Extending Formal Concept Analysis using Intuitionistic L-fuzzy Sets

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***Abstract*—A two-fold general approach to the theory of formal concept analysis is introduced by considering intuitionistic fuzzy sets valued on a residuated lattice as underlying structure for the construction.**

I. Introduction

Formal Concept Analysis (FCA) arose in the eighties of last century from the pioneering work of Rudolph Wille, developed together with the collaboration of Bernhard Ganter [1]. Since then, FCA has proved to be a very fruitful research line both from the theoretical and from the practical standpoint. Several extensions to the fuzzy case have been proposed, the first attempt by Burusco and Fuentes-González [2] using a complete lattice, and later by Bělohlávek using a complete residuated lattice [3]. A number of other extensions have been introduced so far, for instance, the (one-sided) generalized FCA [4], multi-adjoint FCA [5]–[7], heterogeneous and higher-order FCA [8], [9], etc.

We focus on the extension of FCA to the so-called intuitionistic fuzzy sets (IF-sets for short). IF-sets were introduced by Atanassov [10] by considering for all element x a membership degree $\mu(x)$ together with a non-membership degree $\nu(x)$ such that $\mu(x)+\nu(x)\leq 1$, somehow allowing an *indetermination degree* about x in the case of strict inequality. This construction was later generalized when allowing a complete residuated lattice instead of the unit interval as underlying set of truth-values [11], [12].

Some approaches to intuitionistic formal concept analysis have been already introduced in the bibliography, and often using different terms: for instance, in [13] the construction of the concept-forming operators is given as the fuzzy dilation and fuzzy erosion operators of bipolar fuzzy mathematical morphology (on the unit interval).

Other authors have introduced intuitionistic extensions of FCA [14], [15] focusing just on the one-sided approach given by Krajči, or [16] which focuses on an interval-valued intuitionistic fuzzy rough approach. All three previous approaches are based on the unit interval.

We introduce, for the first time as far as we know, a definition of concept-forming operators purely based on intuitionistic fuzzy sets valued on a complete residuated lattice. Two different constructions are given, the first one gives a pair of operators which form a (antitone) Galois connection, whereas the second one gives an isotone Galois connection. Both constructions are based on the assumption that the residuated negation is involutive.

II. Preliminary definitions and results

As stated above, we will be primary dealing with truth-values not necessarily belonging to the unit interval, but to a complete residuated lattice (see [17] for further details).

Definition 1: An algebra $\mathcal{L}=\langle L,\wedge,\vee,\otimes,\rightarrow,0,1\rangle$ is said to be a *complete residuated lattice* if

1) $\langle L,\wedge,\vee,0,1\rangle$ is a complete lattice with least element 0 and greatest element 1,
2) $\langle L,\otimes,1\rangle$ is a commutative monoid,
3) $\langle\otimes,\rightarrow\rangle$ is an adjoint pair, i.e. $a\otimes b\leq c$ if and only if $a\leq b\rightarrow c$, for all $a,b,c\in L$, where $\leq$ is the ordering generated by $\wedge$ and $\vee$.

Remark 1: By commutativity of fuzzy conjunction $\otimes$, the following equivalence holds for any $a,b,c\in L$

$$a\leq b\rightarrow c \qquad\Longleftrightarrow\qquad b\leq a\rightarrow c\,.$$

Example 1: We will consider in our examples a three-element set $\{1,0.5,0\}$ together with the Łukasiewicz logic operations

- $k\otimes m=\max\{a+b-1,0\}$
- $k\rightarrow m=\min\{1-a+b,1\}$

We now recall the basics of fuzzy formal concept analysis (see [18, chapter 5] for more details).

Definition 2: A triple $\langle B,A,r\rangle$ where $r\in\mathcal{L}^{B\times A}$ is said to be an $\mathcal{L}$-fuzzy *formal context*. B is the set of *objects*, A the set of *attributes* and r the *incidence relation*.

