, B \rangle \in \mathcal{B}^{\uparrow\downarrow}(X_2, Y_2, I_2)$  we have  $A = B^{\downarrow I_2}$ , which is equivalent to  $A = I_2 \triangleright B$  by Remark 2.2. Then we have

$$\begin{aligned} f(A) &= S_e \triangleleft A = S_e \triangleleft (I_2 \triangleright B) = (S_e \triangleleft I_2) \triangleright B = \\ &= \beta \triangleright B = B^{\downarrow\beta} \in \text{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta) \subseteq \text{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1). \end{aligned}$$

We have just shown how to construct the associated c-morphism and the associated a-morphism for a given a-bond. Now we show the inverse procedures.

Given an a-morphism  $f$  from  $\text{Int}^{\uparrow\downarrow}(X_1, Y_1, I_1)$  to  $\text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ , by using Lemma 2.6 (c), there is an  $\mathbf{L}$ -relation  $S_i \in \mathbf{L}^{Y_1 \times Y_2}$  such that  $f(B) = B \triangleleft S_i$  for each  $B \in \mathbf{L}^{Y_1}$ . Now, we consider  $\beta_f = I_1 \triangleleft S_i$ , and we need to show that it is an a-bond from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ .

Firstly, by Theorem 2.3 (e) we have  $\text{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta_f) \subseteq \text{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)$ .

Now, all the elements in  $\text{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta_f)$  have the form  $A^{\uparrow\beta_f}$  for some  $A \in \mathbf{L}^{X_1}$ . Thus, we can write

$$A^{\uparrow\beta_f} = A \triangleleft \beta_f = A \triangleleft (I_1 \triangleleft S_i) = (A \circ I_1) \triangleleft S_i$$

and, since  $A \circ I_1 = A^{\wedge I_1} \in \text{Int}^{\uparrow\downarrow}(X_1, Y_1, I_1)$ ,

$$(A \circ I_1) \triangleleft S_i = A^{\wedge I_1} \triangleleft S_i = f(A^{\wedge I_1}) \in \text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2),$$

proving that  $\text{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta_f) \subseteq \text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ , and  $\beta_f$  is an a-bond from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ .

Similarly, let  $g$  be a c-morphism from  $\text{Ext}^{\uparrow\downarrow}(X_2, Y_2, I_2)$  to  $\text{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)$ . By Lemma 2.6 (b) there is an  $\mathbf{L}$ -relation  $R \in \mathbf{L}^{X_2 \times X_1}$  such that  $g(A) = A \triangleright R$  for each  $A \in \mathbf{L}^{X_2}$ . That is equivalent to  $g(A) = R^T \triangleleft A$ . Denoting  $S_e = R^T$  we get  $g(A) = S_e \triangleleft A$  for each  $A \in \mathbf{L}^{X_2}$ . We consider  $\beta_g = S_e \triangleleft I_2$  and we need to show that it is an a-bond from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ .

By Theorem 2.3 (e) we directly have  $\text{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta_g) \subseteq \text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ .

Now, all the elements in  $\text{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta_g)$  have the form  $B^{\downarrow\beta_g}$  for some  $B \in \mathbf{L}^{Y_2}$ . Thus, we can write

$$B^{\downarrow\beta_g} = (S_e \triangleleft I_2) \triangleright B = S_e \triangleleft (I_2 \triangleright B)$$

and, since  $I_2 \triangleright B = B^{\downarrow I_2} \in \text{Ext}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ ,

$$S_e \triangleleft (I_2 \triangleright B) = S_e \triangleleft B^{\downarrow I_2} = g(B^{\downarrow I_2}) \in \text{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1),$$

proving that  $\text{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta_g) \subseteq \text{Ext}^{\cup}(X_1, Y_1, I_1)$ , and  $\beta_g$  is an a-bond from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ .

The fact that the two pairs of mappings between bonds and a-morphisms (resp. c-morphisms) are mutually inverse can be checked as in the proof of Theorem 3.3.

