An adjoint pair for intuitionistic *L*-fuzzy values

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Abstract. We continue our prospective study of the generalization of formal concept analysis in terms of intuitionistic L-fuzzy sets. The main contribution here is an adjoint pair in the set $\mathcal{L}_{\text{ILF}}$ of intuitionistic L-fuzzy values associated to a complete residuated lattice $\mathcal{L}$, which allows the definition of a pair of derivation operators which form an antitone Galois connection.

Keywords: formal concept analysis, complete residuated lattice, intuitionistic fuzzy sets

1 Introduction

In this work, we continue our study of the extension of Formal Concept Analysis (FCA) to the so-called intuitionistic fuzzy sets (IF-sets), introduced in [1] by considering for all element x a membership degree $\mu(x)$ together with a non-membership degree $\nu(x)$ such that $\mu(x) + \nu(x) \leq 1$, somehow allowing an *indetermination degree* about x in the case of strict inequality. This construction was later generalized when allowing a complete residuated lattice instead of the unit interval as underlying set of truth-values [2, 5]. Although some authors have already introduced intuitionistic extensions of FCA (for instance [10, 12] or [11]), all of them are based on the unit interval.

In [7], we introduced for the first time a definition of concept-forming operators purely based on intuitionistic L-fuzzy (ILF) sets valued on a complete residuated lattice. In order to get a Galois connection in the antitone case, the ILF-formal context had to provide values without indetermination, i.e. $\mu(x) = \neg(\mu(x))$, which are essentially equivalent to (usual) L-fuzzy sets. Then in [8] an alternative approach was presented, in terms of isotone Galois connection and an adjoint triple.

In this paper, we construct an adjoint pair in order to generate (by standard means) a Galois connection in the set of intuitionistic L-fuzzy sets which, contrariwise to [7], need not be indetermination-free.

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2 Preliminary definitions

As stated above, we will be primary dealing with truth-values not necessarily belonging to the unit interval, but to a complete residuated lattice (see [6] for further details).

Definition 1. *An algebra* $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, \otimes, \rightarrow \rangle$ *is said to be a* complete residuated lattice *if*

1. $\langle L, \wedge, \vee, 0, 1 \rangle$ *is a complete lattice where* 0 *and* 1 *are the bottom and top elements (resp.).*
2. $\langle L, \otimes, 1 \rangle$ *is a commutative monoid.*
3. $\langle \otimes, \rightarrow \rangle$ *is an adjoint pair, i.e.* $k \otimes m \leq n$ *if and only if* $k \leq m \rightarrow n$*, for all* $k, m, n \in L$*, where* $\leq$ *is the ordering generated by* $\wedge$ *and* $\vee$*.*

Let us recall the notion of intuitionistic fuzzy set defined on a complete lattice, as introduced in [2].

Definition 2. *Given a complete lattice* L *together with an involutive order reversing operation* $N\colon L \rightarrow L$*, and a universe set* E*: An intuitionistic* L*-fuzzy set (ILF set)* A *in* E *is defined as an object having the form:*

$$A = \{\langle \mu_A(x), \nu_A(x) \rangle / x \mid x \in E\}$$

where the functions $\mu_A\colon E \rightarrow L$ *and* $\nu_A\colon E \rightarrow L$ *define the degree of membership and the degree of non-membership, respectively, to* A *of the elements* $x \in E$*, and for every* $x \in E$*:*

$$\mu_A(x) \leq N(\nu_A(x))\,.$$

When the previous inequality is strict, there is a certain indetermination degree on the knowledge about x.

Note that, when the underlying lattice is residuated, we already have a negation operator defined by $\neg x = x \rightarrow 0$. As a result, we can define the ILF-lattice associated with a given residuated lattice $\mathcal{L}$ as follows:

Definition 3. *Given a complete residuated lattice* $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, \otimes, \rightarrow \rangle$*, we can consider the lattice of intuitionistic truth values*

$$\mathcal{L}_{\rm ILF} = \langle \{\langle k_1, k_2 \rangle \in L \times L \mid k_2 \leq \neg k_1\}, \leq \rangle$$

where ordering $\leq$ *on* $\mathcal{L}_{\rm ILF}$ *is defined as follows* $\langle k_1, k_2 \rangle \leq \langle m_1, m_2 \rangle$ *when* $k_1 \leq m_1$ *and* $k_2 \geq m_2$*.*

Note that $\mathcal{L}_{\rm ILF}$ is just the construction of the Pareto ordering, as used in [4], considering $\mathcal{L}$ as the underlying set of truth-values instead of the unit interval. Consider also the following notation for any element of $\mathcal{L}_{\rm ILF}$ as follows $\overline{a} = \langle a_1, a_2 \rangle$.