Definition 3: Given an $\mathcal{L}$-fuzzy formal context $\langle B,A,r\rangle$, two pairs of concept-forming operators $\langle\uparrow,\downarrow\rangle$ and $\langle\nearrow,\swarrow\rangle$

can be defined between the $\mathcal{L}$-fuzzy powersets $\langle \mathcal{L}^B, \subseteq\rangle$ and $\langle \mathcal{L}^A, \subseteq\rangle$. Let $f \in \mathcal{L}^B$ and $g \in \mathcal{L}^A$ be two arbitrary $\mathcal{L}$-sets.

$$\uparrow f(a) = \bigwedge_{b\in B} \big(f(b) \to r(b,a)\big)$$
$$\downarrow g(b) = \bigwedge_{a\in A} \big(g(a) \to r(b,a)\big)$$
$$\nearrow f(a) = \bigvee_{b\in B} f(b) \otimes r(b,a)$$
$$\swarrow g(b) = \bigwedge_{a\in A} \big(r(b,a) \to g(a)\big)$$

Lemma 1: Let $\mathcal{L}$ be a complete residuated lattice with adjoint pair $\langle \otimes, \to\rangle$. Let $\langle B, A, r\rangle$ be an $\mathcal{L}$-fuzzy formal context. Then the pair of mappings $\langle \uparrow, \downarrow\rangle$ forms a Galois connection and the pair $\langle \nearrow, \swarrow\rangle$ forms an isotone Galois connection (adjunction) between $\langle \mathcal{L}^B, \subseteq\rangle$ and $\langle \mathcal{L}^A, \subseteq\rangle$, i.e. for all $\mathcal{L}$-fuzzy sets $f \in \mathcal{L}^B$ and $g \in \mathcal{L}^A$ it holds that

1) $f \subseteq \downarrow g \Leftrightarrow g \subseteq \uparrow f$
2) $\nearrow f \subseteq g \Leftrightarrow f \subseteq \swarrow g$.

Definition 4: An *$\mathcal{L}$-fuzzy concept* of an $\mathcal{L}$-context $\mathcal{C} = \langle B, A, r\rangle$ is a pair $\langle f, g\rangle \in L^B \times L^A$ such that $\uparrow f = g$ and $\downarrow g = f$. The first component f is said to be the *extent* of the concept, whereas the second component g is the *intent* of the concept. The set of all L-fuzzy concepts associated to a fuzzy context $\langle B, A, r\rangle$ will be denoted as $L\text{-FCL}(B, A, r)$.

Example 2: Consider the following example of $\mathcal{L}$-context related to the ice-cream preferences of two girls.

	chocolate	vanilla	stracciatella
Ester	1	0	0.5
Lydia	1	0.5	0

Here are some of obtained concepts after applying the derivation operators $\langle \uparrow, \downarrow\rangle$.

- $\{1/\text{Ester}; 0.5/\text{Lydia}\}; \{1/\text{choc.}; 0/\text{vanilla}; 0.5/\text{strac.}\}$
- $\{0.5/\text{Ester}; 1/\text{Lydia}\}; \{1/\text{choc.}; 0.5/\text{vanilla}; 0/\text{strac.}\}$
- $\{1/\text{Ester}; 1/\text{Lydia}\}; \{1/\text{choc.}; 0/\text{vanilla}; 0/\text{strac.}\}$
- $\{0.5/\text{Ester}; 0.5/\text{Lydia}\}; \{1/\text{choc.}; 0.5/\text{vanilla}; 0.5/\text{strac.}\}$
- $\{0/\text{Ester}; 0/\text{Lydia}\}; \{1/\text{choc.}; 1/\text{vanilla}; 1/\text{strac.}\}$

The results can be interpreted in terms of the satisfaction of customers of patisserie. Ester and Lydia are both fully satisfied only with full dose of chocolate ice-cream. If they would obtain lower dose of vanilla ice cream over full dose of chocolate ice cream then Lydia was still fully satisfied but good mood of Ester went little down. Similarly with Lydia and stracciatella ice cream.

Definition 5: Let $\mathcal{L}$ be a complete residuated lattice. The residuated *negation* $\neg$ and residuated *disjunction* $\oplus$ on $\mathcal{L}$ are defined as follows:

- $\neg k = k \to 0$
- $k \oplus m = \neg k \to m$

for any $k, m \in L$.

Lemma 2: If a residuated lattice $\mathcal{L}$ satisfies the law of double negation, i.e. $\neg\neg k = k$ for any $k \in L$ then it also satisfies the following conditions:

1) $l \to k = \neg(k \otimes \neg l)$
2) $\neg(\bigwedge_{i\in I} l_i) = \bigvee_{i\in I} \neg l_i$
3) $l \to k = \neg k \to \neg l$

Remark 2: Note that if the negation $\neg$ satisfies the double negation law on $\mathcal{L}$ then $\oplus$ is a commutative operation, and we can use it as the disjunction operation on $\mathcal{L}$. Specifically, Łukasiewicz implication satisfies the double negation law.

