On the existence of isotone Galois connections between preorders

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Abstract. Given a mapping $f\colon A \to B$ from a preordered set A into an unstructured set B, we study the problem of defining a suitable preordering relation on B such that there exists a mapping $g\colon B \to A$ such that the pair (f, g) forms an adjunction between preordered sets.

1 Introduction

Galois connections were introduced by Ore [30] in 1944 as a pair of antitone mappings aimed at generalizing Birkhoff's theory of polarities to the framework of complete lattices. Later, in 1958, Kan [23] introduced the notion of pair of adjoint functors in a categorical context. It is not surprising to find a plethora of examples of adjunction in several disparate research areas, ranging from the most theoretical to the most applied. It is remarkable to note that the importance of adjunctions quickly increased to an extent that, for instance, the interest of category theorists moved from universal mapping properties and natural transformations to adjointness.

When instantiating an adjunction to categories of ordered sets, it can be seen that both constructions, adjunctions and Galois connections, are fairly similar and, to some extent, are interdefinable: in some sense, an adjunction between A and B is a Galois connection in which the order relation on B is reversed (this leads to the use of the term *isotone Galois connection* which is exactly that of adjunction between ordered structures).

Nowadays, one can often find publications concerning Galois connections, both isotone and antitone, focused on either theoretical developments or theoretical applications [7,9,24]. Another term for adjunction, frequently used in the context of ordered sets, is that of pair of residuated mappings [5].

Concerning applications to informatics, we can find a first survey [28] on computer science applications published back in 1986. Of course, a number of

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more specific references on certain topics can be found, for instance, to programming [29], data analysis [34], logic [12, 21]. It is specially remarkable that the research topic of approximate reasoning using rough sets has benefitted specially from the use of the theory of Galois connections [13, 20, 31, 32].

It is worth to recall that many recent works on Galois connections use them in the framework of Formal Concept Analysis (FCA), either theoretically or applicatively.This is not surprising, since the operators used to build concepts form a Galois connection. In [33] one can find an extension of conceptualization modes, [1] describes a general approach to fuzzy FCA, [6] studies two previously existing frameworks and proved them equivalent, [10] use them for solving multi-adjoint relation equations, [27] provides new generalizations for FCA, [11] relates FCA and possibility theory, [3] stress on the "duality" between isotone and antitone Galois connections in showing a case of mutual reducibility of the concept lattices generated by using each type of connection, etcetera.

Being able to define a Galois connection between two ordered structures is a matter of major importance, and not only for FCA. For instance, [8] establishes a Galois connection between valued constraint languages and sets of weighted polymorphisms in order to develop an algebraic theory of complexity for valued constraint languages.

Browsing the related literature, one can find several publications concerning sufficient or necessary conditions for the existence of Galois connections between ordered structures. The main results of this paper are related to the existence and construction of the adjoint pair to a given mapping f, but *in a more general framework*.

Our initial setting is to consider a mapping $f\colon A \to B$ from a preordered set A into an unstructured set B, and then characterize those situations in which the set B can be preordered and an isotone mapping $g\colon B \to A$ can be built such that the pair (f, g) is an adjunction. (*Note that hereafter we will use exclusively this term since is shorter than isotone Galois connection*).

The structure of the paper is as follows: in Section 2, given $f\colon A \to B$ we introduce the preliminary definitions, and recall the necessary and sufficient conditions for the existence of a unique partial ordering on B and a mapping g such that (f, g) is an adjunction; then, in Section 3 we study the existence of preordering in B and the existence of g such that (f, g) is an adjunction between preordered structures; at this point, the absence of antisymmetry makes that both the statements and the proofs of the results to be much more involved. Finally, in Section 5, we draw some conclusions and discuss future work.

2 Preliminary definitions and results

We assume basic knowledge of the properties and constructions related to a partially ordered and preordered sets. Anyway, for the sake of self-completion, we include below the formal definitions of the main concepts to be used in this work.

