A flexible logic-based approach to closeness using order of magnitude qualitative reasoning *

Alfredo Burrieza, Emilio Muñoz-Velasco, Manuel Ojeda-Aciego

[1] Dept. Filosofía. Universidad de Málaga. Spain
burrieza@uma.es
[2] Dept. Matemática Aplicada. Universidad de Málaga. Spain
{ejmunoz,aciego}@uma.es

Abstract. In this paper, we focus on a logical approach to the important notion of closeness, which has not received much attention in the literature. Our notion of closeness is based on the so-called *proximity intervals*, which will be used to decide the elements that are close to each other. Some of the intuitions of this definition are explained on the basis of examples. We prove the decidability of the recently introduced multimodal logic for closeness and, then, we show some capabilities of the logic with respect to expressivity in order to denote particular positions of the proximity intervals.

1 Introduction

Qualitative reasoning (QR) is very useful for searching solutions to problems about the behavior of physical systems without using exact numerical data. This way, it is possible to reason on incomplete knowledge by providing an abstraction of the numerical values in order to be able to solve problems that cannot be dealt with using just a quantitative approach. QR has many applications in AI such as Robot Kinematics [12], and dealing with movements [13, 14]. Concerning logics for QR, some papers have been focused on Spatio-Temporal Reasoning [4]; more recently, we can find proposals of logics to deal with movement, for instance [14].

Another interesting approach to QR is to reason with orders of magnitude [15], in which the management of exact values is substituted by reasoning on qualitative classes and relations among them. There are some multimodal logics for order of magnitude reasoning dealing with the relations of negligibility and comparability, see for instance [7, 10]; as far as we know, the only published references on the notion of closeness in a logic-based context are [3, 5, 6].

In [5], the notions of closeness and distance are treated using Propositional Dynamic Logic, and their definitions are based on the concept of qualitative sum; specifically, two values are assumed to be close if one of them can be obtained from the other by adding a small number, and small numbers are defined as those belonging to a fixed interval. This approach has a number of potential applications but might not be so useful in other situations, for instance, when

* Partially supported by the Spanish research project TIN15-70266-C2-P-1

considering physical spaces where there are natural or artificial barriers: you can be very close to a place, but if they are separated by a river, this place is not really so close in terms of time consumed and distance traveled (since we should look for a bridge to cross the river); a similar situation arises for a robot moving in a house between two points separated by a wall. Similarly, one can consider time barriers, such as a deadline to submit an article: if the deadline is, for instance, May 31, the date May 30 can be considered close to the deadline, from the author's point of view, but Jun 1 is not so, because it is already over.

On the other hand, in [6] a multi-modal logic approach was considered to deal with closeness and negligibility, in which the notion of closeness stems from the fact that two values were considered to be *close* if they are within a prescribed area or *proximity interval*; in some sense, this approach resembles the notion of granularity as given in [8]. This idea is useful to deal with the situations described in the previous paragraph since the set of proximity intervals can be established according to the existence of the barriers. This approach is purely crisp and does not use fuzzy set-related ideas as in [1], where a fuzzy set-based approach for handling relative orders of magnitude was introduced.

We continue the study of the multimodal logic for order of magnitude reasoning introduced in [6], where an axiom system was included together with the proof of soundness and completeness. This work is two-fold: from a theoretical standpoint, we prove the decidability of the logic and, on the practical side, we elaborate on the capability of the logic in order to express different properties of the closeness relation in terms of the main technical novelty of this logic, namely, proximity intervals represented by using finitely many constants. This approach can be seen as hybrid in many ways: on the one hand, it considers jointly multimodal logics and order-of-magnitude qualitative reasoning, in the line of [11]; on the other hand, it considers both the absolute and the relative approaches to order-of-magnitude, in which the former is a purely qualitative approach based on abstractions of quantitative values, whereas the latter is based on relations between quantitative values. Moreover, it can also be seen as hybrid in the sense of hybrid logics, [2], because of the use of nominals.

2 On the notions of closeness and neglibility

We will consider a strictly ordered set of real numbers $(\mathbb{S}, <)$ divided into the following qualitative classes:

$$\begin{array}{lll} \textsc{nl} = (-\infty, -\gamma) & & \textsc{ps} = (+\alpha, +\beta] \\ \textsc{nm} = [-\gamma, -\beta) & \textsc{inf} = [-\alpha, +\alpha] & \textsc{pm} = (+\beta, +\gamma] \\ \textsc{ns} = [-\beta, -\alpha) & & \textsc{pl} = (+\gamma, +\infty) \end{array}$$

The elements $-\alpha, +\alpha, -\beta, +\beta, -\gamma, +\gamma$ will be called *milestones*. Note that all the intervals are considered relative to $\mathbb{S}$.

The labels correspond to "negative large" (NL), "negative medium"(NM), "negative small"(NS), "infinitesimals"(INF), "positive small" (PS), "positive medium" (PM) and "positive large" (PL). Note that this classification is slightly

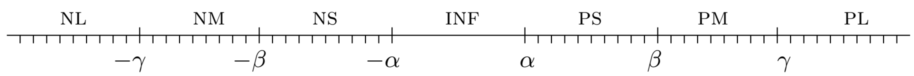

Fig. 1. Proximity intervals.

more general than the standard one [15], since the qualitative class containing the element 0, i.e. INF, need not be a singleton; this allows for considering values very close to zero as null values in practice, which is more in line with a qualitative approach where accurate measurements are not always possible.

Let us now introduce the notion of closeness. As stated in the introduction, the intuitive idea underlying our notion of closeness is that, in real life problems, there are situations in which we consciously choose not to distinguish between *certain* pairs of elements (for instance, two cars priced 19 000 € and 18 000 € might be both acceptable, but perhaps 20 000 € is considered too expensive for our budget). Somehow, there exist some areas of indistinguishability so that x is said to be close to y if and only if both x and y belong to the same area (although, in the example, 18 000 and 20 000 are equidistant to 19 000, the psychological perception[3] is that 20 000 might be too expensive and, therefore, it is not considered close to 19 000).