III. Intuitionistic fuzzy sets based on complete residuated lattices

Let us recall the notion of intuitionistic fuzzy set defined on a complete lattice, as introduced in [11].

Definition 6: Given a complete lattice L together with an involutive order reversing operation $N\colon L \to L$, and a universe set E: An intuitionistic L-fuzzy set (IF set) A in E is defined as an object having the form:

$$A = \big\{\langle x, \mu_A(x), \nu_A(x) \mid x \in E\big\}$$

where the functions $\mu_A\colon E \to L$ and $\nu_A\colon E \to L$ define the degree of membership and the degree of non-membership, respectively, to A of the elements $x \in E$, and for every $x \in E$:

$$\mu_A(x) \le N(\nu_A(x))\,.$$

When the previous inequality is strict, there is certain indetermination degree on the knowledge about x.

The IF-lattice associated to a given residuated lattice $\mathcal{L}$ is defined as follows:

Definition 7: Given $\mathcal{L} = \big\langle L, 0, 1, \otimes, \to, \wedge, \vee\big\rangle$ a complete residuated lattice, we can consider the lattice of intuitionistic truth values

$$\mathcal{L}_{\text{IF}} = \big\langle \{\langle k_1, k_2\rangle \in L \times L \mid k_2 \le \neg k_1\}, \le \big\rangle$$

where ordering $\le$ on $\mathcal{L}_{\text{IF}}$ is defined as follows $\langle k_1, k_2\rangle \le \langle m_1, m_2\rangle$ when $k_1 \le m_1$ and $k_2 \ge m_2$.

Note that $\mathcal{L}_{\text{IF}}$ is just the construction of the Pareto ordering [19] taking $\mathcal{L}$ as the underlying set of truth-values instead of the unit interval.

Remark 3: Note that the elements of $\mathcal{L}_{\text{IF}}$ will represent the membership (and non-membership) degrees and, hence, will be denoted in terms of μ and ν whenever necessary, for instance in the statement and proof of Theorem 1.

Example 3: Considering $\mathcal{L} = \langle \{1, 0.5, 0\}, \otimes, \to\rangle$ we have $\mathcal{L}_{\text{IF}} = \langle \{\langle 1, 0\rangle, \langle 0.5, 0\rangle, \langle 0.5, 0.5\rangle, \langle 0, 0\rangle, \langle 0, 0.5\rangle, \langle 0, 1\rangle\}, \le\rangle$. All the pairs $\langle p_1, p_2\rangle$ satisfy $p_2 \le \neg p_1$, where $\neg p_1 = p_1 \to 0$ and, in particular, $\neg 0.5 = 0.5$. A possible interpretation of such a new richer set of truth-values could be as follows:

- $\langle 1, 0\rangle$ absolutely YES, sure, satisfied, full dose, ...
- $\langle 0.5, 0\rangle$ more yes than no, not completely sure, ...
- $\langle 0.5, 0.5\rangle$ user in general agree but has some doubts, ...
- $\langle 0, 0\rangle$ not interested, but not disagree, ...
- $\langle 0, 0.5\rangle$ more no than yes, ...
- $\langle 0, 1\rangle$ absolutely NO, ...

Definition 8: Let $\mathcal{L}$ be a complete residuated lattice and consider its associated lattice of truth degrees of intuitionistic

$\rightrightarrows$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$
$\langle 0.5,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$
$\langle 0.5,0.5\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$
$\langle 0,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0\rangle$
$\langle 0,0.5\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$
$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$

Fig. 1. Truth-table of $\rightrightarrows$ for the examples.

fuzzy sets $\mathcal{L}_{\mathrm{IF}}$. We define the operator $\rightrightarrows\colon \mathcal{L}_{\mathrm{IF}}\times\mathcal{L}_{\mathrm{IF}}\to\mathcal{L}_{\mathrm{IF}}$ as follows

$$\langle p_1,p_2\rangle \rightrightarrows \langle q_1,q_2\rangle = \langle p_2\oplus q_1, p_1\otimes q_2\rangle$$

Lemma 3: $\rightrightarrows$ is well-defined.

