Towards Intuitionistic L-fuzzy Formal t-Concepts

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***Abstract*—We continue our study of intuitionistic L-fuzzy formal concept analysis by presenting a construction of an adjoint triple based on a non-commutative conjunctor, so that it enables the construction of intuitionistic L-fuzzy t-formal concepts.**

I. Introduction

In this paper we keep investigating the additional flexibility that the framework of intuitionistic fuzzy sets and provide to the research area of Formal Concept Analysis (FCA).

FCA arose some thirty years ago as an applied lattice theory from the seminal work by Ganter and Wille [1]. Originally, the theory was developed in the crisp but was soon extended to the fuzzy case by Burusco and Fuentes-González [2] and, since then, a great number of different generalizations have been developed: by Bělohlávek using a complete residuated lattice [3], the (one-sided) generalized FCA by Krajči [4], multi-adjoint FCA by Medina et al [5]–[8], heterogeneous and higher-order FCA by Krídlo et al [9], [10], etc.

Our approach here will be based on the intuitionistic fuzzy sets (IF-sets for short) introduced by Atanassov [11]. His original construction was later adapted to the L-fuzzy case, in which a complete residuated lattice was used instead of the unit interval as underlying set of truth-values [12], [13].

Although some authors have already introduced IF-based ideas within the FCA framework: for instance, [15], [16] define a generalization based on Krajči's one-sided approach; on the other hand, [17] focuses on an interval-valued intuitionistic fuzzy rough approach. And all three previous approaches are based on the unit interval.

We introduce for the first time, as far as we know, an adjoint triple of operators defined on an IF complete residuated lattice of truth-values which, naturally, allows for introducing the notion of IF-L-fuzzy formal t-concept.

II. Preliminary definitions and results

As stated above, we will be primary dealing with truth-values not necessarily belonging to the unit interval, but to a complete residuated lattice (see [18] for further details).

Definition 1: An algebra $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ is said to be a *complete residuated lattice* if

1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with least element 0 and greatest element 1,
2) $\langle L, \otimes, 1\rangle$ is a commutative monoid,
3) $\langle \otimes, \to\rangle$ is an adjoint pair, i.e. $a \otimes b \leq c$ if and only if $a \leq b \to c$, for all $a, b, c \in L$, where $\leq$ is the ordering generated by $\wedge$ and $\vee$.

Example 1: We will consider in our examples a three-element set $\{1, 0.5, 0\}$ together with the Łukasiewicz logic operations

- $k \otimes m = \max\{a + b - 1, 0\}$
- $k \to m = \min\{1 - a + b, 1\}$

We now recall the basics of fuzzy formal concept analysis (see [19, chapter 5] for more details).

Definition 2: A triple $\langle B, A, r\rangle$ where $r \in \mathcal{L}^{B\times A}$ is said to be an $\mathcal{L}$-fuzzy *formal context*. B is the set of *objects*, A the set of *attributes* and r the *incidence relation*.

Definition 3: Given an $\mathcal{L}$-fuzzy formal context $\langle B, A, r\rangle$, two pairs of concept-forming operators $\langle \uparrow, \downarrow\rangle$ and $\langle \nearrow, \swarrow\rangle$ can be defined between the $\mathcal{L}$-fuzzy powersets $\langle \mathcal{L}^B, \subseteq\rangle$ and $\langle \mathcal{L}^A, \subseteq\rangle$. Let $f \in \mathcal{L}^B$ and $g \in \mathcal{L}^A$ be two arbitrary $\mathcal{L}$-sets.

$$\uparrow f(a) = \bigwedge_{b\in B} \big(f(b) \to r(b, a)\big)$$
$$\downarrow g(b) = \bigwedge_{a\in A} \big(g(a) \to r(b, a)\big)$$
$$\nearrow f(a) = \bigvee_{b\in B} f(b) \otimes r(b, a)$$
$$\swarrow g(b) = \bigwedge_{a\in A} \big(r(b, a) \to g(a)\big)$$

It is well-known that the previous constructions lead to two Galois connections between $\langle \mathcal{L}^B, \subseteq\rangle$ and $\langle \mathcal{L}^A, \subseteq\rangle$,

1) The pair of mappings $\langle \uparrow, \downarrow\rangle$ forms a Galois connection, i.e. for all $\mathcal{L}$-fuzzy sets $f \in \mathcal{L}^B$ and $g \in \mathcal{L}^A$ it holds that
$$f \subseteq \downarrow g \Leftrightarrow g \subseteq \uparrow f$$
2) The pair $\langle \nearrow, \swarrow\rangle$ forms an isotone Galois connection (also called adjunction), i.e.
$$\nearrow f \subseteq g \Leftrightarrow f \subseteq \swarrow g\,.$$

Definition 4: An *$\mathcal{L}$-fuzzy concept* of an $\mathcal{L}$-context $\mathcal{C} = \langle B, A, r\rangle$ with respect to $\langle \uparrow, \downarrow\rangle$ is a pair $\langle f, g\rangle \in L^B \times L^A$ such that $\uparrow f = g$ and $\downarrow g = f$. The first component f is said to be the *extent* of the concept, whereas the second component g is the *intent* of the concept. An *$\mathcal{L}$-fuzzy concept* of an

$\mathcal{L}$-context $\mathcal{C} = \langle B, A, r\rangle$ with respect to $\langle \nearrow, \swarrow\rangle$ is a pair $\langle f, g\rangle \in L^B \times L^A$ such that $\nearrow f = g$ and $\swarrow g = f$.