We will consider each qualitative class to be divided into disjoint intervals called *proximity intervals*, as shown in Figure 1. The qualitative class INF is itself one proximity interval.

Definition 1. *Let $(\mathbb{S}, <)$ be a strictly linear ordering divided into the qualitative classes defined above.*

- *An* (r-)proximity structure *is a finite set of* proximity intervals *in $\mathbb{S}$ of cardinal r, $\mathcal{I}(\mathbb{S}) = \{I_1, I_2, \ldots, I_r\}$, satisfying that:*
  1. *For all $I_i, I_j \in \mathcal{I}(\mathbb{S})$, if $i \neq j$, then $I_i \cap I_j = \varnothing$.*
  2. *$I_1 \cup I_2 \cup \cdots \cup I_r = \mathbb{S}$.*
  3. *For all $x, y \in \mathbb{S}$ and $I_i \in \mathcal{I}(\mathbb{S})$, if $x, y \in I_i$, then x, y belong to the same qualitative class.*
  4. INF $\in \mathcal{I}(\mathbb{S})$.
- *Given a proximity structure $\mathcal{I}(\mathbb{S})$, the binary relation of closeness $\mathfrak{c}$ is defined, for all $x, y \in \mathbb{S}$, as follows: $x \,\mathfrak{c}\, y$ if and only if there exists $I_i \in \mathcal{I}(\mathbb{S})$ such that $x, y \in I_i$.*

Notice that, as a consequence of item 3 above, each proximity interval is included in some qualitative class; this feature will be used later.

It is also worth to notice that, by definition, the number of proximity intervals is finite, regardless of the cardinality of the set $\mathbb{S}$. This choice is justified by the nature of the measuring devices that after reaching a certain limit, they do not

[3] This is a well-known effect in marketing.

distinguish among nearly equal amounts; for instance, consider the limits to represent numbers in a pocket calculator, thermometer, speedometer, etc.

As a result of considering just finitely many proximity intervals, it can be the case that there exist two elements whose magnitudes are not comparable but, according to this approach, turn out to be comparable. In everyday life, we often face similar situations where excessively large quantities are no longer considered to have an appreciable difference. For instance, if the limit of users simultaneously connected to a server is, say, 1 000 000 users, it is clear that the response would be the same whether 10 000 000 users or 100 000 000 are connected to the server. In this case, although these quantities may not be comparable in absolute terms, they turn out to be comparable from the point of view of the response of the server. Nevertheless, we need these quantities to be not comparable, we have just to change the choice of the qualitative classes in our approach.

From now on, we will denote by $\mathcal{Q} = \{\text{NL}, \text{NM}, \text{NS}, \text{INF}, \text{PS}, \text{PM}, \text{PL}\}$ the set of qualitative classes, and we will use QC to refer to any element of $\mathcal{Q}$.

The following proposition is an immediate consequence of the definition of the closeness relation.

Proposition 1. *The relation* $\mathfrak{c}$ *defined above has the following properties:*

1. $\mathfrak{c}$ *is an equivalence relation on* $\mathbb{S}$.
2. *For all* $x, y, z \in \mathbb{S}$*, the following holds:*
   (a) *If* $x, y \in \text{INF}$*, then* $x \,\mathfrak{c}\, y$.
   (b) *For every* $\text{QC} \in \mathcal{Q}$*, if* $x \in \text{QC}$ *and* $x \,\mathfrak{c}\, y$*, then* $y \in \text{QC}$.

The informal notion of negligibility we will use in this paper is the following: x is said to be *negligible* with respect to y if and only if either (i) x is infinitesimal and y is not, or (ii) x is small (but not infinitesimal) and y is *sufficiently large*.

Definition 2. *Let* $(\mathbb{S}, <)$ *be a strictly linear divided into the qualitative classes defined above. The binary relation of negligibility* $\mathfrak{n}$ *is defined on* $\mathbb{S}$ *as* $x \,\mathfrak{n}\, y$ *if and only if one of the following situations holds:*

(i) $x \in \text{INF}$ *and* $y \notin \text{INF}$,
(ii) $x \in \text{NS} \cup \text{PS}$ *and* $y \in \text{NL} \cup \text{PL}$.

The following result states some interesting properties about the interaction between the relations of closeness and negligibility.

Proposition 2. *For all* $x, y, z \in \mathbb{S}$ *we have:*

(i) *If* $x \,\mathfrak{c}\, y$ *and* $y \,\mathfrak{n}\, z$*, then* $x \,\mathfrak{n}\, z$.
(ii) *If* $x \,\mathfrak{n}\, y$ *and* $y \,\mathfrak{c}\, z$*, then* $x \,\mathfrak{n}\, z$.

Proof. (i). Let $x \,\mathfrak{c}\, y$ and $y \,\mathfrak{n}\, z$. Then, by Definition 2, we have two possibilities:

- $y \in \text{INF}$ and $z \notin \text{INF}$. So, as $x \,\mathfrak{c}\, y$, by Proposition 1(2b), we have $x \in \text{INF}$ and as $z \notin \text{INF}$, we obtain $x \,\mathfrak{n}\, z$.
- $y \in \text{NS} \cup \text{PS}$ and $z \in \text{NL} \cup \text{PL}$. So, as $x \,\mathfrak{c}\, y$, by Proposition 1(2b), we have $x \in \text{NS} \cup \text{PS}$, then $x \,\mathfrak{n}\, z$ again.