Lemma 1. $\langle \mathcal{L}_{\mathrm{ILF}}, \leq \rangle$ *forms a complete lattice in which the meet and join are defined by*

$$\bigwedge_{i\in I} \overline{a_i} = \Big\langle \bigwedge_{i\in I} a_{i1}; \bigvee_{i\in I} a_{i2} \Big\rangle \qquad \bigvee_{i\in I} \overline{a_i} = \Big\langle \bigvee_{i\in I} a_{i1}; \bigwedge_{i\in I} a_{i2} \Big\rangle$$

Proof. It is enough to check that the above defined meet and join actually are elements of $\mathcal{L}_{\mathrm{ILF}}$, since the rest is straightforward.

Given $\{\overline{a_i} \mid i \in I\} \subseteq \mathcal{L}_{\mathrm{ILF}}$, recall that for any $\overline{a_i} \in \mathcal{L}_{\mathrm{ILF}}$ it holds that $a_{i2} \leq \neg a_{i1}$. Hence $\bigwedge_{i\in I} a_{i2} \leq \bigwedge_{i\in I} \neg a_{i1} = \bigwedge_{i\in I}(a_{i1} \to 0) = (\bigvee_{i\in I} a_{i1} \to 0) = \neg \bigvee_{i\in I} a_{i1}$.

On the other hand, we also have that $a_{i1} \leq \neg a_{i2}$ for all $i \in I$. Hence $\bigwedge_{i\in I} a_{i1} \leq \bigwedge_{i\in I} \neg a_{i2} = \neg \bigvee_{i\in I} a_{i2}$, which is equivalent to $\bigvee_{i\in I} a_{i2} \leq \neg \bigwedge_{i\in I} a_{i1}$. □

The definition of the conjunctor in $\mathcal{L}_{\mathrm{ILF}}$ (to be introduced in the next section) will make use of the following operator:

Definition 4. *The operator* $\oplus \colon L \times L \to L$ *is defined by*

$$a \oplus b = \neg a \to b = (a \to 0) \to b.$$

Assuming an involutive negation, it is not difficult to check the De Morgan laws between $\otimes$ and $\oplus$, contraposition, and associativity and commutativity of $\oplus$:

Lemma 2. *The following equalities hold*

$$\neg(a \otimes b) = \neg a \oplus \neg b \qquad \neg(a \oplus b) = \neg a \otimes \neg b \qquad a \to b = \neg b \to \neg a$$

Proof. It is straightforward checking; note that double negation is only used in the second and third equalities.

$$\begin{aligned}\neg(a \otimes b) &= (a \otimes b) \to 0 = a \to (b \to 0)\\ &= a \to \neg b = \neg a \oplus \neg b\\ \neg(a \oplus b) &= \neg(\neg\neg a \oplus \neg\neg b) = \neg\neg(\neg a \otimes \neg b)\\ &= \neg a \otimes \neg b\end{aligned}$$

$$\begin{aligned}\neg b \to \neg a &= (b \to 0) \to (a \to 0) = ((b \to 0) \otimes a) \to 0\\ &= (a \otimes (b \to 0)) \to 0 = a \to ((b \to 0) \to 0)\\ &= a \to \neg\neg b = a \to b\end{aligned}$$

□

If we think of $a \to b$ as $\neg a \oplus b$, then it is easy to see that $\neg(a \to b) = (a \otimes \neg b)$.

Lemma 3. *Let* $\mathcal{L}$ *be a complete residuated lattice endowed with an involutive negation (i.e.* $\neg\neg a = a$*). Then* $\oplus$ *is commutative and associative.*

Proof. Firstly,

From $a \to b = \neg b \to \neg a$ we obtain commutativity of $\oplus$

$$a \oplus b = \neg a \to b = \neg b \to a = b \oplus a.$$

Associativity is straightforward

$$\begin{aligned}(a \oplus b) \oplus c &= \neg(a \oplus c) \to c = (\neg a \otimes \neg b) \to c \\ &= \neg a \to (\neg b \to c) = \neg a \to (b \oplus c) = a \oplus (b \oplus c)\end{aligned}$$

□

Hereafter we will assume that $\mathcal{L}$ satisfies the double negation law.