Example 2: Consider the following example of $\mathcal{L}$-context related to the ice-cream preferences of two girls.

	chocolate	vanilla	stracciatella
Ester	1	0	0.5
Lydia	1	0.5	0

Here are some of obtained concepts after applying the derivation operators $\langle\uparrow, \downarrow\rangle$.

- $\{1/\text{Ester}; 0.5/\text{Lydia}\}; \{1/\text{choc.}; 0/\text{vanilla}; 0.5/\text{strac.}\}$
- $\{0.5/\text{Ester}; 1/\text{Lydia}\}; \{1/\text{choc.}; 0.5/\text{vanilla}; 0/\text{strac.}\}$
- $\{1/\text{Ester}; 1/\text{Lydia}\}; \{1/\text{choc.}; 0/\text{vanilla}; 0/\text{strac.}\}$
- $\{0.5/\text{Ester}; 0.5/\text{Lydia}\}; \{1/\text{choc.}; 0.5/\text{vanilla}; 0.5/\text{strac.}\}$
- $\{0/\text{Ester}; 0/\text{Lydia}\}; \{1/\text{choc.}; 1/\text{vanilla}; 1/\text{strac.}\}$

The results can be interpreted in terms of the satisfaction of customers of patisserie. Ester and Lydia are both fully satisfied only with full dose of chocolate ice-cream. If they would obtain lower dose of vanilla ice cream over full dose of chocolate ice cream then Lydia was still fully satisfied but good mood of Ester went little down. Similarly with Lydia and stracciatella ice cream.

Definition 5: Let $\mathcal{L}$ be a complete residuated lattice. The residuated *negation* $\neg$ and residuated *disjunction* $\oplus$ on $\mathcal{L}$ are defined as follows:

- $\neg k = k \to 0$
- $k \oplus m = \neg k \to m$

for any $k, m \in L$.

Remark 1: Note that if the negation $\neg$ satisfies the double negation law on $\mathcal{L}$ then $\oplus$ is a commutative operation, and we can use it as the disjunction operation on $\mathcal{L}$. Specifically, Łukasiewicz implication satisfies the double negation law.

III. Intuitionistic fuzzy sets based on complete residuated lattices

We start by recalling the notion of intuitionistic fuzzy set defined on a complete lattice, as introduced in [12].

Definition 6: Given a complete lattice L together with an involutive order reversing operation $N\colon L \to L$, and a universe set E: An intuitionistic L-fuzzy set (IF set) A in E is defined as an object having the form:

$$A = \big\{\langle x, \mu_A(x), \nu_A(x) \mid x \in E\big\}$$

where the functions $\mu_A\colon E \to L$ and $\nu_A\colon E \to L$ define the degree of membership and the degree of non-membership, respectively, to A of the elements $x \in E$, and for every $x \in E$:

$$\mu_A(x) \leq N(\nu_A(x)) .$$

When the previous inequality is strict, there is certain indetermination degree on the knowledge about x.

The IF-lattice associated to a given residuated lattice $\mathcal{L}$ was introduced in [14] as a natural extension of the Pareto ordering [20] taking $\mathcal{L}$ as the underlying set of truth-values instead of the unit interval. The formal definition can be seen below:

Definition 7: Given $\mathcal{L} = \big\langle L, 0, 1, \otimes, \to, \wedge, \vee\big\rangle$ a complete residuated lattice, we can consider the lattice of intuitionistic truth values

$$\mathcal{L}_{\mathrm{IF}} = \big\langle\{\langle k_1, k_2\rangle \in L \times L \mid k_2 \leq \neg k_1\}, \leq\big\rangle$$

where ordering $\leq$ on $\mathcal{L}_{\mathrm{IF}}$ is defined as follows $\langle k_1, k_2\rangle \leq \langle m_1, m_2\rangle$ when $k_1 \leq m_1$ and $k_2 \geq m_2$.

The negation operator in $\mathcal{L}$ can be easily generalized to elements in $\mathcal{L}_{\mathrm{IF}}$ by

$$\neg\langle k_1, k_2\rangle = \langle k_2, k_1\rangle .$$

Remark 2: Note that the elements of $\mathcal{L}_{\mathrm{IF}}$ will represent the membership (and non-membership) degrees and, hence, will be denoted in terms of μ and ν whenever necessary.