Proof: We have to check that $p_1\otimes q_2\le\neg(p_2\oplus q_1)$. Since $\langle p_1,p_2\rangle,\langle q_1,q_2\rangle\in\mathcal{L}_{\mathrm{IF}}$ we have that $p_1\le\neg p_2$ and $q_1\le\neg q_2$, thus we can write

$$\begin{aligned} p_1\otimes q_2 &\le \neg p_2\otimes\neg q_1 = \neg\neg(\neg p_2\otimes\neg q_1)\\ &= \neg(\neg p_2\to q_1) = \neg(p_2\oplus q_1)\end{aligned}$$

■

The definition of $\rightrightarrows$ is inspired by two equivalences well known in classical logic

- $(A\Rightarrow B)\Leftrightarrow(\neg A\vee B)$
- $\neg(A\Rightarrow B)\Leftrightarrow(A\wedge\neg B)$

where instead of negation of some membership degree is used false part of the degree. The second component is inspired by the negation of classical implication.

The truth-table of $\rightrightarrows$ on our running example is given in Figure 1.

The following remarks are in order now:

1) $\langle 1,0\rangle\rightrightarrows\langle p_1,p_2\rangle=\langle p_1,p_2\rangle$ to any $\langle p_1,p_2\rangle\in\mathcal{L}_{\mathrm{IF}}$
2) $\langle p_1,p_2\rangle\rightrightarrows\langle 0,1\rangle=\langle p_2,p_1\rangle=\neg\langle p_1,p_2\rangle$
3) if $\langle p_1,p_2\rangle\le\langle q_1,q_2\rangle$ then $\langle p_1,p_2\rangle\rightrightarrows\langle q_1,q_2\rangle=\langle r_1,r_2\rangle$ where $r_1\ge r_2$.
4) $\bigwedge_{i\in I}(\langle k_{1i},k_{2i}\rangle\rightrightarrows\langle m_{1i},m_{2i}\rangle)=$

$$\begin{aligned} &=\bigwedge_{i\in I}\langle k_{2i}\oplus m_{1i}, k_{1i}\otimes m_{2i}\rangle\\ &=\left\langle\bigwedge_{i\in I}(k_{2i}\oplus m_{1i}),\bigvee_{i\in I}(k_{1i}\otimes m_{2i})\right\rangle\end{aligned}$$

IV. IF-FCA WITH ANTITONE CONCEPT FORMING OPERATORS

To begin with, the notion of IF-formal context is given.

Definition 9: Let $\mathcal{L}$ be a complete residuated lattice and $\mathcal{L}_{\mathrm{IF}}$ be its associated lattice of intuitionistic degrees. A triple $\langle B,A,r\rangle$ where $r\colon B\times A\to\mathcal{L}_{\mathrm{IF}}$ is said to be an *IF-formal context*.

We are now in position to define the first pair of concept forming operators associated to an IF-formal context.

Definition 10: Given an IF-formal context $\langle B,A,r\rangle$ be the, we define a pair of mappings $\langle\upuparrows,\downdownarrows\rangle$ between the intuitionistic $\mathcal{L}_{\mathrm{IF}}$-fuzzy powersets $\langle{\mathcal{L}_{\mathrm{IF}}}^B,\subseteq\rangle$ and $\langle{\mathcal{L}_{\mathrm{IF}}}^A,\subseteq\rangle$ as follows

- $\upuparrows f(a)=\bigwedge_{b\in B}(f(b)\rightrightarrows r(b,a))$, for any $f\in{\mathcal{L}_{\mathrm{IF}}}^B$
- $\downdownarrows g(b)=\bigwedge_{a\in A}(g(a)\rightrightarrows r(b,a))$, for any $g\in{\mathcal{L}_{\mathrm{IF}}}^A$.

The pair of mappings $\langle\upuparrows,\downdownarrows\rangle$ are the *concept forming operators* for the IF-formal context $\langle B,A,r\rangle$.

Following the construction in the non-intuitionistic case, one would expect that the two previous operators form a Galois connection. This is not the case in general; however, assuming the extra assumption that the incidence relation r should assign values (p,q) for which $q=\neg p$. The formal statement is given and proved below (where we make use of μ and ν following the notational convention of Remark 3).

Theorem 1: Let $\mathcal{L}$ be a complete residuated lattice and $\mathcal{L}_{\mathrm{IF}}$ its associated lattice of intuitionistic degrees. Let $\langle B,A,r\rangle$ be an IF-formal context where $\nu\big(r(b,a)\big)=\neg\mu(r(b,a))$ for any $(b,a)\in B\times A$. Then $\langle\upuparrows,\downdownarrows\rangle$ forms a Galois connection between powersets $\langle\mathcal{L}_{\mathrm{IF}}^B,\subseteq\rangle$ and $\langle\mathcal{L}_{\mathrm{IF}}^A,\subseteq\rangle$.