The proof of item (ii) is analogous. QED

3 Syntax and semantics

In this section, we define a logic for multimodal qualitative reasoning based on proximity intervals, whose language will be denoted by $\mathcal{L}(MQ)^{\mathcal{P}}$. We will use the modal connectives $\overrightarrow{\Box}$ and $\overleftarrow{\Box}$ to deal with the usual ordering $<$, so $\overrightarrow{\Box}A$ and $\overleftarrow{\Box}A$ have the informal readings A *is true for all numbers greater than the current one* and A *is true for all numbers less than the current one*, respectively. We will also use $\boxed{c}$ for closeness, where the informal reading of $\boxed{c}A$ is A *is true for all numbers close to the current one*, and $\boxed{n}$ for negligibility, where $\boxed{n}A$ means A *is true for all numbers with respect to the current one is negligible.*

The alphabet of the language $\mathcal{L}(MQ)^{\mathcal{P}}$ is defined by using a stock of atoms or propositional variables, $\mathcal{V}$, the classical connectives $\neg, \wedge, \vee$ and $\rightarrow$; the constants for milestones $\alpha^-, \alpha^+, \beta^-, \beta^+, \gamma^-, \gamma^+$; a finite set $\mathcal{C}$ of constants for proximity intervals, $\mathcal{C} = \{c_1, \ldots, c_r\}$ [4]; the unary modal connectives $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\boxed{n}$, $\boxed{c}$, and the parentheses '(' and ')'. We define the formulas of $\mathcal{L}(MQ)^{\mathcal{P}}$ as follows:

$$A = p \mid \xi \mid c_i \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid \overrightarrow{\Box}A \mid \overleftarrow{\Box}A \mid \boxed{n}A \mid \boxed{c}A$$

where $p \in \mathcal{V}$, $\xi \in \{\alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^+, \gamma^-\}$ and $c_i \in \mathcal{C}$.

The *mirror image* of a formula A is the result of replacing in A each occurrence of $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, α^+, β^+ and γ^+, respectively, by $\overleftarrow{\Box}$, $\overrightarrow{\Box}$, α^-, β^- and γ^- and reciprocally. We will use the symbols $\overrightarrow{\Diamond}, \overleftarrow{\Diamond}, \Diamond_c, \Diamond_n$ as abbreviations, respectively, of $\neg\overrightarrow{\Box}\neg$, $\neg\overleftarrow{\Box}\neg$, $\neg\boxed{c}\neg$ and $\neg\boxed{n}\neg$. Moreover, we will introduce nl, ..., pl as abbreviations for qualitative classes, for instance, ps for $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}\beta^+) \vee \beta^+$.

The cardinality r of the set $\mathcal{C}$ of constants for proximity intervals will play an important role since it, somehow, encodes the granularity of the underlying logic. This implies that, actually, *we are introducing a family of logics which depend parametrically on* r.

Definition 3. *A* multimodal qualitative frame *for* $\mathcal{L}(MQ)^{\mathcal{P}}$ *(a* frame*, for short) is a tuple* $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$*, where:*

1. $(\mathbb{S}, <)$ *is a strict linearly ordered set.*
2. $\mathcal{D} = \{+\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$ *is a set of milestones in* $\mathbb{S}$.
3. $\mathcal{I}(\mathbb{S})$ *is an r-proximity structure.*
4. $\mathcal{P}$ *is a bijection (called* proximity function*),* $\mathcal{P}\colon \mathcal{C} \longrightarrow \mathcal{I}(\mathbb{S})$*, that assigns to each proximity constant c a proximity interval.*

Notice that item 4 above means that every proximity interval corresponds to one and only one proximity constant.

Definition 4. *Let* Σ *be a frame for* $\mathcal{L}(MQ)^{\mathcal{P}}$*, a* multimodal qualitative model on Σ *(an MQ-*model*, for short) is an ordered pair* $\mathcal{M} = (\Sigma, h)$*, where h is a* meaning function *(or,* interpretation*)* $h\colon \mathcal{V} \longrightarrow 2^{\mathbb{S}}$*. Any interpretation can be uniquely extended to the set of all formulas in* $\mathcal{L}(MQ)^{\mathcal{P}}$ *(also denoted by h)*

[4] There are at least as many elements in $\mathcal{C}$ as qualitative classes.

by means of the usual conditions for the classical Boolean connectives and the following conditions:

$$h(\overrightarrow{\Box}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x < y\}$$
$$h(\overleftarrow{\Box}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } y < x\}$$
$$h(\boxed{c}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x \,\mathfrak{c}\, y\}$$
$$h(\boxed{n}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x \,\mathfrak{n}\, y\}$$
$$h(\alpha^+) = \{+\alpha\} \qquad h(\beta^+) = \{+\beta\} \qquad h(\gamma^+) = \{+\gamma\}$$
$$h(\alpha^-) = \{-\alpha\} \qquad h(\beta^-) = \{-\beta\} \qquad h(\gamma^-) = \{-\gamma\}$$
$$h(c_i) = \{x \in \mathbb{S} \mid x \in \mathcal{P}(c_i)\}$$

The definitions of *truth*, *satisfiability* and *validity* are the usual ones.

Example 1. The aim of this example is to specify in $\mathcal{L}(MQ)^{\mathcal{P}}$ the behavior of a device to automatically control the speed of a car. Assume the system has, ideally, to maintain the speed close to some speed limit v. For practical purposes, any value in an interval $[v - \varepsilon, v + \varepsilon]$ for small ε is admissible. The extreme points of this interval can then be considered as the milestones $-\alpha$ and $+\alpha$ of our frames; on the other hand, we will consider different levels of velocity in a qualitative approach ranging from *very slow* to *very fast*. We will introduce consequently the atoms $\mathtt{v}_{-3}, \mathtt{v}_{-2}, \mathtt{v}_{-1}, \mathtt{v}_0, \mathtt{v}_1, \mathtt{v}_2, \mathtt{v}_3$ associated to them (which are interpreted, respectively, as the qualitative classes NL, NM, NS, INF, PS, PM, PL and, moreover, $\mathtt{v}_0$ represents the interval $[v - \varepsilon, v + \varepsilon]$).