Example 3: Considering $\mathcal{L} = \langle\{1, 0.5, 0\}, \otimes, \to\rangle$ we have $\mathcal{L}_{\mathrm{IF}} = \langle\{\langle 1, 0\rangle, \langle 0.5, 0\rangle, \langle 0.5, 0.5\rangle, \langle 0, 0\rangle, \langle 0, 0.5\rangle, \langle 0, 1\rangle\}, \leq\rangle$. All the pairs $\langle p_1, p_2\rangle$ satisfy $p_2 \leq \neg p_1$, where $\neg p_1 = p_1 \to 0$ and, in particular, $\neg 0.5 = 0.5$. A possible interpretation of such a new richer set of truth-values could be as follows:

- $\langle 1, 0\rangle$ absolutely YES, sure, satisfied, full dose, ...
- $\langle 0.5, 0\rangle$ more yes than no, not completely sure, ...
- $\langle 0.5, 0.5\rangle$ user in general agree but has some doubts, ...
- $\langle 0, 0\rangle$ not interested, but not disagree, ...
- $\langle 0, 0.5\rangle$ more no than yes, ...
- $\langle 0, 1\rangle$ absolutely NO, ...

The two well-known equivalences in classical logic below

$$(A \Rightarrow B) \Leftrightarrow (\neg A \vee B) \quad \text{and} \quad \neg(A \Rightarrow B) \Leftrightarrow (A \wedge \neg B)$$

inspired the introduction in [14] of the following binary operator $\rightrightarrows_1\colon \mathcal{L}_{\mathrm{IF}} \times \mathcal{L}_{\mathrm{IF}} \to \mathcal{L}_{\mathrm{IF}}$ in the lattice of truth degrees of intuitionistic fuzzy sets $\mathcal{L}_{\mathrm{IF}}$. Its definition is as follows, for all pair of elements $\langle p_1, p_2\rangle, \langle q_1, q_2\rangle \in \mathcal{L}_{\mathrm{IF}}$

$$\langle p_1, p_2\rangle \rightrightarrows_1 \langle q_1, q_2\rangle = \langle p_2 \oplus q_1, p_1 \otimes q_2\rangle$$

Notice that the nonmembership value (second component of the pairs) plays the role of the negated atoms in the previous construction.

The proof that $\rightrightarrows_1$ is well-defined, namely, that their results are in $\mathcal{L}_{\mathrm{IF}}$ was given in [14].

The behavior of $\rightrightarrows_1$ with respect to the top and bottom elements of $\mathcal{L}_{\mathrm{IF}}$ directly follows that of $0, 1 \in \mathcal{L}$. Namely, for all $\langle p_1, p_2\rangle, \langle q_1, q_2\rangle \in \mathcal{L}_{\mathrm{IF}}$, we have that

$$\langle 1, 0\rangle \rightrightarrows_1 \langle p_1, p_2\rangle = \langle 0 \oplus p_1, 1 \otimes p_2\rangle = \langle p_1, p_2\rangle$$
$$\langle 0, 1\rangle \rightrightarrows_1 \langle p_1, p_2\rangle = \langle 1 \oplus p_1, 0 \otimes p_2\rangle = \langle 1, 0\rangle$$
$$\langle p_1, p_2\rangle \rightrightarrows_1 \langle 1, 0\rangle = \langle p_2 \oplus 1, p_1 \otimes 0\rangle = \langle 1, 0\rangle$$
$$\langle p_1, p_2\rangle \rightrightarrows_1 \langle 0, 1\rangle = \langle p_2 \oplus 0, p_1 \otimes 1\rangle = \langle p_2, p_1\rangle = \neg\langle p_1, p_2\rangle$$

Moreover, we have that if $\big(\langle p_1, p_2\rangle \rightrightarrows_1 \langle q_1, q_2\rangle\big) = \langle r_1, r_2\rangle$ and the inequality $\langle p_1, p_2\rangle \leq \langle q_1, q_2\rangle$ holds, then $r_1 \geq r_2$.

In order to prove this property, we have to show that $p_1 \otimes q_2 \leq p_2 \oplus q_1$:

Firstly, since $p_1 \leq q_1$, we have

$$p_1 \otimes q_2 \leq q_1 \otimes q_2 \tag{1}$$

$\rightrightarrows_1$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$
$\langle 0.5,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$
$\langle 0.5,0.5\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$
$\langle 0,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$
$\langle 0,0.5\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$
$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$

Fig. 1. Truth-table of $\rightrightarrows_1$ for the examples.

Then

$$q_1 \otimes q_2 \leq q_2 \oplus q_1 \tag{2}$$

because of the following equivalence

$$\begin{aligned} q_1 \otimes q_2 \leq q_2 \oplus q_1 = \neg q_2 \to q_1 \\ \iff \text{(by the adjoint property)} \\ q_1 \otimes q_2 \otimes \neg q_2 \leq q_1 \text{ (holds trivially)} \end{aligned}$$

Finally, since $q_2 \leq p_2$ or, equivalently, $\neg p_2 \leq \neg q_2$ we have

$$q_2 \oplus q_1 = \neg q_2 \to q_1 \leq \neg p_2 \to q_1 = p_2 \oplus q_1 \tag{3}$$

The desired result follows by inequalities (1), (2), (3).