Definition 1. *Given a partially ordered set $\mathbb{A} = (A, \leq_A)$, $X \subseteq A$, and $a \in A$.*

- *An element u is said to be an* upper bound *of X, if $x \leq u$ for all $x \in X$. We write $UB(X)$ to refer to the set of upper bounds of X.*
- *An element a is said to be the* maximum *of X, denoted* $\max X$*, if $a \in X$ and $x \leq a$ for all $x \in X$.*
- *The* downset *$a^{\downarrow}$ of a is defined as $a^{\downarrow} = \{x \in A \mid x \leq_A a\}$.*
- *The* upset *$a^{\uparrow}$ of a is defined as $a^{\uparrow} = \{x \in A \mid x \geq_A a\}$.*

A mapping $f\colon (A, \leq_A) \to (B, \leq_B)$ between partially ordered sets is said to be

- isotone *if $a_1 \leq_A a_2$ implies $f(a_1) \leq_B f(a_2)$, for all $a_1, a_2 \in A$.*
- antitone *if $a_1 \leq_A a_2$ implies $f(a_2) \leq_B f(a_1)$, for all $a_1, a_2 \in A$.*

In the particular case in which $A = B$,

- *f is* inflationary *(also called extensive) if $a \leq_A f(a)$ for all $a \in A$.*
- *f is* deflationary *if $f(a) \leq_A a$ for all $a \in A$.*

As we are including the necessary definitions for the development of the construction of adjunctions, we state below the definition of adjunction we will be working with.

Definition 2. *Let $\mathbb{A} = (A, \leq_A)$ and $\mathbb{B} = (B, \leq_B)$ be posets, $f\colon A \to B$ and $g\colon B \to A$ be two mappings. The pair (f, g) is said to be an* adjunction between $\mathbb{A}$ and $\mathbb{B}$*, denoted by $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$, whenever for all $a \in A$ and $b \in B$ we have that*

$$f(a) \leq_B b \qquad \textit{if and only if} \qquad a \leq_A g(b)$$

The mapping f is called left adjoint *and g is called* right adjoint*.*

As we will not be working with partially ordered sets but with preordered sets, some of the previous notions have to be adapted to this more general setting.

The definitions of downset (resp. upset) of an element in a preordered set, and those of isotone, antitone, inflationary and deflationary mapping between preordered sets are exactly the same as those given for posets.

The notion of maximum or minimum element of a subset of a preordered set is defined as usual. Note, however, that due to the absence of antisymmetry, these elements need not be unique. This is an important difference which justifies the introduction of special terminology in this context.

Definition 3. *Given a preordered set $(A, \leq_A)$ and a subset $X \subseteq A$, an element $a \in A$ is said to be a* p-maximum *(resp.,* p-minimum *) of X if $a \in X$ and $x \leq_A a$ (resp., $a \leq_A x$) for all $x \in X$. The set of p-maxima (resp., p-minima) of X will be denoted as* p-max(X) *(resp.,* p-min(X)*).*

Notice that p-max(X) (resp., p-min(X)) need not be a singleton. In the event that, say $a, b \in$ p-max(X), then the two relations $a \leq b$ and $b \leq a$ hold. As this situation will repeat several times, we introduce the equivalence relation $\approx_A$ in any preordered set $(A, \leq_A)$, defined as follows for $a_1, a_2 \in A$:

$$a_1 \approx_A a_2 \quad \text{if and only if} \quad a_1 \leq_A a_2 \ \text{ and } \ a_2 \leq_A a_1 \tag{1}$$

The equivalence class of element a wrt an equivalence relation R will be written, as usual, as $[a]_R$. If there is no risk of ambiguity, the subscript will be omitted.

In this work we will assume a mapping $f \colon A \to B$ such that the original set is preordered. In order to study the existence of adjoints in this framework, we will need to use the previously defined relation $\approx_A$, together with the kernel relation $\equiv_f$, defined as $a \equiv_f b$ if and only if $f(a) = f(b)$.

The two relations above are used together in the definition of the *p-kernel* relation defined below:

Definition 4. *Let $\mathbb{A} = (A, \leq_A)$ be a preordered set, and $f \colon A \to B$ a mapping. The* p-kernel *relation $\cong_A$ is the equivalence relation obtained as the transitive closure of the union of the relations $\approx_A$ and $\equiv_f$.*

It is well-known that the transitive closure in the definition above can be described as follows: given $a_1, a_2 \in A$, we have that $a_1 \cong_A a_2$ if and only if there exists a finite chain $\{x_i\}_{i \in \{1,\dots,n\}} \subseteq A$ such that $x_1 = a_1$, $x_n = a_2$ and, for all $i \in \{1, \dots, n-1\}$, either $x_i \equiv_f x_{i+1}$ or $x_i \approx_A x_{i+1}$.