We will introduce also the atoms `accelerate`, `maintain`, `release` and `brake` to describe actions of the system with their intuitive meaning.

Now we represent how the system works:

1. Whenever the speed is below the intended limit, then the engine is accelerated, whereas when the speed is within the admissible limits, the speed is maintained. Thus, we have the two formulas below

$$(\mathtt{v}_{-3} \vee \mathtt{v}_{-2} \vee \mathtt{v}_{-1}) \to \mathtt{accelerate} \qquad\qquad \mathtt{v}_0 \to \mathtt{maintain}$$

2. It can happen that the speed increases more than the limit allowed due to external factors, for instance when the road has negative slope, this way some rules are required to maintain the speed. Usually, when the car reaches the speed limit, the driver does not brake immediately but releases the accelerator instead, so that the air friction helps to recover an admissible speed. We accomplish this action precisely in the proximity interval immediately after $\mathtt{v}_0$, which we will call, say, c. As a result, we have the two formulas

$$c \to (\overleftarrow{\Diamond}\alpha^+ \wedge \overleftarrow{\Box}(\overleftarrow{\Diamond}\alpha^+ \to c)) \qquad\qquad c \to \mathtt{release}$$

3. When we are beyond the limit imposed by the interval c, then the system has to *actively* brake:

$$(\neg c \wedge \overleftarrow{\Diamond} c) \to \mathtt{brake}$$

According to the intended meaning of the previous formulas, the atoms `accelerate`, `maintain`, `release`, and `brake` are true, respectively, at

$$\mathrm{NL}\cup\mathrm{NM}\cup\mathrm{NS} \qquad \mathrm{INF} \qquad I_c \qquad (\mathrm{PS}\setminus I_c)\cup\mathrm{PM}\cup\mathrm{PL}$$

where I_c is the proximity interval represented by c. Note that the length of this interval depends on the granularity of the system.

Some consequences of the behavior of the system (specifically, valid formulas in the model) are the following:

$\mathtt{brake} \to \overrightarrow{\Box}\mathtt{brake}$
(If the system brakes at a specific speed, then it brakes at higher speeds)

$\mathtt{release} \to \boxed{\scriptstyle c}(\mathtt{v}_1 \wedge \neg\mathtt{brake})$
(If the throttle is released at certain speed, then any small variation implies that the speed is still slightly fast and the system does not brake)

$\mathtt{release} \to \boxed{\scriptstyle n}(\mathtt{v}_{-3} \to \mathtt{accelerate})$
(If the throttle is released at certain speed and, by any circumstances, the speed decreases excessively, then it has to accelerate again)

$\mathtt{accelerate} \to \boxed{\scriptstyle c}\neg\mathtt{brake}$
(If the system accelerates, it will not brake immediately)

$\overrightarrow{\Box}(\mathtt{v}_2 \to \overleftarrow{\Diamond}\mathtt{release})$
(The throttle will be released before reaching a fast speed)

4 On the expressivity of $\mathcal{L}(MQ)^{\mathcal{P}}$ wrt proximity intervals

In this section, we consider the question whether it is possible to express the positions, with respect to the underlying ordering, of the proximity intervals within a qualitative class by means of formulas of $\mathcal{L}(MQ)^{\mathcal{P}}$. Specifically, we consider the existence of formulas which are true, just in a given proximity interval. For instance, the following formula

$$\bigvee_{i=1}^{r} \big(c_i \wedge \overleftarrow{\Diamond}\alpha^+ \wedge \overleftarrow{\Box}(\overleftarrow{\Diamond}\alpha^+ \to c_i)\big) \qquad (1)$$

expresses the first proximity interval after $+\alpha$, in the sense that it is true just in the points belonging to that interval. The intuitive meaning of each disjunct of formula (1), is that there are not other proximity intervals after $+\alpha$ and before the current one. In order to prove this, assume a model $\mathcal{M} = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P}, h)$ and let K be the first proximity interval after $+\alpha$ in $\mathcal{M}$. Such an interval exists, since PS contains at least one proximity interval. Let us prove that the formula (1) is true in a point x (with respect to the model $\mathcal{M}$) if and only if $x \in K$. Assume that c_k, for some $k \in \{1, \ldots, r\}$, is assigned by the function $\mathcal{P}$ to the interval K, that is, $\mathcal{P}(c_k) = K$ (see Definition 3).

Given $x \in K$, by definition of model, we have that $x \in h(c_k)$. Moreover, since $+\alpha < x$ and $+\alpha \in h(\alpha^+)$, then $x \in h(\overleftarrow{\Diamond}\alpha^+)$. For all y smaller than x, we trivially have $y \in h(\overleftarrow{\Diamond}\alpha^+ \to c_k)$, then $x \in h(\overleftarrow{\Box}(\overleftarrow{\Diamond}\alpha^+ \to c_k))$. Thus, we have

$$x \in h(c_k \wedge \overleftarrow{\Diamond}\alpha^+ \wedge \overleftarrow{\Box}(\overleftarrow{\Diamond}\alpha^+ \to c_k))$$

and, hence, the formula (1) is true in x.

Conversely, assume $x \notin K$. By Definition 1, we have that x belongs to some proximity interval $J \neq K$, and assume that $\mathcal{P}(c_j) = J$. In these conditions, the only disjunct of formula (1) that could be true at x would be the following

$$c_j \wedge \overleftarrow{\Diamond}\alpha^+ \wedge \overleftarrow{\Box}(\overleftarrow{\Diamond}\alpha^+ \to c_j)$$

Now, there are two possibilities:

- If x is to the left of K, then clearly $x \notin h(\overleftarrow{\Diamond}\alpha^+)$
- If x is to the right of K, then $x \notin h(\overleftarrow{\Box}(\overleftarrow{\Diamond}\alpha^+ \to c_j))$, because $\overleftarrow{\Diamond}\alpha^+ \to c_j$ is not true at the points of K.

either of which possibilities above proves that formula (1) is not true in x.