The proof of (b) is similar.  $\square$

**Theorem 4.5 :**

- (a) *The system of all a-bonds from  $\mathbb{K}_1$  to  $\mathbb{K}_2$  is an  $\mathbf{L}$ -closure system.*
- (b) *The system of all c-bonds from  $\mathbb{K}_1$  to  $\mathbb{K}_2$  is an  $\mathbf{L}$ -closure system.*

**Proof:** (a) Consider a family  $\{\beta_j \in L^{X_1 \times X_2} \mid j \in J\}$  of a-bonds from  $\mathbb{K}_1$  to  $\mathbb{K}_2$  and let us show that  $\bigcap_j \beta_j$  is an a-bond. By Theorem 4.3 a-bonds  $\beta_j$  are of the form

$$\beta_j = I_1 \triangleleft S_{ij} = S_{ej} \triangleleft I_2 \quad \text{for each } j \in J.$$

We have the following two expressions for  $\bigcap_{j \in J} \beta_j$

$$\begin{aligned} \bigcap_{j \in J} \beta_j &= \bigcap_{j \in J} (I_1 \triangleleft S_{ij}) = I_1 \triangleleft \left( \bigcap_{j \in J} S_{ij} \right); \\ \bigcap_{j \in J} \beta_j &= \bigcap_{j \in J} (S_{ej} \triangleleft I_2) = \left( \bigcup_{j \in J} S_{ej} \right) \triangleleft I_2. \end{aligned}$$

Thus, by Theorem 4.3,  $\bigcap_{j \in J} \beta_j$  is an a-bond.

Similarly, consider an a-bond  $\beta$ , hence  $\beta = I_1 \triangleleft S_i = S_e \triangleleft I_2$ . Let us show that  $a \rightarrow \beta$  is an a-bond as well:

$$\begin{aligned} a \rightarrow \beta &= \beta \triangleright \text{Id}_a = (I_1 \triangleleft S_i) \triangleright \text{Id}_a = I_1 \triangleleft (S_i \triangleright \text{Id}_a); \\ a \rightarrow \beta &= \text{Id}_a \triangleleft \beta = \text{Id}_a \triangleleft (S_e \triangleleft I_2) = (\text{Id}_a \circ S_e) \triangleleft I_2. \end{aligned}$$

Thus,  $a \rightarrow \beta$  is an a-bond from  $\mathbb{K}_1$  to  $\mathbb{K}_2$  by Theorem 4.3. We showed that the system of all a-bonds is closed under intersections and shifts, whence it is an  $\mathbf{L}$ -closure system.

Proof of (b) is similar.  $\square$

#### 4.1 Direct $\triangleleft$ -product and direct $\triangleright$ -product

In this part, we focus on direct products of  $\mathbf{L}$ -contexts which are related to a-bonds and c-bonds.

**Definition 4.6:** Let  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle, \mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$  be  $\mathbf{L}$ -contexts.

- (a) A *direct  $\triangleleft$ -product* of  $\mathbb{K}_1$  and  $\mathbb{K}_2$  is defined as the  $\mathbf{L}$ -context  $\mathbb{K}_1 \boxtimes \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta \rangle$  with  $\Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) = I_1(x_1, y_1) \rightarrow I_2(x_2, y_2)$  for all  $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$ .
- (b) A *direct  $\triangleright$ -product* of  $\mathbb{K}_1$  and  $\mathbb{K}_2$  is defined as the  $\mathbf{L}$ -context  $\mathbb{K}_1 \boxtimes \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta \rangle$  with  $\Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) = I_2(x_2, y_2) \rightarrow I_1(x_1, y_1)$  for all  $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$ .

The following theorem shows that  $\mathbb{K}_1 \boxtimes \mathbb{K}_2$  (resp.  $\mathbb{K}_1 \boxtimes \mathbb{K}_2$ ) induces a-bonds (resp. c-bonds) as its intents.

**Theorem 4.7:**

- (a) The intents of  $\mathbb{K}_1 \boxplus \mathbb{K}_2$  w.r.t  $\langle \uparrow, \downarrow \rangle$  are a-bonds from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ , i.e for each  $\phi \in L^{X_2 \times Y_1}$ ,  $\phi^\uparrow$  is an a-bond from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ .
- (b) The intents of  $\mathbb{K}_1 \boxminus \mathbb{K}_2$  w.r.t  $\langle \uparrow, \downarrow \rangle$  are c-bonds from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ , i.e for each  $\phi \in L^{X_2 \times Y_1}$ ,  $\phi^\uparrow$  is a c-bond from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ .

**Proof:** (a) For  $\phi \in L^{X_2 \times Y_1}$  we have

$$\begin{aligned}
\phi^\uparrow(x_1, y_2) &= \bigwedge_{\langle x_2, y_1 \rangle \in X_2 \times Y_1} \phi(x_2, y_1) \rightarrow \Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) \\
&= \bigwedge_{x_2 \in X_2} \bigwedge_{y_1 \in Y_1} \phi(x_2, y_1) \rightarrow (I_1(x_1, y_1) \rightarrow I_2(x_2, y_2)) \\
&= \bigwedge_{x_2 \in X_2} \bigwedge_{y_1 \in Y_1} I_1(x_1, y_1) \rightarrow (\phi(x_2, y_1) \rightarrow I_2(x_2, y_2)) \\
&= \bigwedge_{y_1 \in Y_1} I_1(x_1, y_1) \rightarrow \bigwedge_{x_2 \in X_2} \phi(x_2, y_1) \rightarrow I_2(x_2, y_2) \\
&= \bigwedge_{y_1 \in Y_1} I_1(x_1, y_1) \rightarrow \bigwedge_{x_2 \in X_2} \phi^\top(y_1, x_2) \rightarrow I_2(x_2, y_2) \\
&= \bigwedge_{y_1 \in Y_1} I_1(x_1, y_1) \rightarrow (\phi^\top \triangleleft I_2)(y_1, y_2) \\
&= [I_1 \triangleleft (\phi^\top \triangleleft I_2)](x_1, y_2) \\
&= [(I_1 \circ \phi^\top) \triangleleft I_2](x_1, y_2).
\end{aligned}$$

Thus  $\phi^\uparrow$  is an a-bond by Theorem 4.3 (a). Proof of (b) is similar.  $\square$

A similar proposition can be stated also for extents of direct  $\triangleleft$ -products and direct  $\triangleright$ -products. More exactly, extents of  $\mathbb{K}_1 \boxminus \mathbb{K}_2$  are c-morphisms from  $\mathbb{K}_2$  to  $\mathbb{K}_1$ , and extents of  $\mathbb{K}_1 \boxplus \mathbb{K}_2$  are a-morphisms from  $\mathbb{K}_2$  to  $\mathbb{K}_1$ .

**Remark 4.8** It is worth to note that not all a-bonds need to be intents of the direct product as the following examples shows.

**Example 4.9** Consider the  $\mathbf{L}$ -context  $\mathbb{K} = \langle \{x\}, \{y\}, \{^{0.5}/\langle x, y \rangle\} \rangle$  with  $\mathbf{L}$  being the three-element Lukasiewicz chain. Consider  $\beta$  to be the  $\mathbf{L}$ -relation  $\{^{0.5}/\langle x, y \rangle\}$ . We have

$$\text{Ext}^{\cup}(\{x\}, \{y\}, \beta) = \{^{0.5}/x, x\} = \text{Ext}^{\downarrow}(\{x\}, \{y\}, \{^{0.5}/\langle x, y \rangle\}),$$

and  $\text{Int}^{\downarrow}(\{x\}, \{y\}, \beta) \subseteq \text{Int}^{\downarrow}(\{x\}, \{y\}, \{^{0.5}/\langle x, y \rangle\})$  is trivial. Thus  $\beta$  is an a-bond from  $\mathbb{K}$  to  $\mathbb{K}$ . We have  $\mathbb{K} \boxplus \mathbb{K} = \langle \{\langle x, y \rangle\}, \{\langle x, y \rangle\}, \{\langle x, y \rangle, \langle x, y \rangle\} \rangle$ . The only intent of  $\mathbb{K} \boxplus \mathbb{K}$  is  $\{\langle x, y \rangle\}$ ; thus the a-bond  $\beta = \{^{0.5}/\langle x, y \rangle\}$  is not among its intents.