The following theorem [17] states different equivalent characterizations of the notion of adjunction *between preordered sets* that will be used in the main construction of the right adjoint. As expected, the general structure of the definitions is preserved, but those concerning the actual definition of the adjoints have to be modified by using the notions of p-maximum and p-minimum.

Theorem 1. *Let $\mathbb{A} = (A, \leq_A), \mathbb{B} = (B, \leq_B)$ be two preordered sets, and $f \colon \mathbb{A} \to \mathbb{B}$ and $g \colon \mathbb{B} \to \mathbb{A}$ be two mappings. The following statements are equivalent:*

1. $(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$.
2. *f and g are isotone maps, and $g \circ f$ is inflationary map, $f \circ g$ is deflationary map.*
3. *$f(a)^\uparrow = g^{-1}(a^\uparrow)$ for all $a \in A$.*
4. *$g(b)^\downarrow = f^{-1}(b^\downarrow)$ for all $b \in B$.*
5. *f is isotone and $g(b) \in$ p-max $f^{-1}(b^\downarrow)$ for all $b \in B$.*
6. *g is isotone and $f(a) \in$ p-min $g^{-1}(a^\uparrow)$ for each $a \in A$.*

Once again, the absence of antisymmetry leads to slight modifications of some well-known properties of adjunctions, as that stated in the result below and its corollary.

Theorem 2. *Let $\mathbb{A} = (A, \leq), \mathbb{B} = (B, \leq)$ be two preordered sets, and $f\colon A \to B$ and $g\colon B \to A$ be two mappings. If $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$ then, $(f \circ g \circ f)(a) \approx_B f(a)$ for all $a \in A$, and $(g \circ f \circ g)(b) \approx_A g(b)$ for all $b \in B$.*

Corollary 1. *Let $\mathbb{A} = (A, \leq_A), \mathbb{B} = (B, \leq_B)$ be two preordered sets, and $f\colon A \to B$ and $g\colon B \to A$ be two mappings. If $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$ then, $(g \circ f \circ g \circ f)(a) \approx_A (g \circ f)(a)$ for all $a \in A$, and $(f \circ g \circ f \circ g)(b) \approx_B (f \circ g)(b)$ for all $b \in B$.*

The following definition recalls the notion of Hoare ordering between subsets of a preordered set, and then introduces an alternative statement in the subsequent lemma.

Definition 5. *Let $(A, \leq)$ be a preordered set, and consider $X, Y \subseteq A$.*

- *We will denote by $\sqsubseteq_H$ the Hoare relation, $X \sqsubseteq_H Y$ if and only if, for all $x \in X$, there exists $y \in Y$ such that $x \leq y$.*
- *We define $X \sqsubseteq Y$ if and only if there exist $x \in X$ and $y \in Y$ such that $x \leq y$.*

Lemma 1. *Let $(A, \leq)$ be a preordered set, and consider $X, Y \subseteq A$ such that $\text{p-min}(X) \neq \varnothing$ and $\text{p-min}(Y) \neq \varnothing$. The following statements are equivalent:*

1. $\text{p-min}(X) \sqsubseteq_H \text{p-min}(Y)$
2. $\text{p-min}(X) \sqsubseteq \text{p-min}(Y)$
3. *For all $x \in \text{p-min}(X)$ and for all $y \in \text{p-min}(Y)$, $x \leq y$.*

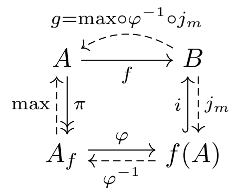
Proof. The implications 1) $\Rightarrow$ 2) and 3) $\Rightarrow$ 1) are straightforward. Let us prove, 2) $\Rightarrow$ 3). For this, consider any $x \in \text{p-min}(X)$ and $y \in \text{p-min}(Y)$. Using the hypothesis and $x \in \text{p-min}(X)$, we have that, there exists $y_1 \in \text{p-min}(Y)$ such that $x \leq y_1$. Since $y_1 \in \text{p-min}(Y)$, we have that $y_1 \leq y$ for all $y \in Y$. Therefore, $x \leq y$ for all $x \in \text{p-min}(X)$ and $y \in \text{p-min}(Y)$. $\square$