Moreover, the second interval after $+\alpha$ can be expressed as:

$$\bigvee_{i=1}^{r}\Big(c_i \wedge \bigvee_{j=1}^{r}(\neg c_j \wedge \overleftarrow{\Box}((\neg c_j \wedge \overleftarrow{\Diamond}c_j) \to c_i) \wedge \overleftarrow{\Diamond}L_j^1)\Big) \tag{2}$$

where $L_j^1 = c_j \wedge \overleftarrow{\Box}(\overleftarrow{\Diamond}\alpha^+ \to c_j) \wedge \overleftarrow{\Diamond}\alpha^+$. In this case, the intuitive meaning of formula (2) is as follows: the second proximity interval after $+\alpha$ is, obviously, some c_i which is after the first one (denoted by some c_j and represented by L_j^1), and there are no other proximity intervals between them.

In order to express the n-th proximity interval after $+\alpha$, it will be convenient to consider the following generalization of L_i^1:

$$\begin{aligned}
L_i^1 &:= c_i \wedge \overleftarrow{\Diamond}\alpha^+ \wedge \overleftarrow{\Box}(\overleftarrow{\Diamond}\alpha^+ \to c_i)\\
L_i^2 &:= c_i \wedge \bigvee_{j=1}^{r}(X_{i,j} \wedge \overleftarrow{\Diamond}L_j^1)\\
&\vdots \qquad\qquad \vdots\\
L_i^n &:= c_i \wedge \bigvee_{j=1}^{r}(X_{i,j} \wedge \overleftarrow{\Diamond}L_j^n)
\end{aligned}$$

where $X_{i,j} := \neg c_j \wedge \overleftarrow{\Box}((\neg c_j \wedge \overleftarrow{\Diamond}c_j) \to c_i)$, for any $i, j \geq r$, is read as "c_i represents the first proximity interval after c_j."

It is not difficult to check that L_i^k states that c_i denotes the k-th proximity interval after $+\alpha$. Therefore, and similarly to the formulas (1) and (2), in order to represent the n-th interval after $+\alpha$, it is sufficient to consider the disjunction of the formulas L_i^n, namely

$$\bigvee_{i=1}^{r} (c_i \wedge \bigvee_{j=1}^{r} (X_{i,j} \wedge \overleftarrow{\Diamond} L_j^n)) \tag{3}$$

Note that the previous construction has been given for the landmark $+\alpha$ just for the sake of an example but, in fact, can be used to represent any interval after either a given landmark or proximity constant.

Focusing on the hybrid features of our logic, in the following construction we consider simultaneously both the proximity intervals and the qualitative classes. In the following, we introduce the formulas which allow to express the granularity of a given qualitative class, i.e., the number of proximity intervals it contains. For the particular case of PS $= (+\alpha, +\beta]$, we consider the following formula

$$L_i^k \wedge (\beta^+ \vee \overrightarrow{\Diamond}\beta^+) \wedge \overrightarrow{\Box}((\beta^+ \vee \overrightarrow{\Diamond}\beta^+) \to c_i)$$

which can be read as c_i is the k-th proximity interval after $+\alpha$ (this is L_i^k) and the rest of the formula expresses that it is the last interval before $+\beta$; therefore, its meaning is that there are exactly k intervals in the qualitative class PS.

Example 2. Continuing with the previous example, we will introduce two new atoms representing modified versions of the braking action, namely, `gentle-brake` and `hard-brake`. The possibility to detect how many proximity intervals have been passed after the speed limit determines whether the system will choose either a gentle or hard brake. We will use the notation c_{PS}^n to denote the n-th proximity interval in PS, as introduced in formula (3).

Now we can provide a more detailed response to the situation in which the speed increases more than the limit allowed in terms of the position of the proximity interval detected after the speed limit:

1. If we are over the speed limit but only slightly (and this is interpreted as that we are in the first proximity interval after the speed limit) we simply release the throttle. As a result, we have the formula: $c_{\text{PS}}^1 \to$ `release`.
2. When we are immediately beyond the limit imposed by the first proximity interval in PS, then the system has to *actively* brake: $c_{\text{PS}}^2 \to$ `gentle-brake`.
3. When we exceed the second proximity interval after the speed limit, then we must brake hard. Namely, $(\neg c_{\text{PS}}^2 \wedge \overleftarrow{\Diamond} c_{\text{PS}}^2) \to$ `hard-brake`.

5 Decidability

In order to prove the decidability, we show the *strong finite model property*, following the strategy used in [9]. For this, we will work with a weaker class of models than those stated in Definition 4. This is justified by the fact that *MQ*-models do not serve our purposes in order to prove the strong finite model property; this is because there are formulas which are satisfiable just in infinite MQ-models due to the fact that MQ-models are strict linear orders. The definition of the MQC-models is a generalization of that of MQ-models in which the irreflexivity is restricted just to the milestones.

Definition 5. *A* multimodal qualitative cluster frame *for $\mathcal{L}(MQ)^{\mathcal{P}}$ (or simply an MQC-frame) is a tuple $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{K}(\mathbb{S}), \mathcal{P})$, where:*

1. *$\mathbb{S}$ is a set containing a subset $\mathcal{D} = \{+\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$ of designated elements (milestones).*[5]
2. *$<$ is a binary relation on $\mathbb{S}$ which is transitive and connected. Moreover, $\xi \not< \xi$ for the milestones $\xi \in \mathcal{D}$ and $-\gamma < -\beta < -\alpha < +\alpha < +\beta < +\gamma$.*
3. *$\mathcal{K}(\mathbb{S}) = \{K_1, K_2, \ldots, K_n\}$ is a partition of $\mathbb{S}$ such that:*
   (a) For all $x, y \in \mathbb{S}$ and $K_i \in \mathcal{K}(\mathbb{S})$, if $x, y \in K_i$, then x, y belong to the same qualitative class defined by the milestones.
   (b) $\text{INF} \in \mathcal{K}(\mathbb{S})$.
4. *$\mathcal{P} \colon \mathcal{C} \longrightarrow \mathcal{K}(\mathbb{S})$ is a bijection.*

Given an MQC-frame Σ, an MQC-model on Σ *is an ordered pair $\mathcal{M} = (\Sigma, h)$, where h is a* meaning function *defined as in Definition 4.*

The concepts of satisfiability, truth and validity of a formula in a MQC-model are defined as usual.