**Example 4.10** Consider the following  $\mathbf{L}$ -context with  $\mathbf{L}$  being three-element Lukasiewicz chain.

$$\mathbb{K}_1 = \overline{\begin{array}{cccc} 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}} \quad \mathbb{K}_2 = \overline{\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{array}}.$$

There are 11 a-bonds from  $\mathbb{K}_1$  to  $\mathbb{K}_2$ , but  $\mathbb{K}_1 \boxplus \mathbb{K}_2$  has only 9 concepts; see Figure 1.

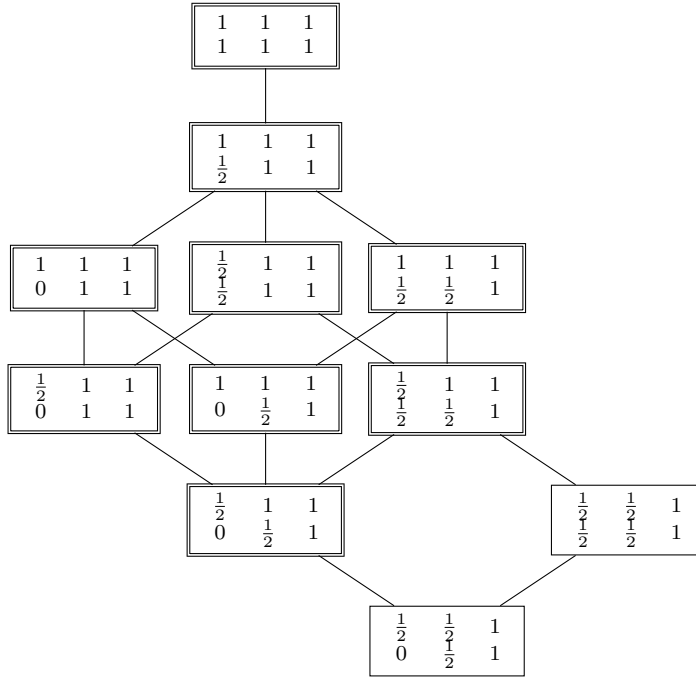


Figure 2. System of a-bonds between  $\mathbb{K}_1$  and  $\mathbb{K}_2$  from Example 4.10. The a-bonds with double border are those which are intents of  $\mathbb{K}_1 \boxtimes \mathbb{K}_2$ .

#### 4.2 Relationship to homogeneous bonds with respect to $\langle \uparrow, \downarrow \rangle$

In this section we establish a relationship of a-bonds and c-bonds with homogeneous bonds with respect to  $\langle \uparrow, \downarrow \rangle$ . Firstly, we will introduce the notion of strong heterogeneous bond and, then, will prove that they are a special case of homogeneous bond wrt  $\langle \uparrow, \downarrow \rangle$ . Then we study equality of homogeneous bonds wrt  $\langle \uparrow, \downarrow \rangle$  with a-bonds and c-bonds under special conditions.

**Definition 4.11:** An **L**-relation  $\beta$  is called *strong heterogeneous bond* from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  if it is both a-bond and c-bond from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$ .

Let us start with the analogous version of Lemma 3.7 (with alternative characterizations) for homogeneous bonds wrt  $\langle \uparrow, \downarrow \rangle$ .

**Lemma 4.12:** *The following statements are equivalent:*

- (1)  $\beta$  is a homogeneous bond wrt  $\langle \uparrow, \downarrow \rangle$  from  $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$  to  $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$ .
- (2)  $\beta$  satisfies both  $\{y\}^{\downarrow\beta} \in \text{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)$  and  $\{x\}^{\uparrow\beta} \in \text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$  for each  $x \in X_1, y \in Y_2$ .
- (3)  $\beta = S_e \triangleleft I_2 = I_1 \triangleright S_i$  for some  $S_e \in L^{X_1 \times X_2}$  and  $S_i \in L^{Y_1 \times Y_2}$ .

**Proof:**

(1)  $\Rightarrow$  (2): Trivially, we have that  $\{y\}^{\downarrow\beta} \in \text{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta)$  and  $\{x\}^{\uparrow\beta} \in \text{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta)$ .

(2)  $\Rightarrow$  (3): Each **L**-set  $A$  in  $L^{X_1}$  can be written in the form  $\bigcup_{x \in X} A(x) \otimes \{x\}$ .