Proof: We have to prove the following equivalence

$$f\subseteq\downdownarrows g\quad\Leftrightarrow\quad g\subseteq\upuparrows f$$

for arbitrary intuitionistic fuzzy sets $f\in\mathcal{L}_{\mathrm{IF}}^B$ and $g\in\mathcal{L}_{\mathrm{IF}}^A$.

Firstly, assume $f\subseteq\downdownarrows g$. This means that, for any $b\in B$, it holds

$$f(b)\le\downdownarrows g(b)=\bigwedge_{a\in A}\big(g(a)\rightrightarrows r(b,a)\big)$$

By the previous definitions, this means that the two following inequalities hold:

$$\mu\big(f(b)\big)\le\bigwedge_{a\in A}\Big(\nu\big(g(a)\big)\oplus\mu\big(r(b,a)\big)\Big)\qquad(1)$$

$$\nu\big(f(b)\big)\ge\bigvee_{a\in A}\Big(\mu\big(g(a)\big)\otimes\nu\big(r(b,a)\big)\Big)\qquad(2)$$

In the rest of the proof we will obtain an equivalent expression of the previous inequalities which will turn out to prove the intended inclusion, namely, $g\subseteq\upuparrows f$. The details are the following:

(1) Unfolding the definition of the operator $\oplus$ we obtain

$$\mu\big(f(b)\big)\le\bigwedge_{a\in A}\big(\neg\nu\big(g(a)\big)\to\mu\big(r(b,a)\big)\big)$$

for all $b\in B$ which, by Remark 1, is equivalent to

$$\neg\nu(g(a))\le\bigwedge_{b\in B}\big(\mu\big(f(b)\big)\to\mu(r(b,a))\big)$$

for all $a \in A$. Hence

$$\begin{aligned}\neg\nu\big(g(a)\big) &\leq \bigwedge_{b\in B}\Big(\mu\big(f(b)\big)\to\mu\big(r(b,a)\big)\Big)\\ &= \bigwedge_{b\in B}\Big(\mu\big(f(b)\big)\to\neg\nu\big(r(b,a)\big)\Big)\\ &= \bigwedge_{b\in B}\Big(\mu\big(f(b)\big)\to\big(\nu(r(b,a))\to 0\big)\Big)\\ &= \bigwedge_{b\in B}\Big(\big(\mu\big(f(b)\big)\otimes\nu\big(r(b,a)\big)\to 0\big)\Big)\\ &= \Big(\bigvee_{b\in B}\big(\mu\big(f(b)\big)\otimes\nu\big(r(b,a)\big)\to 0\big)\Big)\\ &= \neg\bigvee_{b\in B}\mu\big(f(b)\big)\otimes\nu\big(r(b,a)\big)\end{aligned}$$

From the antitonicity of negation we finally obtain

$$\nu(g(a)) \geq \bigvee_{b\in B}\mu\big(f(b)\big)\otimes\nu\big(r(b,a)\big)$$

(2) By Remark 1, this inequality is equivalent to

$$\mu\big(g(a)\big) \leq \bigwedge_{b\in B}\Big(\nu\big(r(b,a)\big)\to\nu\big(f(b)\big)\Big) \text{ for all } a\in A.$$

Therefore, we have

$$\begin{aligned}\mu\big(g(a)\big) &\leq \bigwedge_{b\in B}\Big(\nu\big(r(b,a)\big)\to\nu\big(f(b)\big)\Big)\\ &= \bigwedge_{b\in B}\Big(\neg\nu\big(f(b)\big)\to\neg\nu\big(r(b,a)\big)\Big)\\ &= \bigwedge_{b\in B}\Big(\neg\nu\big(f(b)\big)\to\mu\big(r(b,a)\big)\Big)\\ &= \bigwedge_{b\in B}\Big(\nu\big(f(b)\big)\oplus\mu\big(r(b,a)\big)\Big)\end{aligned}$$

As a result, we obtain that

$$g(a) \leq \bigwedge_{b\in B}\big(f(b)\rightrightarrows r(b,a)\big)$$

holds for all $a \in A$, i.e. $g \subseteq \upuparrows f$ holds.