The cornerstone of this section is that validity wrt MQC-models is equivalent to validity wrt MQ-models. Before including the proof, let us recall that an axiom system, denoted $MQ^{\mathcal{P}}$, was introduced for our logic, together with a proof of the completeness theorem (see [6]).

Proposition 3. *For every formula A of $\mathcal{L}(MQ)^{\mathcal{P}}$, it holds that A is MQC-valid if and only if A is MQ-valid.*

Proof. If A is MQC-valid, then it is MQ-valid, since every MQ-model is an MQC-model. If A is MQ-valid, then A is a theorem of $MQ^{\mathcal{P}}$, by the completeness theorem of [6]. If A is a theorem of $MQ^{\mathcal{P}}$, then A is MQC-valid. QED

We will use the well-known *filtration method*, showing that each formula which is satisfiable in an MQC-model is satisfiable also in a *finite* MQC-model with bounded size. In order to obtain this finite model, we will define an equivalence relation in the original (non-necessarily finite) model. Due to the fact that our logic contains *finitely* many constants and milestones, this equivalence relation will be based on the set of subformulas of a suitable modification A^* of the formula A. Specifically, given a formula A written only in terms of the primitive operators we define

$$A^* =_{def} A \wedge \bigvee_{c_i \in \mathcal{C}} c_i \wedge \bigwedge_{\xi \in \mathcal{D}} (\xi \rightarrow \overrightarrow{\Box} \neg \xi)$$

[5] Note that these milestones induce the qualitative classes as usual.

In what follows, we denote by Γ the set of subformulas of A^*. Given any MQC-model $\mathcal{M} = (\mathbb{S}, \mathcal{D}, <, \mathcal{K}(\mathbb{S}), \mathcal{P}, h)$ of A^* and $x, y \in \mathbb{S}$, we define $x \sim_\Gamma y$ iff $\{B \in \Gamma \mid x \in h(B)\} = \{B \in \Gamma \mid y \in h(B)\}$. Clearly $\sim_\Gamma$ is an equivalence relation on $\mathbb{S}$. So, for every $x \in \mathbb{S}$, we denote $[x] = \{y \in \mathbb{S} \mid y \sim_\Gamma x\}$.

Definition 6. *Given A^*, Γ, and $\sim_\Gamma$ as defined above, and given an MQC-model $\mathcal{M} = (\mathbb{S}, \mathcal{D}, <, \mathcal{K}(\mathbb{S}), \mathcal{P}, h)$ of A^*, the* Γ-filtration of $\mathcal{M}$ *is a structure of the form $\mathcal{M}_\Gamma = (\mathbb{S}_\Gamma, \mathcal{D}_\Gamma, <_\Gamma, \mathcal{K}(\mathbb{S})_\Gamma, \mathcal{P}_\Gamma, h_\Gamma)$, where:*

1. $\mathbb{S}_\Gamma = \{[x] \mid x \in \mathbb{S}\}$.
2. $\mathcal{D}_\Gamma = \{[+\alpha], [+\beta], [+\gamma], [-\alpha], [-\beta], [-\gamma]\}$.
3. $\mathcal{K}(\mathbb{S})_\Gamma = \{K_\Gamma \mid K \in \mathcal{K}(\mathbb{S})\}$, *where K_Γ denotes the set $\{[x] \in \mathbb{S}_\Gamma \mid x \in K\}$.*
4. $\mathcal{P}_\Gamma(c_i) = \{[x] \mid x \in \mathcal{P}(c_i)\}$
5. $<_\Gamma \subseteq \mathbb{S}_\Gamma \times \mathbb{S}_\Gamma$, *so that for every $[x], [y] \in \mathbb{S}_\Gamma$ we have $[x] <_\Gamma [y]$ iff:*
   - *for every $\overrightarrow{\Box} A \in \Gamma$: if $x \in h(\overrightarrow{\Box} A)$, then $y \in h(A) \cap h(\overrightarrow{\Box} A)$;*
   - *for every $\overleftarrow{\Box} A \in \Gamma$: if $y \in h(\overleftarrow{\Box} A)$, then $x \in h(A) \cap h(\overleftarrow{\Box} A)$.*
6. $h_\Gamma(p) = \{[x] \mid x \in h(p)\}$, *for every atom $p \in \Gamma$ (if $p \notin \Gamma$, $h_\Gamma(p) = \varnothing$).*
7. $h_\Gamma(\xi) = \{[\xi]\}$.
8. $h_\Gamma(c_i) = \mathcal{P}_\Gamma(c_i)$.