Finally, we also have

$$\begin{aligned} &\bigwedge_{i\in I}(\langle k_{1i}, k_{2i}\rangle \rightrightarrows_1 \langle m_{1i}, m_{2i}\rangle) = \\ &\qquad = \bigwedge_{i\in I}\langle k_{2i} \oplus m_{1i}, k_{1i} \otimes m_{2i}\rangle \\ &\qquad = \left\langle \bigwedge_{i\in I}(k_{2i} \oplus m_{1i}), \bigvee_{i\in I}(k_{1i} \otimes m_{2i}) \right\rangle . \end{aligned}$$

Let us finally show the behavior of $\rightrightarrows_1$ with respect to monotonicity.

Lemma 1: $\rightrightarrows_1$ is decreasing in first and increasing in second argument.

Proof: Assume $\langle p_1, p_2\rangle, \langle p_1', p_2'\rangle, \langle q_1, q_2\rangle \in \mathcal{L}_{\rm IF}$ be arbitrary intuitionistic truth values, such that $\langle p_1, p_2\rangle \leq \langle p_1', p_2'\rangle$, i.e. $p_1 \leq p_1'$ and $p_2 \geq p_2'$. Now, we have

$$p_2 \oplus q_1 \geq p_2' \oplus q_1 \quad \text{and} \quad p_1 \otimes q_2 \leq p_1' \otimes q_2$$

and, as a result,

$$\begin{aligned} \langle p_1, p_2\rangle \rightrightarrows_1 \langle q_1, q_2\rangle &= \langle p_2 \oplus q_1, p_1 \otimes q_2\rangle \\ &\geq \langle p_2' \oplus q_1, p_1' \otimes q_2\rangle \\ &= \langle p_1', p_2'\rangle \rightrightarrows_1 \langle q_1, q_2\rangle . \end{aligned}$$

Similarly, we have

$$q_2 \oplus p_1 \leq q_2 \oplus p_1' \quad \text{and} \quad q_1 \otimes p_2 \geq q_1 \otimes p_2'$$

therefore

$$\begin{aligned} \langle q_1, q_2\rangle \rightrightarrows_1 \langle p_1, p_2\rangle &= \langle q_2 \oplus p_1, q_1 \otimes p_2\rangle \\ &\leq \langle q_2 \oplus p_1', q_1 \otimes p_2'\rangle \\ &= \langle q_1, q_2\rangle \rightrightarrows_1 \langle p_1', p_2'\rangle \end{aligned}$$

■

The truth-table of $\rightrightarrows_1$ on our running example is given in Figure 1.

The mere consideration of the intuitionistic lattice $\mathcal{L}_{\rm IF}$ instead of $\mathcal{L}$, provides more expressiveness to the obtained concept lattice.

Example 4: By considering an $\mathcal{L}_{\rm IF}$-based formal context in our previous example, replacing values $1, 0.5, 0$ by $\langle 1,0\rangle, \langle 0.5,0.5\rangle, \langle 0,1\rangle$ one can notice that the behaviour of such intuitionistic values are the same as the original values with respect to implication $\to$ in $\mathcal{L}$, as can be seen using the table of $\rightrightarrows_1$ (see Fig. 1). But, after applying the new concept forming operators some new $\mathcal{L}_{\rm IF}$-concepts are obtained. $\langle 0,0\rangle$ can be interpreted on the standpoint of customers as to be not interested, whereas from the ice cream standpoint should be read more likely as "I don't know how much I'll obtain".

- The pair with extent $\{\text{Ester}/\langle 0,0\rangle; \text{Lydia}/\langle 0,0\rangle\}$, and intent $\{\text{choco.}/\langle 1,0\rangle, \text{vanilla}/\langle 0,0\rangle, \text{strac.}/\langle 0,0\rangle\}$ should be interpreted as Ester and Lydia would not be interested in any ice cream portion where is no certainty of how much unpopular ice cream they will obtain.

IV. IF-Formal Concept Analysis

The introduction of the lattice of IF-degrees allows for extending the constructions of FCA to this more general framework. Following the methodology and constructions in the fuzzy case, the definitions of IF-formal context and its associated concept-forming operations were given in [14] based on the operator $\rightrightarrows_1$.

Definition 8: Let $\mathcal{L}$ be a complete residuated lattice and $\mathcal{L}_{\rm IF}$ be its associated lattice of intuitionistic degrees. A triple $\langle B, A, r\rangle$ where $r\colon B \times A \to \mathcal{L}_{\rm IF}$ is said to be an *IF-formal context*.