3 The complete residuated lattice $\mathcal{L}_{\text{ILF}}$

We will define an intuitionistic conjunctor on $\mathcal{L}_{\text{ILF}}$ with the help of the operators $\otimes$ and $\oplus$.

Definition 5. *Let $\mathcal{L}_{\text{ILF}}$ be the ILF-lattice associated to a residuated lattice $\mathcal{L}$. We define two binary operations on $\mathcal{L}_{\text{ILF}}$ by*

$$\begin{aligned}\langle a_1; a_2\rangle \boxtimes \langle b_1; b_2\rangle &= \langle a_1 \otimes b_1; a_2 \oplus b_2\rangle \\ \langle a_1; a_2\rangle \rightrightarrows \langle b_1; b_2\rangle &= \langle (a_1 \to b_1) \wedge (\neg a_2 \to \neg b_2); (\neg a_2 \otimes b_2)\rangle\end{aligned}$$

for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in \mathcal{L}_{\text{ILF}}$.

The following lemma shows that both operations are well defined. Formally,

Lemma 4. $\boxtimes$ *and* $\rightrightarrows$ *are internal binary operations in $\mathcal{L}_{\text{ILF}}$.*

Proof. We have just to check the condition for belonging to $\mathcal{L}_{\text{ILF}}$, namely, the non-membership degree is less or equal than the negation of the membership degree. In the following chain of equalities we will use the De Morgan laws from Lemma 2.

1. $a_2 \le \neg a_1$ and $b_2 \le \neg b_1$. Hence because of the monotonicity of $\oplus$ we have $a_2 \oplus b_2 \le \neg a_1 \oplus \neg b_1 = \neg(a_1 \otimes b_1)$.
2. $\neg(\neg b_2 \otimes c_2) = b_2 \oplus \neg c_2 = \neg b_2 \to \neg c_2 \ge (\neg b_2 \to \neg c_2) \wedge (b_1 \to c_1)$. Hence $\neg b_2 \otimes c_2 \le \neg((\neg b_2 \to \neg c_2) \wedge (b_1 \to c_1))$. □

We can now state and prove the main contribution of this work.

Theorem 1. $\langle \mathcal{L}_{\text{ILF}}, \langle 1, 0\rangle, \langle 0, 1\rangle, \boxtimes, \rightrightarrows\rangle$ *is complete residuated lattice.*

Proof. Firstly, $\mathcal{L}_{\text{ILF}}$ is a complete lattice, by Lemma 1.

$\langle \mathcal{L}_{\text{ILF}}, \boxtimes, \langle 1, 0\rangle\rangle$ forms a commutative monoid. This is straightforward, by Lemma 3 and the definition of $\boxtimes$.

Finally, let us know prove that $\langle \boxtimes, \rightrightarrows\rangle$ is an adjoint pair on $\mathcal{L}_{\text{ILF}}$ which, in our case, means the following:

$$\langle a_1 \otimes b_1, a_2 \oplus b_2\rangle \le \langle c_1, c_2\rangle \Longleftrightarrow \langle a_1, a_2\rangle \le \langle (b_1 \to c_1) \wedge (\neg b_2 \to \neg c_2), \neg b_2 \otimes c_2\rangle$$

⇒: Let us assume that $\langle a_1 \otimes b_1, a_2 \oplus b_2\rangle \leq \langle c_1, c_2\rangle$.
From the second component we have that $a_2 \oplus b_2 \geq c_2$ but $a_2 \oplus b_2 = \neg b_2 \to a_2 \geq c_2$, and that is equivalent to $a_2 \geq \neg b_2 \otimes c_2$.
From the first component we have $a_1 \otimes b_1 \leq c_1$, which is equivalent to $a_1 \leq b_1 \to c_1$. Moreover, using $\langle a_1, a_2\rangle \in \mathcal{L}_{\mathrm{ILF}}$ and the previous inequality, we obtain $a_1 \leq \neg a_2 \leq \neg(\neg b_2 \otimes c_2) = \neg\neg b_2 \oplus \neg c_2 = \neg b_2 \to \neg c_2$. Hence, $a_1 \leq (b_1 \to c_1) \wedge (\neg b_2 \to \neg c_2)$.
As a result, we obtain

$$\langle a_1, a_2\rangle \leq \langle (b_1 \to c_1) \wedge (\neg b_2 \to \neg c_2), \neg b_2 \otimes c_2\rangle$$