We finish this preliminary section by stating the characterization theorem of existence of a suitable *partial ordering* on B so that a right adjoint exists. The core of this work is to develop a generalized version of the theorem below:

Theorem 3 ([18]). *Given a poset $(A, \leq_A)$ and a map $f\colon A \to B$, let $\equiv_f$ be the kernel relation. Then, there exists an ordering $\leq_B$ in B and a map $g\colon B \to A$ such that $(f, g)\colon A \leftrightharpoons B$ if and only if*

1. *There exists $\max([a])$ for all $a \in A$.*
2. *For all $a_1, a_2 \in A$, $a_1 \leq_A a_2$ implies $\max([a_1]) \leq_A \max([a_2])$.*

Roughly speaking, the proof of the previous theorem is done by using the canonical decomposition theorem via the quotient set A_f wrt the kernel relation, and building right adjoints to any of the arrows in the path.



The tricky part of the proof was to extend the ordering on $f(A)$ to the whole set B so that it is still compatible with the existence of right adjoint j_m, obviously when f is not surjective. The underlying idea here is related to the definition of an order-embedding of the image into the codomain set; more generally, the idea is to extend a partial ordering defined just on a subset of a set to the whole set.

Definition 6. *Given a subset $X \subseteq B$, and a fixed element $m \in X$, any preordering $\leq_X$ in X can be extended to a preordering $\leq_m$ on B, defined as the reflexive and transitive closure of the relation $\leq_X \cup \{(m, y) \mid y \notin X\}$.*

Note that the relation above can be described as, for all $x, y \in B$, $x \leq_m y$ if and only if some of the following holds:

(a) $x, y \in X$ and $x \leq_X y$
(b) $x \in X, y \notin X$ and $x \leq_X m$
(c) $x, y \notin X$ and $x = y$

3 Building adjunctions between preordered sets

Given a mapping $f\colon \mathbb{A} \to B$ from a preordered set $\mathbb{A} = (A, \leq)$ to an unstructured set B, our first goal is to find sufficient conditions to define a suitable preordering on B such that a right adjoint exists. Notice that there is much more than a mere adaptation of the result for posets.

Lemma 2. *Let $\mathbb{A} = (A, \leq_A)$ be a preordered set and $f\colon \mathbb{A} \to B$ a surjective map. Let $S \subseteq \bigcup_{a\in A} \text{p-max}[a]_{\cong_A}$ such that the following conditions hold:*

- $\text{p-min}(UB[a]_{\cong_A} \cap S) \neq \varnothing$, *for all $a \in A$.*
- *If $a_1 \leq_A a_2$, then* $\text{p-min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq \text{p-min}(UB[a_2]_{\cong_A} \cap S)$.

Then, there exists a preorder $\leq_B$ in B and a map g such that $(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$.

Proof. The definition of the preorder $\leq_B$ in B, given $b_1, b_2 \in B$, is as follows:
$b_1 \leq_B b_2$ if and only if there exist $a_1 \in f^{-1}(b_1)$ and $a_2 \in f^{-1}(b_2)$ such that

$$\text{p-min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq \text{p-min}(UB[a_2]_{\cong_A} \cap S).$$

Let us prove that it is a preordering:

Reflexivity: By the first hypothesis, we have that p-min($UB[a]_{\cong_A} \cap S$) $\neq \varnothing$.
Now, trivially, p-min($UB[a]_{\cong_A} \cap S$) $\sqsubseteq$ p-min($UB[a]_{\cong_A} \cap S$) holds for any $a \in f^{-1}(b)$. Therefore, $b \leq_B b$ for any $b \in B$.