Definition 9: Given an IF-formal context $\langle B, A, r\rangle$, its *concept forming operators* are a pair of mappings $\langle \upuparrows_1, \downdownarrows_1\rangle$ between the intuitionistic $\mathcal{L}_{\rm IF}$-fuzzy powersets $\langle {\mathcal{L}_{\rm IF}}^B, \subseteq\rangle$ and $\langle {\mathcal{L}_{\rm IF}}^A, \subseteq\rangle$ defined as follows

- $\upuparrows_1 f(a) = \bigwedge_{b\in B}(f(b) \rightrightarrows_1 r(b,a))$, for any $f \in {\mathcal{L}_{\rm IF}}^B$
- $\downdownarrows_1 g(b) = \bigwedge_{a\in A}(g(a) \rightrightarrows_1 r(b,a))$, for any $g \in {\mathcal{L}_{\rm IF}}^A$.

Unfortunately, and contrariwise to what one would expect, the concept-forming operators need not form a Galois connection unless the extra assumption that the incidence relation r should assign values (p, q) satisfying $q = \neg p$ is assumed, but this means that the IF-formal context has to provide values without indetermination, which are essentially equivalent to (usual) L-fuzzy sets. As a result, not much is gained in this approach.

In [14] this problem was circumvented by providing an alternative construction in terms of an isotone Galois connection which did not require so strong extra assumptions. In this paper, we will follow an alternative path by providing a construction in terms of adjoint triples, and giving rise to the so-called t-formal concepts [6].

To begin with, let us recall the notion of adjoint triple.

Definition 10: Let $\langle P_1, \leq_1\rangle$, $\langle P_2, \leq_2\rangle$, $\langle P_3, \leq_3\rangle$ be posets and consider three mappings $\& : P_1 \times P_2 \longrightarrow P_3$, $\nearrow: P_1 \times P_3 \longrightarrow P_2$, $\searrow: P_2 \times P_3 \longrightarrow P_1$, then $\langle \&, \nearrow, \searrow\rangle$ is a *adjoint triple* with respect to P_1, P_2, P_3 if:

1) $\&$ is increasing in both arguments.
2) $\nearrow, \searrow$ are decreasing in first argument and increasing in second argument.
3) $x \leq_1 y \searrow z \Leftrightarrow x \& y \leq_3 z \Leftrightarrow y \leq_2 x \nearrow z$ for any $x \in P_1$, $y \in P_2$, $z \in P_3$.

We have just proved that $\rightrightarrows_1$ is an implication operators, so property 2) above holds for it. In order to build an adjoint triple we need a conjunction and a second implication operator satisfying the mutual adjoint property. The conjunction is introduced as follows:

Definition 11: Let $\mathcal{L}$ be a complete residuated lattice and consider its associated lattice of intuitionistic truth degrees $\mathcal{L}_{\text{IF}}$. The operator $\boxtimes : \mathcal{L}_{\text{IF}} \times \mathcal{L}_{\text{IF}} \to \mathcal{L}_{\text{IF}}$ is defined as follows

$$\langle p_1, p_2\rangle \boxtimes \langle q_1, q_2\rangle = \big\langle p_1 \otimes \neg q_2, (\neg p_1 \oplus q_2) \wedge (p_2 \oplus \neg q_1)\big\rangle.$$

The very construction of $\boxtimes$ leads to the fact that its output is in $\mathcal{L}_{\text{IF}}$, so it is well-defined. Its properties with respect to the distinguished elements of $\mathcal{L}_{\text{IF}}$ are given below:

$$\begin{aligned}
\langle 0, 1\rangle \boxtimes \langle q_1, q_2\rangle &= \langle 0, 1\rangle\\
\langle p_1, p_2\rangle \boxtimes \langle 0, 1\rangle &= \langle 0, 1\rangle\\
\langle 1, 0\rangle \boxtimes \langle q_1, q_2\rangle &= \langle \neg q_2, q_2\rangle\\
\langle p_1, p_2\rangle \boxtimes \langle 1, 0\rangle &= \langle p_1, p_2\rangle
\end{aligned}$$

Furthermore, the following monotonicity properties hold:

Lemma 2: $\boxtimes$ is increasing in both arguments.

Proof:

Let $\langle p_1, p_2\rangle, \langle p_1', p_2'\rangle, \langle q_1, q_2\rangle \in \mathcal{L}_{\text{IF}}$ be arbitrary intuitionistic truth values, such that $\langle p_1, p_2\rangle \leq \langle p_1', p_2'\rangle$, i.e. $p_1 \leq p_1'$ and $p_2 \geq p_2'$. Then

- $p_1 \otimes \neg q_2 \leq p_1' \otimes \neg q_2$
- $\neg p_1 \oplus q_2 \geq \neg p_1' \oplus q_2$
- $p_2 \oplus \neg q_1 \geq p_2' \oplus \neg q_1$
- $(\neg p_1 \oplus q_2) \wedge (p_2 \oplus \neg q_1) \geq (\neg p_1' \oplus q_2) \wedge (p_2' \oplus \neg q_1)$