Thus, we have

$$\begin{aligned}
A^{\uparrow\beta}(y) &= \bigwedge_{x' \in X} \left( \bigcup_{x \in X} A(x) \otimes \{x\} \right)(x') \rightarrow \beta(x', y) \\
&= \bigwedge_{x' \in X} \left( \bigvee_{x \in X} A(x) \otimes \{x\} \right)(x') \rightarrow \beta(x', y) \\
&= \bigwedge_{x' \in X} \bigwedge_{x \in X} A(x) \rightarrow (\{x\}(x') \rightarrow \beta(x', y)) \\
&= \bigwedge_{x \in X} A(x) \rightarrow \bigwedge_{x' \in X} \{x\}(x') \rightarrow \beta(x', y) \\
&= \bigwedge_{x \in X} A(x) \rightarrow \{x\}^{\uparrow\beta}(y) \\
&= \left( \bigcap_{x \in X} A(x) \rightarrow \{x\}^{\uparrow\beta} \right)(y).
\end{aligned}$$

As a result we have  $A^{\uparrow\beta} \in \text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$  since  $\{x\}^{\uparrow\beta} \in \text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$  for each  $x \in X_1$  and  $\text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$  is an  $\mathbf{L}$ -closure system. Because each intent in  $\text{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta)$  has the form  $A^{\uparrow\beta}$ , we get  $\text{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta) \subseteq \text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ . The existence of  $S_e$  now follows from Theorem 2.3. Similarly, the existence of  $S_i$  can be proved.

(c)  $\Rightarrow$  (a): From Theorem 2.3 (c) and (d).  $\square$

One can easily observe that each strong heterogeneous bond is a homogeneous bond wrt  $\langle \uparrow, \downarrow \rangle$ . The following example shows that the converse is not true in general.

**Example 4.13** Use  $\mathbf{L} = \mathbf{2}$ ; obviously the empty relation is a homogeneous bond wrt  $\langle \uparrow, \downarrow \rangle$  between two formal contexts with empty incidence relation. On the other hand, it is not an a-bond because  $|\text{Ext}^{\text{ou}}(X_1, Y_1, \emptyset)| = 1 < 2 = |\text{Ext}^{\uparrow\downarrow}(X_1, Y_2, \emptyset)|$ . Specifically, the only concept in  $\mathcal{B}^{\text{ou}}(X_1, Y_1, \emptyset)$  is  $\langle X_1, \emptyset \rangle$ , whereas the two concepts in  $\mathcal{B}^{\uparrow\downarrow}(X_1, Y_2, \emptyset)$  are  $\langle X_1, \emptyset \rangle$  and  $\langle \emptyset, Y_2 \rangle$ .

#### 4.2.1 Assuming the double negation law

If the double negation law holds true in  $\mathbf{L}$ , each pair of the concept-forming operators we have been using so far (namely,  $\langle \uparrow, \downarrow \rangle$ ,  $\langle \wedge, \vee \rangle$ , and  $\langle \wedge, \vee \rangle$ ) can define the other two.

As a consequence, for instance, we have that  $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$  and  $\mathcal{B}^{\text{ou}}(X, Y, \neg I)$  are isomorphic as lattices with  $\langle A, B \rangle \mapsto \langle A, \neg B \rangle$  being the isomorphism.

In order to prove this, note that  $A \in \text{Ext}^{\uparrow\downarrow}(X, Y, I)$  iff  $A = A^{\uparrow I \downarrow I}$  and that  $A \in \text{Ext}^{\text{ou}}(X, Y, \neg I)$  iff  $A = A^{\wedge \neg I \vee \neg I}$ . We have

$$\begin{aligned}
A^{\wedge \neg I \vee \neg I} &= \neg I \triangleleft (A \circ \neg I) \\
&= \neg I \triangleleft (\neg \text{Id} \triangleright ((A \circ \neg I) \triangleleft \neg \text{Id})) \\
&= \neg I \triangleleft (\neg \text{Id} \triangleright (A \triangleleft (\neg I \triangleleft \neg \text{Id}))) \\
&= \neg I \triangleleft (\neg \text{Id} \triangleright (A \triangleleft I)) \\
&= (\neg I \triangleleft \neg \text{Id}) \triangleright (A \triangleleft I) \\
&= (I \triangleright (A \triangleleft I)) = A^{\uparrow I \downarrow I}.
\end{aligned}$$