Transitivity: Assume $b_1 \leq_B b_2$ and $b_2 \leq_B b_3$.
From $b_1 \leq_B b_2$, there exist $a_i \in f^{-1}(b_i)$, and $c_i \in$ p-min($UB[a_i]_{\cong_A} \cap S$) for $i = 1, 2$ such that $c_1 \leq_A c_2$.
From $b_2 \leq_B b_3$, there exist $a'_i \in f^{-1}(b_i)$, and $c'_i \in$ p-min($UB[a'_i]_{\cong_A} \cap S$) for $i = 2, 3$ such that $c'_2 \leq_A c'_3$.
As $a_2, a'_2 \in f^{-1}(b_2)$, we have that $[a_2]_{\cong_A} = [a'_2]_{\cong_A}$, which implies that $c_2 \approx c'_2$. Therefore, $c_1 \leq_A c_2 \approx_A c'_2 \leq_A c'_3$ and, as a result, $b_1 \leq_B b_3$.

In order to define $g\colon B \to A$, firstly notice that, as f is onto, given $b \in B$ there exists $x_b \in A$ with $f(x_b) = b$. By hypothesis, p-min($UB[x_b]_{\cong_A} \cap S$) $\neq \varnothing$ for all $b \in B$ and, therefore, there exists a choice function. Any of these functions can be used to define g, in such a manner that $g(b) \in$ p-min($UB[x_b]_{\cong_A} \cap S$).

To finish the proof, we have just to check that $(f, g)\colon (A, \leq_A) \leftrightharpoons (B, \leq_B)$.

Assume $f(a) \leq_B b$, then there exist $a_1 \in f^{-1}(f(a))$, $a_2 \in f^{-1}(b)$, $c_1 \in$ p-min($UB[a_1]_{\cong_A} \cap S$) and $c_2 \in$ p-min($UB[a_2]_{\cong_A} \cap S$) with $c_1 \leq_A c_2$; as $[a_1]_{\cong_A} = [a]_{\cong_A}$, and $c_1 \in UB[a_1]$, we also have $a \leq_A c_1$. By definition, we have that $g(b) \in$ p-min($UB[x]_{\cong_A} \cap S$) for $x \in f^{-1}(b)$, then $[a_2]_{\cong_A} = [x]_{\cong_A}$, and p-min($UB[a_2]_{\cong_A} \cap S$) = p-min($UB[x]_{\cong_A} \cap S$). Thus, $c_2 \approx g(b)$ and, as $a \leq_A c_1 \leq_A c_2$, then $a \leq_A g(b)$.

Assuming now that $a \leq_A g(b)$, let us prove $f(a) \leq_B b$. For this, consider $a \in f^{-1}(f(a))$ and $x \in f^{-1}(b)$ where x is the element in $f^{-1}(b)$ used in the definition of $g(b)$, and let us prove that p-min($UB[a]_{\cong_A} \cap S$) $\sqsubseteq$ p-min($UB[x]_{\cong_A} \cap S$). For this, it is enough to see that for all $z \in$ p-min($UB[a]_{\cong_A} \cap S$) the inequality $z \leq_A g(b)$ holds, since obviously $g(b) \in$ p-min($UB[x]_{\cong_A} \cap S$).

Fixed $z \in$ p-min($UB[a]_{\cong_A} \cap S$), firstly consider that from $g(b) \in S$, using the hypothesis on S, we have that $g(b) \in$ p-max$[g(b)]_{\cong_A}$, which means that $g(b) \in UB[g(b)]_{\cong_A}$ as well; that is, $g(b) \in (UB[g(b)]_{\cong_A} \cap S)$. On the other hand, from $a \leq_A g(b)$ and the second hypothesis we have p-min($UB[a]_{\cong_A} \cap S$) $\sqsubseteq$ p-min($UB[g(b)]_{\cong_A} \cap S$). By Lemma 1, we have that $z \leq_A t$ for all $t \in$ p-min($UB[g(b)]_{\cong_A} \cap S$). Since, obviously $t \leq_A g(b)$, we obtain $z \leq_A g(b)$. $\square$

The following lemma gets rid of the condition of f being surjective, and will be used in the proof of the main theorem of this work, stated as Theorem 4.

Lemma 3. *Consider $(A, \leq_A)$ a preordered set, B a set, and $f\colon A \to B$. Then, there exist both a preorder $\leq_B$ and an adjunction $(f, g)\colon (A, \leq_A) \leftrightharpoons (B, \leq_B)$ if and only if there exist a preorder $\leq_{f(A)}$ and an adjunction $(f, g')\colon (A, \leq_A) \leftrightharpoons (f(A), \leq_{f(A)})$.*

Proof. The direct implication is trivial, by considering $\leq_{f(A)}$ and g' as the restrictions to $f(A)$ of $\leq_B$ and g, respectively.