From now on, we follow the standard technique of the filtration methods for decidability. Firstly, we obtain the following results:

Lemma 1. *Given an MQC-model $\mathcal{M}$ of A^*, the* Γ-filtration of $\mathcal{M}$ *has at most 2^n elements in $\mathbb{S}_\Gamma$, where n is the cardinal of Γ.*

Lemma 2. *Let $\mathcal{M}_\Gamma = (\mathbb{S}_\Gamma, \mathcal{D}_\Gamma, <_\Gamma, \mathcal{K}(\mathbb{S})_\Gamma, \mathcal{P}_\Gamma, h_\Gamma)$ be the Γ-filtration of a MQC-model $\mathcal{M} = (\mathbb{S}, \mathcal{D}, <, \mathcal{K}(\mathbb{S}), \mathcal{P}, h)$. Then, $x < y$ implies $[x] <_\Gamma [y]$ for every $x, y \in \mathbb{S}$.*

Lemma 3. *If $\mathcal{M}_\Gamma = (\mathbb{S}_\Gamma, \mathcal{D}_\Gamma, \mathcal{P}_\Gamma, <_\Gamma, h_\Gamma)$ is the Γ-filtration of an MQC-model $\mathcal{M} = (\mathbb{S}, \mathcal{D}, \mathcal{P}, <, h)$, then $\mathcal{M}_\Gamma$ is also an MQC-model.*

Proof. We prove that $<_\Gamma$ is a transitive and connected relation. The transitivity is an immediate consequence of the definition of $<_\Gamma$. On the other hand, connectedness arises from the fact that $<$ is connected and by Lemma 2.

Let us prove that $[\xi] \not<_\Gamma [\xi]$ for all ξ. Assume, by contradiction, that $[\xi] <_\Gamma [\xi]$. Now, given $\xi \in h(\xi)$ and the validity of $\xi \to \overrightarrow{\Box}\neg\xi$, we obtain $\xi \in h(\overrightarrow{\Box}\neg\xi)$. So, taking into account that $\overrightarrow{\Box}\neg\xi \in \Gamma$ (by definition of Γ) and the assumption $[\xi] <_\Gamma [\xi]$, then $\xi \in h(\neg\xi)$, that is, $\xi \notin h(\xi)$, a contradiction.

We also have that $[-\gamma] <_\Gamma [-\beta] <_\Gamma [-\alpha] <_\Gamma [+\alpha] <_\Gamma [+\beta] <_\Gamma [+\gamma]$. This is immediate by the order of milestones in $\mathcal{M}$ and Lemma 2. This enables us to build the sets: $\mathrm{NL}_\Gamma = \{[x] \in \mathbb{S}_\Gamma \mid [x] <_\Gamma [-\gamma]\}$ and the same for the rest of qualitative classes.

Let us now see that $\mathcal{K}(\mathbb{S})_\Gamma = \{K_\Gamma \mid K \in \mathcal{K}(\mathbb{S})\}$ is a partition of $\mathbb{S}$ such that:

1. For all $[x], [y] \in \mathbb{S}_\Gamma$ and $K_\Gamma \in \mathcal{K}(\mathbb{S})_\Gamma$, if $[x], [y] \in K_\Gamma$, then $[x], [y]$ belong to the same qualitative class.

2. $\text{INF}_\Gamma \in \mathcal{K}(\mathbb{S})_\Gamma$.

Let us prove first that $\mathcal{K}(\mathbb{S})_\Gamma$ is a partition of $\mathbb{S}$. Assume $K_\Gamma, K'_\Gamma \in \mathcal{K}(\mathbb{S})_\Gamma$ such that $K_\Gamma \neq K'_\Gamma$, so we have $K \neq K'$. This means by Definition 1 that K and K' are disjoint. Now, if there exists $[x] \in K_\Gamma \cap K'_\Gamma$, then $x \in K \cap K'$, a contradiction. On the other hand, if $[x] \in \mathbb{S}_\Gamma$, then $x \in \mathbb{S}$, and thus there exists $K \in \mathbb{S}$ such that $x \in K$, so $[x] \in K_\Gamma$. Therefore $\mathcal{K}(\mathbb{S})_\Gamma$ is a partition of $\mathbb{S}$.

For 1, assume that $[x], [y] \in K_\Gamma$, then $x, y \in K$ and, as a result, x, y belong to the same qualitative class. Now, we have to consider several cases: for instance, if $x, y \in \text{PS}$, we have $+\alpha < x \leq +\beta$, which means, by Lemma 2, that $[+\alpha] <_\Gamma [x] \leq_\Gamma [+\beta]$, that is, $[x] \in \text{PS}_\Gamma$. Similarly, we obtain $[y] \in \text{PS}_\Gamma$. The other cases can be dealt with in a similar way.

For 2, let us prove that $\text{INF}_\Gamma \in \mathcal{K}(\mathbb{S})_\Gamma$. To this end, we prove that $\{[x] \mid x \in \text{INF}\} = \{[x] \mid [-\alpha] \leq_\Gamma [x] \leq_\Gamma [+\alpha]\}$. Assume $[-\alpha] \leq_\Gamma [x] \leq_\Gamma [+\alpha]$, let us prove that $x \in \text{INF}$. By contradiction, if $x \notin \text{INF}$ then obviously $x \in \text{NL} \cup \text{NM} \cup \text{NS} \cup \text{PS} \cup \text{PM} \cup \text{PS}$. Assume $x \in \text{NL}$ (the remainder cases can be treated in a similar way). As $x < -\gamma$, by Lemma 2 we obtain $[x] <_\Gamma [-\gamma]$ and as $[-\gamma] <_\Gamma [-\alpha]$ we also obtain, by transitivity of $<_\Gamma$ previously proved, that $[x] <_\Gamma [-\alpha]$. Hence, by the assumption $[-\alpha] \leq_\Gamma [x]$, we obtain $[-\alpha] <_\Gamma [-\alpha]$, a contradiction. The other inclusion is straightforward by Lemma 2.