That shows that

$$\text{Ext}^{\uparrow\downarrow}(X, Y, I) = \text{Ext}^{\cup}(X, Y, \neg I) \quad (28)$$

As the ordering between the extents is defined to be the fuzzy subsethood ordering (which is independent from the concept-forming pair used to build the concept lattice), one obtain that both lattices are isomorphic.

To justify the intent part of the isomorphism, note that for each  $A \in \mathbf{L}^X, B \in \mathbf{L}^Y$  we have

$$\begin{aligned} \neg B &= \neg(A^{\uparrow\downarrow}) = \neg(A^{\uparrow\neg\neg I}) = \neg(A \triangleleft \neg\neg I) = \neg(A \triangleleft (\neg I \triangleleft \neg\text{Id})) = \\ &= \neg((A \circ \neg I) \triangleleft \neg\text{Id}) = \neg\neg(A \circ \neg I) = (A \circ \neg I) = A^{\wedge\neg I}. \end{aligned}$$

Similarly,  $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$  and  $\mathcal{B}^{\wedge\nu}(X, Y, \neg I)$  are isomorphic as lattices with  $\langle A, B \rangle \mapsto \langle \neg A, B \rangle$  being the isomorphism. The proof follows the line of the previous one, but showing

$$\text{Int}^{\wedge\nu}(X, Y, \neg I) = \text{Int}^{\uparrow\downarrow}(X, Y, I). \quad (29)$$

Using that we can state the following theorem.

**Theorem 4.14:** *Assume that the double negation holds true in  $\mathbf{L}$ . Then homogeneous bonds wrt  $\langle \uparrow, \downarrow \rangle$  from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  are exactly a-bonds from  $\langle X_1, Y_1, \neg I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$ ; and c-bonds from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, \neg I_2 \rangle$ .*

**Proof:** Directly from the definitions and Equations (28) and (29).  $\square$

Note that with the double negation law, the incidence relation  $\Delta$  in  $\triangleleft$ -direct product  $\langle X_1, Y_1, I_1 \rangle \boxtimes \langle X_2, Y_2, I_2 \rangle$  becomes

$$\Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) = \neg I_1(x_1, y_1) \rightarrow I_2(x_2, y_2)$$

and the incidence relation  $\Delta$  in direct  $\triangleright$ -product  $\mathbb{K}_1 \boxtimes \mathbb{K}_2$  becomes

$$\Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) = \neg I_2(x_2, y_2) \rightarrow I_1(x_1, y_1),$$

which coincides with the direct product (25) from (Křídlo *et al.* 2012).

#### 4.2.2 Using an alternative notion of complement

The mutual reducibility of concept-forming operators (17)–(19) does not hold generally. In (Belohlavek and Konecny 2012a), a new notion of complement of  $\mathbf{L}$ -relation was proposed in order to overcome that. Using this notion we showed that each for each  $I \in L^{X \times Y}$ , and fixed an element  $a \in L$ , one can define  $\neg I \in L^{X \times (Y \times L)}$  as

$$\neg I(x, \langle y, a \rangle) = I(x, y) \rightarrow a,$$

and obtain

$$\text{Ext}^{\uparrow\downarrow}(X, Y \times L, \neg I) = \text{Ext}^{\cup}(X, Y, I), \quad (30)$$

and similarly,

$$\text{Int}^{\uparrow\downarrow}(X \times L, Y, (\neg I^{\text{T}})^{\text{T}}) = \text{Int}^{\wedge\nu}(X, Y, I). \quad (31)$$