As all the steps in the previous chains are reversible, the converse implication is already proven. ■

We show below that the equality $\mu = \neg\nu$ cannot be dropped from the statement of the previous theorem.

Example 4: Consider the following IF-formal context

$$\big\langle\{b\},\{a\},\{\langle 0,0\rangle/(b,a)\}\big\rangle$$

for which, obviously, we have that $\mu(\langle 0,0\rangle) \neq \neg\nu(\langle 0,0\rangle)$.

Consider the following equality

$$\begin{aligned}\upuparrows\downdownarrows\big(\{\langle 1,0\rangle/b\}\big)(b) &= \big(\langle 1,0\rangle\rightrightarrows\langle 0,0\rangle\big)\rightrightarrows\langle 0,0\rangle\\ &= \langle 0,0\rangle\rightrightarrows\langle 0,0\rangle = \langle 0,0\rangle\end{aligned}$$

As a result, the composition $\upuparrows\downdownarrows$ (and also $\downdownarrows\upuparrows$) is not a closure operator since it is not extensive. Hence the pair of mappings $\langle\upuparrows,\downdownarrows\rangle$ cannot be a Galois connection between complete lattices of intuitionistic fuzzy powersets of the set of objects and attributes of the input formal context.

The mere consideration of the intuitionistic lattice $\mathcal{L}_{\text{IF}}$ instead of $\mathcal{L}$, provides more expressiveness to the obtained concept lattice.

Example 5: By considering an $\mathcal{L}_{\text{IF}}$-based formal context in our previous example, replacing values $1, 0.5, 0$ by $\langle 1,0\rangle, \langle 0.5,0.5\rangle, \langle 0,1\rangle$ one can notice that the behaviour of such intuitionistic values are the same as the original values with respect to implication $\to$ in $\mathcal{L}$, as can be seen using the table of $\rightrightarrows$ (see Fig. 1). But, after applying the new concept forming operators some new $\mathcal{L}_{\text{IF}}$-concepts are obtained. $\langle 0,0\rangle$ can be interpreted on the standpoint of customers as to be not interested, whereas from the ice cream standpoint should be read more likely as "I don't know how much I'll obtain".

- The pair with extent $\{\text{Ester}/\langle 0,0\rangle;\text{Lydia}/\langle 0,0\rangle\}$, and intent $\{\text{choco.}/\langle 1,0\rangle,\text{vanilla}/\langle 0,0\rangle,\text{strac.}/\langle 0,0\rangle\}$ should be interpreted as Ester and Lydia would not be interested in any ice cream portion where is no certainty of how much unpopular ice cream they will obtain.

V. IF-FCA WITH ISOTONE CONCEPT FORMING OPERATORS

We now focus our attention to a suitable definition of concepts based on the isotone concept forming operators.

Definition 11: Let $\mathcal{L}$ be a complete residuated lattice and consider its associated lattice of intuitionistic truth degrees $\mathcal{L}_{\text{IF}}$. The operator $\boxtimes : \mathcal{L}_{\text{IF}} \times \mathcal{L}_{\text{IF}} \to \mathcal{L}_{\text{IF}}$ is defined as follows

$$\langle p_1,p_2\rangle\boxtimes\langle q_1,q_2\rangle = \big\langle p_1\otimes\neg q_2, (\neg p_1\oplus q_2)\wedge(p_2\oplus\neg q_1)\big\rangle.$$

Lemma 4: $\boxtimes$ is well-defined.

Proof: We have to check that

$$(\neg p_1\oplus q_2)\wedge(p_2\oplus\neg q_1)\leq\neg(p_1\otimes\neg q_2)$$

This follows easily from the facts that

$$\neg p_1\oplus q_2 = p_1\to q_2 = \neg(p_1\otimes\neg q_2)$$

■

Remark 4: It is worth to note that, in fact, $\boxtimes$: $\mathcal{L}\times\mathcal{L}\to\mathcal{L}_{\text{IF}}$ since the proof does not use that the arguments were in $\mathcal{L}_{\text{IF}}$ as in the case of $\rightrightarrows$.