Then

$$\begin{aligned}
\langle p_1, p_2\rangle \boxtimes \langle q_1, q_2\rangle &= \langle p_1 \otimes \neg q_2, (\neg p_1 \oplus q_2) \wedge (p_2 \oplus \neg q_1)\rangle\\
&\leq \langle p_1' \otimes \neg q_2, (\neg p_1' \oplus q_2) \wedge (p_2' \oplus \neg q_1)\rangle\\
&= \langle p_1', p_2'\rangle \boxtimes \langle q_1, q_2\rangle
\end{aligned}$$

Similarly,

- $q_1 \otimes \neg p_2 \leq q_1 \otimes \neg p_2'$
- $\neg q_1 \oplus p_2 \geq \neg q_1 \oplus p_2'$
- $q_2 \oplus \neg p_1 \geq q_2 \oplus \neg p_1'$
- $(\neg q_1 \oplus p_2) \wedge (q_2 \oplus \neg p_1) \geq (\neg q_1 \oplus p_2') \wedge (q_2 \oplus \neg p_1')$

then

$$\begin{aligned}
\langle q_1, q_2\rangle \boxtimes \langle p_1, p_2\rangle &= \langle q_1 \otimes \neg p_2, (\neg q_1 \oplus p_2) \wedge (q_2 \oplus \neg p_1)\rangle\\
&\leq \langle q_1 \otimes \neg p_2', (\neg q_1 \oplus p_2') \wedge (q_2 \oplus \neg p_1')\rangle\\
&= \langle q_1, q_2\rangle \boxtimes \langle p_1', p_2'\rangle
\end{aligned}$$

Hence $\boxtimes$ is increasing in both arguments. ∎

The fact that $\boxtimes$ is, in general, non-commutative suggests the existence of two residuals depending on which argument is fixed. This second construction as follows.

Definition 12: Let $\mathcal{L}$ be a complete residuated lattice and consider its associated lattice of intuitionistic truth degrees $\mathcal{L}_{\text{IF}}$, and consider the following implication

$$\langle p_1, p_2\rangle \rightrightarrows_2 \langle q_1, q_2\rangle = \langle(\neg p_1 \oplus q_1) \wedge (p_2 \oplus \neg q_2), p_1 \otimes \neg q_1\rangle$$

defined for all $\langle p_1, p_2\rangle, \langle q_1, q_2\rangle \in \mathcal{L}_{\text{IF}}$.

It is not difficult to check that the output of $\rightrightarrows_2$ actually belongs to $\mathcal{L}_{\text{IF}}$. Its properties with respect to the distinguished elements of $\mathcal{L}_{\text{IF}}$ are given below:

$$\begin{aligned}
\langle 0, 1\rangle \rightrightarrows_2 \langle q_1, q_2\rangle &= \langle 1, 0\rangle\\
\langle p_1, p_2\rangle \rightrightarrows_2 \langle 0, 1\rangle &= \langle p_2, p_1\rangle = \neg\langle p_1, p_2\rangle\\
\langle 1, 0\rangle \rightrightarrows_2 \langle q_1, q_2\rangle &= \langle q_1, \neg q_1\rangle\\
\langle p_1, p_2\rangle \rightrightarrows_2 \langle 1, 0\rangle &= \langle 1, 0\rangle
\end{aligned}$$

Lemma 3: $\rightrightarrows_2$ is decreasing in its first argument and increasing in its second argument.

Proof: Let $\langle p_1, p_2\rangle, \langle p_1', p_2'\rangle, \langle q_1, q_2\rangle \in \mathcal{L}_{\text{IF}}$ be arbitrary intuitionistic truth values, such that $\langle p_1, p_2\rangle \leq \langle p_1', p_2'\rangle$, i.e. $p_1 \leq p_1'$ and $p_2 \geq p_2'$. Then, we have

- $\neg p_1 \oplus q_1 \geq \neg p_1' \oplus q_1$
- $p_2 \oplus \neg q_2 \geq p_2' \oplus \neg q_2$
- $(\neg p_1 \oplus q_1) \wedge (p_2 \oplus \neg q_2) \geq (\neg p_1' \oplus q_1) \wedge (p_2' \oplus \neg q_2)$
- $p_1 \otimes \neg q_2 \leq p_1' \otimes \neg q_2$

And, as a result, we obtain

$$\begin{aligned}
\langle p_1, p_2\rangle \rightrightarrows_2 \langle q_1, q_2\rangle &= \langle(\neg p_1 \oplus q_1) \wedge (p_2 \oplus \neg q_2), p_1 \otimes \neg q_2\rangle\\
&\geq \langle(\neg p_1' \oplus q_1) \wedge (p_2' \oplus \neg q_2), p_1' \otimes \neg q_2\rangle\\
&= \langle p_1', p_2'\rangle \rightrightarrows_2 \langle q_1, q_2\rangle
\end{aligned}$$