Conversely, consider the adjunction $(f, g')\colon (A, \leq_A) \leftrightharpoons (f(A), \leq_{f(A)})$, fix $m \in f(A)$, and choose $\leq_B$ to be its associated preorder, as introduced in Definition 6. It is just a matter of straightforward computation to check that we have an adjunction $(f, g)\colon (A, \leq_A) \leftrightharpoons (B, \leq_B)$ where g is the extension of g' defined as follows:

$$g(x) = \begin{cases} g'(x) & \text{if } x \in f(A) \\ g'(m) & \text{if } x \notin f(A) \end{cases}$$

□

The corresponding version of Theorem 3 is a twofold extension of the statement of Lemma 2 in that, firstly, the mapping f need not be onto and, secondly, gives a necessary and sufficient condition for the existence of adjunction.

Theorem 4. *Given any preordered set $\mathbb{A} = (A, \leq_A)$ and a mapping $f\colon \mathbb{A} \to B$, there exists a preorder $\mathbb{B} = (B, \leq_B)$ and $g\colon B \to A$ such that $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ if and only if there exists $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$ such that*

1. $\text{p-min}(UB[a]_{\cong_A} \cap S) \neq \varnothing$, *for all $a \in A$.*
2. *If $a_1 \leq_A a_2$, then* $\text{p-min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq \text{p-min}(UB[a_2]_{\cong_A} \cap S)$.

Proof. Assume the existence of the preordering in B and the mapping g such that $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$, and let us prove the three properties in the statement.

Define $S = g(f(A))$, consider $g(f(a)) \in S$, and let us show that $g(f(a)) \in \text{p-max}[g(f(a))]_{\cong_A}$. Consider $x \in [g(f(a))]_{\cong_A}$, by a straightforward induction argument we obtain $f(x) \approx_B f(g(f(a)))$; now, using $f(g(f(a))) \approx_B f(a)$ we have $f(x) \approx_B f(a)$. Since $f(x) \leq_B f(a)$, by using the adjunction, we obtain $x \leq_A g(f(a))$, hence $g(f(a)) \in \text{p-max}[g(f(a))]_{\cong_A}$.

For property 1, we will check that $g(f(a)) \in \text{p-min}(UB[a]_{\cong_A} \cap S)$. To begin with, by definition $g(f(a)) \in S$; then, we will prove that $g(f(a)) \in UB[a]_{\cong_A}$. Given $x \in [a]_{\cong_A}$ we have to prove $x \leq_A g(f(a))$; the argument follows by induction on the length of the chain connecting x and a

- For $n = 0$, we have $a \leq_A g(f(a))$ by properties of adjunction.
- Assume the result is true for any chain of length n, and consider $a \cong_A a_2 \cong_A \dots a_n \cong_A x$, then, by induction hypothesis, $a_n \leq_A g(f(a))$. Now, as $a_n \cong_A x$, there are two possibilities:
  - $a_n \approx_A x$ and, trivially $x \leq_A g(f(a))$.
  - $f(a_n) = f(x)$, using the properties of adjunction twice we firstly obtain $f(x) \leq_A f(g(f(a)))$ and, then, $x \leq_A g(f(g(f(a)))) \approx_A g(f(a))$.

We have just proved that $g(f(a)) \in UB[a]_{\cong_A} \cap S$, the remaining point is to prove that it is a p-minimum element. Consider $x \in UB[a]_{\cong_A} \cap S$; then $z \leq_A x$ for all $z \in [a]_{\cong_A}$ and, by definition of S, $x = g(f(a_1))$. Particularly, for $z = a$ we have that, $a \leq_A g(f(a_1))$, by properties of adjunction, $g(f(a)) \leq_A g(f(g(f(a_1)))) \approx_A g(f(a_1)) = x$, i.e. $g(f(a)) \leq_A x$.

For Property 2, assume $a_1 \leq_A a_2$, by adjunction, f and g are isotone maps, then $g(f(a_1)) \leq_A g(f(a_2))$. From this, we directly obtain p-min$(UB[a_1]_{\cong_A} \cap S) \sqsubseteq$ p-min$(UB[a_2]_{\cong_A} \cap S)$ since we just proved above that for all $a \in A$ $g(f(a)) \in$ p-min$(UB[a]_{\cong_A} \cap S)$.