⇐: Conversely, let us assume that $\langle a_1, a_2\rangle \leq \langle (b_1 \to c_1) \wedge (\neg b_2 \to \neg c_2), \neg b_2 \otimes c_2\rangle$.
From the first component we obtain $a_1 \leq (b_1 \to c_1) \wedge (\neg b_2 \to \neg c_2) \leq b_1 \to c_1$, which is equivalent to $a_1 \otimes b_1 \leq c_1$.
From the second component we have $a_2 \geq \neg b_2 \otimes c_2$, which is equivalent to $\neg b_2 \to a_2 = a_2 \oplus b_2 \geq c_2$. Hence

$$\langle a_1 \otimes b_1, a_2 \oplus b_2\rangle \leq \langle c_1, c_2\rangle.$$

□

4 Antitonic ILF Formal Concept Analysis

Theorem 1 is the key to build a consistent version of formal concept analysis interpreted on $\mathcal{L}_{\mathrm{ILF}}$. To begin with, the notion of ILF-formal context is given as follows:

Definition 6. *Let $\mathcal{L}$ be a complete residuated lattice and $\mathcal{L}_{\mathrm{ILF}}$ be its associated lattice of ILF degrees. A triple $\langle B, A, r\rangle$, where $r\colon B \times A \to \mathcal{L}_{\mathrm{ILF}}$, is said to be an* ILF-formal context.

The definition of the concept-forming operators associated with an ILF-formal context is introduced in the standard way in terms of $\rightrightarrows$.

Definition 7. *Let $\mathcal{L}$ be a complete residuated lattice and let $\mathcal{L}_{\mathrm{ILF}}$ be its associated lattice of ILF values. Given an ILF-formal context $\langle B, A, r\rangle$, we define a pair of mappings $\langle \upuparrows, \downdownarrows\rangle$ between the intuitionistic $\mathcal{L}_{\mathrm{ILF}}$-fuzzy powersets $\langle {\mathcal{L}_{\mathrm{ILF}}}^B, \subseteq\rangle$ and $\langle {\mathcal{L}_{\mathrm{ILF}}}^A, \subseteq\rangle$ as follows*

a) $\upuparrows f(a) = \bigwedge_{b\in B}(f(b) \rightrightarrows r(b,a))$, *for all* $f \in {\mathcal{L}_{\mathrm{ILF}}}^B$
b) $\downdownarrows g(b) = \bigwedge_{a\in A}(g(a) \rightrightarrows r(b,a))$, *for all* $g \in {\mathcal{L}_{\mathrm{ILF}}}^A$.

The pair of mappings $\langle \upuparrows, \downdownarrows\rangle$ are the concept forming operators *for the IF-formal context $\langle B, A, r\rangle$.*

Theorem 2. *Let $\mathcal{L}$ be a complete residuated lattice and $\mathcal{L}_{\mathrm{ILF}}$ its associated lattice of intuitionistic degrees. Let $\langle B, A, r\rangle$ be an IF-formal context. Then $\langle \upuparrows, \downdownarrows\rangle$ forms a Galois connection between powersets $\langle \mathcal{L}^B_{\mathrm{ILF}}, \subseteq\rangle$ and $\langle \mathcal{L}^A_{\mathrm{ILF}}, \subseteq\rangle$.*

Proof. Follows from Theorem 1 and the standard construction on a complete residuated lattice (see, for instance, [3]). □

The notion of concept in this framework follows the standard approach, and is defined as a fixpoint of the Galois connection from Theorem 2. Similarly, the set of concepts can be ordered by the suitable extension of the subset/superset hierarchy.

5 Conclusions and future work

An adjoint pair has been defined on the set of ILF values associated to a complete lattice $\mathcal{L}$ and, as a result, an antitone Galois connection can be induced between the powersets of ILF sets. This result improves a previous attempt in which the Galois connection was only obtained under the assumption that the underlying context is indetermination-free (i.e. $\mu(x) + \nu(x) = 1$ in the standard terminology of IF sets).

As future work, we will study the possible existence of different (families of) adjoint pairs so that the multi-adjoint framework of [9] could also be extended to an ILF setting.

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