That is, for any  $I \in \mathbf{L}^{X \times Y}$  one can find a relation which induces the same structure of extents (resp. intents) wrt  $\langle \downarrow, \uparrow \rangle$  as  $I$  induces wrt  $\langle \cup, \cap \rangle$  (resp. wrt  $\langle \vee, \wedge \rangle$ ). Unfortunately, the converse does not hold true in general; i.e. there are relations  $I \in \mathbf{L}^{X \times Y}$  such that no relation induces the same structure of extents wrt  $\langle \cup, \cap \rangle$  (resp. intents wrt  $\langle \vee, \wedge \rangle$ ) as  $I$  induces wrt  $\langle \downarrow, \uparrow \rangle$ . Only for those  $\mathbf{L}$ -relations  $I \in \mathbf{L}^{X \times Y}$  whose set of extents  $\text{Ext}^{\uparrow\downarrow}(X, Y, I)$  is a  $c$ -closure system (Belohlavek and Konecny 2011); i.e. an  $\mathbf{L}$ -closure system generated by a system of all  $a$ -complements of some  $\mathcal{T} \subseteq \mathbf{L}^X$ .

**Theorem 4.15:** *If  $\text{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)$  is a  $c$ -closure system, the  $i$ -bond wrt  $\langle \uparrow, \downarrow \rangle$  from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  are exactly  $a$ -bonds from  $\langle X_1, Y_1 \times L, \neg I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$ . If  $\text{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$  is a  $c$ -closure system, the  $i$ -bonds wrt  $\langle \uparrow, \downarrow \rangle$  from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2, Y_2, I_2 \rangle$  are exactly  $c$ -bonds from  $\langle X_1, Y_1, I_1 \rangle$  to  $\langle X_2 \times L, Y_2, (\neg I_2^T)^T \rangle$ .*

**Proof:** Directly from Definitions and (28) and (29).  $\square$

## 5. Conclusions

Continuing with our study of generalized forms of formal concept analysis, we have focused on the different natural extensions of the notion of bond.

To the best of our knowledge, only (Křídlo *et al.* 2012) had introduced a generalized definition of bond, but it turns out that, in a generalized framework, several alternatives can be considered, depending essentially on the pair(s) of concept-forming operators one relies on. In this paper, we have introduced the notion of homogeneous  $\mathbf{L}$ -bond, namely, a generalized bond wrt a pair of isotone concept-forming operators, and presented a thorough study of them.

Specifically, homogeneous bonds with respect to  $\langle \cap, \cup \rangle$  (resp.  $\langle \wedge, \vee \rangle$ ) have been proved to be in one-to-one correspondence with  $c$ -morphisms from extents (resp. intents) of the corresponding concept lattices. Moreover, the set of all homogeneous bonds (of either case) is proved to form an  $\mathbf{L}$ -interior system. The natural notion of homogeneous bond wrt the two pairs of isotone concept-forming operators simultaneously (strong homogeneous bond) is proved to be in one-to-one correspondence with  $i$ -morphisms between intents of  $\langle \cap, \cup \rangle$  and also with  $i$ -morphisms between extents of  $\langle \wedge, \vee \rangle$ . Obviously, the set of all strong homogeneous bonds is an  $\mathbf{L}$ -interior system. The study is concluded by presenting the existing relationship with the direct  $\circ$ -product of contexts.

A different notion of bond arises when one allows the interaction of isotone and antitone concept-forming operators, and this leads to the so-called heterogeneous bonds, which are proved to be closely related to the  $a$ -morphisms and  $c$ -morphisms. Specific types of product were needed in order to establish the connection between these new types of bonds with the direct product of contexts. It is worth to remark that one can see applied papers in the area of information retrieval, see for instance (Valverde-Albacete 2006), which directly calls for heterogeneous bonds, specifically for some  $a$ -,  $c$ - and  $i$ -morphisms.

The obtained results shed new light on the structure and properties of generalized versions of bond between contexts: on the one hand, the results are abstract versions of those already known in the classical case and, on the other hand, generalize as well those published in (Křídlo *et al.* 2012).

As future work, on the one hand, it seems worth to consider a further generalization in terms of complete idempotent semifields, which satisfy all the properties of residuated lattices except that the multiplicative unit need not be the top element of the lattice. In this new framework, it makes sense to consider the fourth pair of



concept-forming operators not considered in the present work, which can be viewed as a double dualization on the first pair, see (Valverde-Albacete and Peláez-Moreno 2011) for these connections defined over completed idempotent semifields.

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