Lemma 5: Let $\mathcal{L}$ be a complete residuated lattice and consider its associated lattice of intuitionistic truth degrees $\mathcal{L}_{\text{IF}}$. The pair of operations $\langle\boxtimes,\rightrightarrows\rangle$ forms an adjoint pair on $\mathcal{L}_{\text{IF}}$, i.e. for any three values $\langle p_1,p_2\rangle,\langle q_1,q_2\rangle,\langle r_1,r_2\rangle \in \mathcal{L}_{\text{IF}}$ it holds that

$$\langle p_1,p_2\rangle\boxtimes\langle q_1,q_2\rangle\leq\langle r_1,r_2\rangle$$

if and only if

$$\langle p_1,p_2\rangle\leq\langle q_1,q_2\rangle\rightrightarrows\langle r_1,r_2\rangle.$$

Proof: Assume $\langle p_1,p_2\rangle\boxtimes\langle q_1,q_2\rangle\leq\langle r_1,r_2\rangle$. Unfolding the definition of $\boxtimes$ and the order between intuitionistic truth-values we obtain two inequalities:

1) $p_1\otimes\neg q_2\leq r_1$.
   Equivalently, we have $p_1\leq\neg q_2\to r_1 = q_2\oplus r_1$.

$\boxtimes$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0\rangle$	$\langle 1,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,1\rangle$
$\langle 0.5,0\rangle$	$\langle 0.5,0\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,0.5\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0.5,0.5\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0.5,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 0,0.5\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,0.5\rangle$	$\langle 0,0.5\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$

Fig. 2. Truth-table of $\boxtimes$ for the examples.

2) $r_2 \leq (\neg p_1 \oplus q_2) \wedge (p_2 \oplus \neg q_1) \leq$

$$\begin{aligned} &\leq p_2 \oplus \neg q_1 \\ &= \neg p_2 \to \neg q_1 = q_1 \to p_2 \end{aligned}$$

which is equivalent to $q_1 \otimes r_2 \leq p_2$.

Hence $\langle p_1, p_2\rangle \leq \langle q_2 \oplus r_1, q_1 \otimes r_2\rangle = \langle q_1, q_2\rangle \rightrightarrows \langle r_1, r_2\rangle$.

Conversely, assuming $\langle p_1, p_2\rangle \leq \langle q_1, q_2\rangle \rightrightarrows \langle r_1, r_2\rangle$ we have:

1′) $p_1 \leq q_2 \oplus r_1$ which, by reversing the previous steps lead to $p_1 \otimes \neg q_2 \leq r_1$.

2′) $q_1 \otimes r_2 \leq p_2$ which, by reversing the previous steps lead to $r_2 \leq p_2 \oplus \neg q_1$.

To finish with, we have just to show that $r_2 \leq \neg p_1 \oplus q_2$ holds as well or, equivalently, $p_1 \otimes r_2 \leq q_2$. We have

$$p_1 \otimes r_2 \overset{(\star)}{\leq} p_1 \otimes \neg r_1 \overset{(*)}{\leq} q_2$$

where $(\star)$ follows from $\langle r_1, r_2\rangle \in \mathcal{L}_{\mathrm{IF}}$, i.e. $r_2 \leq \neg r_1$, and $(*)$ follows from $p_1 \otimes \neg r_1 \leq q_2$, which is a consequence of 1′) above. ■

Definition 12: Let $\mathcal{L}$ be a complete residuated lattice and consider its associated lattice of intuitionistic truth degrees $\mathcal{L}_{\mathrm{IF}}$. Let $\langle B, A, r\rangle$ be an IF-formal context and define a pair of mappings $\langle \looparrowright, \looparrowleft\rangle$ between intuitionistic $\mathcal{L}_{\mathrm{IF}}$-fuzzy powersets $\langle {\mathcal{L}_{\mathrm{IF}}}^B, \subseteq\rangle$ and $\langle {\mathcal{L}_{\mathrm{IF}}}^A, \subseteq\rangle$ as follows:

- $\looparrowright f(a) = \bigvee_{b\in B} \big(r(b,a) \boxtimes f(b)\big)$, for any $f \in {\mathcal{L}_{\mathrm{IF}}}^B$
- $\looparrowleft g(b) = \bigwedge_{a\in A} \big(r(b,a) \rightrightarrows g(a)\big)$, for any $g \in {\mathcal{L}_{\mathrm{IF}}}^A$.

Theorem 2: The pair $\langle \looparrowright, \looparrowleft\rangle$ forms an isotone Galois connection between complete lattices of powersets $\langle {\mathcal{L}_{\mathrm{IF}}}^B, \subseteq\rangle$ and $\langle {\mathcal{L}_{\mathrm{IF}}}^A, \subseteq\rangle$.