Finally, $\mathcal{P}_\Gamma \colon \mathcal{C} \longrightarrow \mathcal{K}(\mathbb{S})_\Gamma$ is a bijection. For the injectivity of $\mathcal{P}_\Gamma$ assume $c_i \neq c_j$, by the injectivity of $\mathcal{P}$, then we have $\mathcal{P}(c_i) \neq \mathcal{P}(c_j)$. So there exists $x \in \mathbb{S}$ such that $x \in \mathcal{P}(c_i)$ but $x \notin \mathcal{P}(c_j)$. Hence $[x] \in \mathcal{P}_\Gamma(c_i)$ and $[x] \notin \mathcal{P}_\Gamma(c_j)$, that is, $\mathcal{P}_\Gamma(c_i) \neq \mathcal{P}_\Gamma(c_j)$. For the surjectivity of $\mathcal{P}_\Gamma$, consider any $K_\Gamma \in \mathcal{K}(\mathbb{S})_\Gamma$; thus $K \in \mathcal{K}(\mathbb{S})$. Now by the surjectivity of $\mathcal{P}$, there exists $c \in \mathcal{C}$ such that $\mathcal{P}(c) = K$; this means that $K = \{x \in \mathbb{S} \mid x \in \mathcal{P}(c)\}$, hence $K_\Gamma = \{[x] \in \mathbb{S}_\Gamma \mid x \in \mathcal{P}(c)\} = \mathcal{P}_\Gamma(c)$. QED

Lemma 4. *Let $\mathcal{M}_\Gamma = (\mathbb{S}_\Gamma, \mathcal{D}_\Gamma, <_\Gamma, \mathcal{K}(\mathbb{S})_\Gamma, \mathcal{P}_\Gamma, h_\Gamma)$ be a Γ-filtration of an MQC-model $\mathcal{M} = (\mathbb{S}, \mathcal{D}, <, \mathcal{K}(\mathbb{S}), \mathcal{P}, h)$. Then, for every $A \in \Gamma$ and for every $x \in \mathbb{S}$, we have that $x \in h(A)$ if and only if $[x] \in h_\Gamma(A)$.*

Proof. The proof is by induction on the complexity of $A \in \Gamma$. QED

The strong finite model property is a consequence of the previous results:

Proposition 4. *Let A be a formula of $\mathcal{L}(MQ)^\mathcal{P}$. If A^* is satisfiable in a MQC-model, then A^* is satisfiable in a finite MQC-model containing at most 2^n points, where n is the number of subformulas of A^*.*

The previous proposition can be used to define a test for satisfiability of a formula A with respect to MQC-models: since the class of all MQC-models with size at most 2^n is effectively enumerable, in order to determine whether A^* is satisfiable, one has just to traverse the list of such models and check whether A^* gets satisfied. As a consequence, we obtain

Theorem 1 (Decidability). *$MQ^\mathcal{P}$ is decidable.*

6 Conclusions

Logics for order of magnitude reasoning are important to deal with situations where numerical values are either imprecise or unavailable. In this paper, we continue our work with a multimodal logic for order of magnitude reasoning which considers a new approach to closeness based on proximity intervals. In this respect, we have shown that the relative order between the proximity intervals can be reproduced in term of the logic, and some examples have been shown. Last but not least, we have proved the decidability of the logic.

References

1. A. Ali, D. Dubois, and H. Prade. Qualitative reasoning based on fuzzy relative orders of magnitude. *IEEE transactions on Fuzzy Systems*, 11(1):9–23, 2003.
2. C. Areces and B. ten Cate. Hybrid logics. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*. Elsevier, 2006.
3. P. Balbiani. Reasoning about negligibility and proximity in the set of all hyperreals. *Journal of Applied Non-Classical Logics*, 16:14–36, 2016.
4. B. Bennett, A. Cohn, F. Wolter, and M. Zakharyaschev. Multi-dimensional modal logic as a framework for spatio-temporal reasoning. *Applied Intelligence*, 17(3):239–251, 2002.
5. A. Burrieza, E. Muñoz-Velasco, and M. Ojeda-Aciego. Closeness and distance in order of magnitude qualitative reasoning via PDL. *Lecture Notes in Artificial Intelligence*, 5988:71–80, 2010.
6. A. Burrieza, E. Muñoz Velasco, and M. Ojeda-Aciego. A multimodal logic for closeness. *Journal of Applied Non-Classical Logics*, 2018. To appear
7. A. Burrieza and M. Ojeda-Aciego. A multimodal logic approach to order of magnitude qualitative reasoning with comparability and negligibility relations. *Fundamenta Informaticae*, 68(1):21–46, 2005.
8. D. Dubois, A. Hadj-Ali, and H. Prade. Granular computing with closeness and negligibility relations. In T. Y. Lin, Y. Y. Yao, and L. A. Zadeh, editors, *Data Mining, Rough Sets and Granular Computing*, pages 290–307. Physica-Verlag, 2002.
9. J. Golińska-Pilarek. On decidability of a logic for order of magnitude qualitative reasoning with bidirectional negligibility. *Lecture Notes in Artificial Intelligence*, 7519:255–266, 2012.
10. J. Golińska-Pilarek and E. Muñoz-Velasco. Relational approach for a logic for order of magnitude qualitative reasoning with negligibility, non-closeness and distance. *Logic Journal of IGPL*, 17(4):375–394, 2009.
11. J. Golińska-Pilarek and E. Muñoz-Velasco. A hybrid qualitative approach for relative movements. *Logic Journal of IGPL*, 23(3):410–420, 2015.
12. H. Liu, D. J. Brown, and G. M. Coghill. Fuzzy qualitative robot kinematics. *IEEE Transactions on Fuzzy Systems*, 16(3):808–822, 2008.
13. T. Mossakowski and R. Moratz. Qualitative reasoning about relative direction of oriented points. *Artificial Intelligence*, 180:34–45, 2012.
14. E. Muñoz-Velasco, A. Burrieza, and M. Ojeda-Aciego. A logic framework for reasoning with movement based on fuzzy qualitative representation. *Fuzzy Sets and Systems*, 242:114–131, 2014.
15. L. Travé-Massuyès, F. Prats, M. Sánchez, and N. Agell. Relative and absolute order-of-magnitude models unified. *Annals of Mathematics and Artificial Intelligence*, 45:323–341, 2005.