Similarly,

- $\neg q_1 \oplus p_1 \leq \neg q_1 \oplus p_1'$
- $q_2 \oplus \neg p_2 \leq q_2 \oplus \neg p_2'$
- $(\neg q_1 \oplus p_1) \wedge (q_2 \oplus \neg p_2) \leq (\neg q_1 \oplus p_1') \wedge (q_2 \oplus \neg p_2')$
- $q_1 \otimes \neg p_1 \geq q_1 \otimes \neg p_1'$

therefore

$$\begin{aligned}
\langle q_1, q_2\rangle \rightrightarrows_2 \langle p_1, p_2\rangle &= \langle(\neg q_1 \oplus p_1 1) \wedge (q_2 \oplus \neg p_2), q_1 \otimes \neg p_1\rangle\\
&\leq \langle(\neg q_1 \oplus p_1') \wedge (q_2 \oplus \neg p_2'), q_1 \otimes \neg p_1'\rangle\\
&= \langle q_1, q_2\rangle \rightrightarrows_2 \langle p_1', p_2'\rangle
\end{aligned}$$

Hence $\rightrightarrows_2$ is decreasing in its first and increasing in its second argument. ∎

$\boxtimes$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,1\rangle$
$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0.5\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0.5,0.5\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,0.5\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$

Fig. 2. Truth-table of $\boxtimes$ for the examples.

$\rightrightarrows_2$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0.5,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0.5\rangle$
$\langle 0.5,0.5\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0.5,0.5\rangle$
$\langle 0,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0,0\rangle$
$\langle 0,0.5\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$
$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$

Fig. 3. Truth-table of $\rightrightarrows_2$ for the examples.

We are now in position to prove that the operators $\boxtimes$, $\rightrightarrows_1$, and $\rightrightarrows_2$ form an adjoint triple in $\mathcal{L}_{\mathrm{IF}}$ (in this case, the three posets coincide with $\mathcal{L}_{\mathrm{IF}}$).

Theorem 1: $\langle\boxtimes,\rightrightarrows_1,\rightrightarrows_2\rangle$ is an adjoint triple with respect to $\mathcal{L}_{\mathrm{IF}},\mathcal{L}_{\mathrm{IF}},\mathcal{L}_{\mathrm{IF}}$.

Proof: By Lemma we have and Lemmas 1, 2 and 3 we have the monotonicity properties of $\boxtimes$ and $\rightrightarrows_1$ and $\rightrightarrows_2$.

The adjoint properties were proved in [14] for $(\boxtimes,\rightrightarrows_1)$ and $(\boxtimes,\rightrightarrows_2)$ separately. ■

Now, following [6] we have two possibilities to construct concept-forming operators in terms of Galois connections. The first one is the following:

Definition 13: Given an IF-formal context $\langle B,A,r\rangle$ we can define two pairs of mappings between the intuitionistic $\mathcal{L}_{\mathrm{IF}}$-fuzzy powersets $\langle {\mathcal{L}_{\mathrm{IF}}}^B,\subseteq\rangle$ and $\langle {\mathcal{L}_{\mathrm{IF}}}^A,\subseteq\rangle$

1) $\langle\upuparrows_1,\downdownarrows_2\rangle$ is defined by

$$\upuparrows_1 f(a)=\bigwedge_{b\in B}(f(b)\rightrightarrows_1 r(b,a)),\ \text{for all } f\in{\mathcal{L}_{\mathrm{IF}}}^B$$
$$\downdownarrows_2 g(b)=\bigwedge_{a\in A}(g(a)\rightrightarrows_2 r(b,a)),\ \text{for all } g\in{\mathcal{L}_{\mathrm{IF}}}^A.$$

2) $\langle\upuparrows_2,\downdownarrows_1\rangle$ is defined by

$$\upuparrows_2 f(a)=\bigwedge_{b\in B}(f(b)\rightrightarrows_2 r(b,a)),\ \text{for all } f\in{\mathcal{L}_{\mathrm{IF}}}^B$$
$$\downdownarrows_1 g(b)=\bigwedge_{a\in A}(g(a)\rightrightarrows_1 r(b,a)),\ \text{for all } g\in{\mathcal{L}_{\mathrm{IF}}}^A.$$

The pair of mappings $\langle\upuparrows_1,\downdownarrows_2\rangle$ are the *multi adjoint concept forming operators* for the IF-formal context $\langle B,A,r\rangle$.

Theorem 2: Let $\mathcal{L}$ be a complete residuated lattice and $\mathcal{L}_{\mathrm{IF}}$ its associated lattice of intuitionistic degrees. Let $\langle B,A,r\rangle$ be an IF-formal context. Then $\langle\upuparrows_1,\downdownarrows_2\rangle$ and $\langle\upuparrows_2,\downdownarrows_1\rangle$ form Galois connections between powersets $\langle {\mathcal{L}_{\mathrm{IF}}}^B,\subseteq\rangle$ and $\langle {\mathcal{L}_{\mathrm{IF}}}^A,\subseteq\rangle$.