Proof: Follows directly from Lemma 5. ■

It is remarkable that, contrary to Theorem 1, no precondition $\nu\big(r(b,a)\big) = \neg\mu(r(b,a))$ for all $(b,a) \in B \times A$.

The fact that $\boxtimes$ is, in general, noncommutative suggest the existence of two residuals depending on which argument is fixed. We introduce this second construction as follows.

Definition 13: Let $\mathcal{L}$ be a complete residuated lattice and consider its associated lattice of intuitionistic truth degrees $\mathcal{L}_{\mathrm{IF}}$, and consider the following implication

$$\langle p_1, p_2\rangle \rightrightarrows_2 \langle q_1, q_2\rangle = \langle (\neg p_1 \oplus q_1) \wedge (p_2 \oplus \neg q_2), p_1 \otimes \neg q_1\rangle$$

defined for all $\langle p_1, p_2\rangle, \langle q_1, q_2\rangle \in \mathcal{L}_{\mathrm{IF}}$.

It is not difficult to check that the output of $\rightrightarrows_2$ is in $\mathcal{L}_{\mathrm{IF}}$.

Lemma 6: $\rightrightarrows_2$ is well-defined.

Proof: We have to check that

$$p_1 \otimes \neg q_1 \leq \neg\big[(\neg p_1 \oplus q_1) \wedge (p_2 \oplus \neg q_2)\big]$$

but this is straightforward since

$$\begin{aligned} \neg\big[(\neg p_1 \oplus q_1) \wedge (p_2 \oplus \neg q_2)\big] &= \neg(\neg p_1 \oplus q_1) \vee \neg(p_2 \oplus \neg q_2) \\ &= (p_1 \otimes \neg q_1) \vee \neg(p_2 \oplus \neg q_2)\,. \end{aligned}$$

■

Theorem 3: $\langle \boxtimes, \rightrightarrows_2\rangle$ is an adjoint pair.

Proof: We have just to check that

$$\langle p_1, p_2\rangle \boxtimes \langle q_1, q_2\rangle \leq \langle r_1, r_2\rangle$$

if and only if

$$\langle q_1, q_2\rangle \leq \langle p_1, p_2\rangle \rightrightarrows_2 \langle r_1, r_2\rangle.$$

Hence, assume $\langle p_1, p_2\rangle \boxtimes_1 \langle q_1, q_2\rangle \leq \langle r_1, r_2\rangle$. This means that the two following inequalities hold:

- $p_1 \otimes \neg q_2 \leq r_1$. This is equivalent to

$$\neg q_2 \leq p_1 \to r_1 \tag{3}$$

$$q_2 \geq \neg(p_1 \to r_1) = p_1 \otimes \neg r_1 \tag{4}$$

- $r_2 \leq (\neg p_1 \oplus q_2) \wedge (p_2 \oplus \neg q_1)$. In particular, we have

$$\begin{aligned} r_2 \leq p_2 \oplus \neg q_1 = \neg p_2 \to \neg q_1 &\iff \neg p_2 \otimes r_2 \leq \neg q_1 \\ &\iff q_1 \leq p_2 \oplus \neg r_2 \end{aligned}$$

Now, from (3), and $q_1 \leq \neg q_2$ we obtain $q_1 \leq \neg p_1 \oplus r_1$ which, together with the previous result leads to

$$q_1 \leq (\neg p_1 \oplus r_1) \wedge (p_2 \oplus \neg r_2) \tag{5}$$

From (4) and (5) we get $\langle q_1, q_2\rangle \leq \langle p_1, p_2\rangle \rightrightarrows_2 \langle r_1, r_2\rangle$.

The proof of the converse implication is essentially the same as above. ■

VI. Conclusion

We have introduced two possible constructions of the concept-forming operators with intuitionistic L-fuzzy sets. In order to get a Galois connection in the antitone case, the IF-formal context has to provide values without indetermination, i.e. $\mu(x) = \neg(\mu(x))$, which are essentially equivalent to (usual) L-fuzzy sets. As a result, not much is gained in this first approach except the following question: is it possible to relax this restriction or, equivalently, isolate the causes which enable us to build counterexamples?

The second construction is much more interesting, in that it is based on a non-commutative operator $\boxtimes$ which, on a purely IF-framework, leads to two different residuals. As future work, it will be convenient to develop a thorough study of the full structure provided by the given constructions (in the sense of further properties of $\boxtimes$, $\rightrightarrows$ and $\rightrightarrows_2$).

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