Proof: Let $\langle f,g\rangle\in{\mathcal{L}_{\mathrm{IF}}}^B\times{\mathcal{L}_{\mathrm{IF}}}^A$ be an arbitrary pair of $\mathcal{L}_{\mathrm{IF}}$-sets of objects and attributes. Assume $g\subseteq\upuparrows_1 f$, this means

$$g(a)\le\upuparrows_1 f(a)=\bigwedge_{b\in B}(f(b)\rightrightarrows_1 r(b,a))\le f(o)\rightrightarrows_1 r(o,a)$$

for all $(o,a)\in B\times A$. This is equivalent to the inequality $f(o)\le g(a)\rightrightarrows_2 r(o,a)$ and, since $a\in A$ is arbitrary, we obtain $f(o)\le\downdownarrows_2 g(o)$ and, since $o\in B$ is also arbitrary, we have $f\subseteq\downdownarrows_2 g$.

The rest of implications are similar. ■

The previous result allows to (in general) different constructions of an IF-concept lattice associated with an IF-formal context $\langle B,A,r\rangle$. Alternatively, one can think of existing two different sets of IF-L-fuzzy attributes associated to each IF-L-fuzzy set of objects, and this leads to the introduction of the IF-L-fuzzy formal t-concept as a triplet of $\mathcal{L}_{\mathrm{IF}}$-sets $\langle f,g,h\rangle\in\mathcal{L}_{\mathrm{IF}}^B\times\mathcal{L}_{\mathrm{IF}}^A\times\mathcal{L}_{\mathrm{IF}}^A$ where $\upuparrows_1 f=g$, $\downdownarrows_2 g=f$, $\upuparrows_2 f=h$, $\downdownarrows_1 h=f$.

Example 5: Consider the following example of $\mathcal{L}_{\mathrm{IF}}$-context related to the ice-cream preferences of two girls.

	chocolate	vanilla	stracciatella
Ester	$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0.5,0\rangle$
Lydia	$\langle 1,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0\rangle$

Almost all values from $\mathcal{L}_{\mathrm{IF}}$ are used in the previous context. After aplying $\langle\upuparrows_1,\downdownarrows_2\rangle$ concept forming operators the following concepts are obtained

- $\{\text{Ester}/\langle 0.5,0.5\rangle,\text{Lydia}/\langle 1,0\rangle\}$ and $\{\text{choco.}/\langle 1,0\rangle,\text{vanilla}/\langle 0.5,0.5\rangle,\text{strac.}/\langle 0,0\rangle\}$

After aplying $\langle\upuparrows_2,\downdownarrows_1\rangle$ concept forming operators the following concepts are obtained

- $\{\text{Ester}/\langle 0.5,0.5\rangle,\text{Lydia}/\langle 1,0\rangle\}$ and $\{\text{choco.}/\langle 1,0\rangle,\text{vanilla}/\langle 0.5,0.5\rangle,\text{strac.}/\langle 0,1\rangle\}$

Hence the triple

1) $\{\text{Ester}/\langle 0.5,0.5\rangle,\text{Lydia}/\langle 1,0\rangle\}$
2) $\{\text{choco.}/\langle 1,0\rangle,\text{vanilla}/\langle 0.5,0.5\rangle,\text{strac.}/\langle 0,0\rangle\}$

3) $\{\text{choco.}/\langle 1,0\rangle, \text{vanilla}/\langle 0.5,0.5\rangle, \text{strac.}/\langle 0,1\rangle\}$

forms an IF $\mathcal{L}$-formal t-concept.

V. CONCLUSION

We have introduced a (purely L-)fuzzy intuitionistic generalization of the framework of Formal Concept Analysis. Based on the notion of intuitionistic L-fuzzy set, given a complete residuated lattice $\mathcal{L}$, we have worked with its associated lattice of intuitionistic degrees $\mathcal{L}_{\mathrm{IF}}$ (the set of intuitionistic pairs of elements together with the suitable version of the Pareto ordering) in order to construct three binary operators on $\mathcal{L}_{\mathrm{IF}}$ which generate an adjoint triple. The structure of adjoint triple on $\mathcal{L}_{\mathrm{IF}}$ allows to generate to different Galois connections (the concept-forming operators) and, then, introduce the so-called IF-L-fuzzy formal t-concepts.

Having introduced the notion, a number of different directions have to be explored:

1) What is the scope of the new generalization?; does it collapse to some of the existing approaches when the underlying structure changes?
2) Although the usual frontier conditions of $\boxtimes$, $\rightrightarrows_1$, and $\rightrightarrows_2$ are not required to form an adjoint triple, most of them are satisfied and, in those which do not hold, the result actually eliminates the degree of indetermination of the other input argument. This suggests the need to further analyse the underlying meaning of those connectives and relate them to (possibly) existing ones within the fuzzy intuitionistic framework.
3) Some approaches to intuitionistic formal concept analysis have been already introduced in the bibliography, and often using different terms: for instance, in [21] the construction of the concept-forming operators is given as the fuzzy dilation and fuzzy erosion operators of bipolar fuzzy mathematical morphology (on the unit interval).
   Is there any natural interpretation of the notions introduced in this paper within the area of mathematical morphology?

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