Conversely, if we assume properties 1 and 2, then by Lemma 2 and Lemma 3, there exist a preorder $\mathbb{B} = (B, \leq_B)$ and a map g such that $(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$. $\square$

4 On the uniqueness of right adjoints and the inherited ordered structure in the codomains

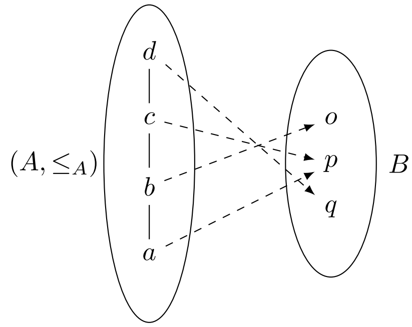
The unicity of the right adjoint between posets is well-known. Specifically, given two posets $\mathbb{A} = (A, \leq_A)$ and $\mathbb{B} = (B, \leq_B)$ and a mapping $f : A \to B$, if there exists $g : B \to A$ such that the pair (f, g) is an adjunction, then it is unique.

This behavior was further analyzed in [18], where the uniqueness property was extended, in the case of surjective mappings, not only to the right adjoint, but also to the ordering relation in the codomain: namely, there exists just one partial ordering on the codomain B such that a right adjoint exists, that is, given a surjective mapping f from a poset $\mathbb{A}$ to an unstructured set B, we introduced necessary and sufficient conditions to ensure the existence of an ordering $\leq_B$ in B and a mapping $g : B \to A$ such that (f, g) is an adjunction. Moreover, both $\leq_B$ and g are uniquely determined by $\leq_A$ and f.

Contrariwise to the partially ordered case, given two preordered sets $\mathbb{A} = (A, \leq_A)$ and $\mathbb{B} = (B, \leq_B)$ and a mapping $f : A \to B$, the unicity of the mapping $g : B \to A$ satisfying $(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$, when it exists, cannot be guaranteed. However, it is well known that if g_1 and g_2 are right adjoints, then $g_1(b) \approx_A g_2(b)$ for all $b \in B$, and one usually says that the right adjoint is *essentially unique*. This scenario is much more similar to what occurs in category theory: if one functor F has two right adjoints G and G', then G and G' are naturally isomorphic.

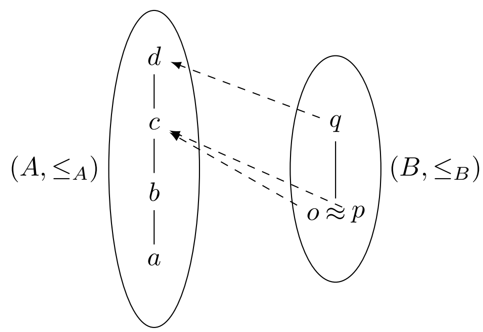
However, and this is the interesting part, the unicity cannot be extended to the case in which the codomain is unstructured. In this section we introduce several examples supporting this statement.

Examples Let $A = \{a, b, c, d\}$, $B = \{o, p, q\}$ be two sets and $f : A \to B$ defined as $f(a) = f(c) = p$, $f(b) = o$ and $f(d) = q$. Consider $(A, \leq_A)$ ordered by $a \leq_A b \leq_A c \leq_A d$. We have $[a]_{\cong_A} = [c]_{\cong_A} = \{a, c\}$, $[b]_{\cong_A} = \{b\}$ and $[d]_{\cong_A} = \{d\}$ and $\bigcup_{x \in A}$ p-max$[x]_{\cong_A} = \{b, c, d\}$.



Notice that f is surjective, and does not fulfill the conditions in Theorem 3, specifically the second one. Thus, there does not exist any *partial ordering* relation in B for which some $g\colon B \to A$ would be a right adjoint to f. Notice, however, that if we relax the requirement to be an adjunction *between preordered* sets, then there exist a preordering (actually more than one) which generates a right adjoint to f. Some examples are worked out below to illustrate the previous situation.

Example 1. Consider $\mathbb{B} = (B, \leq_B)$ preordered with $o \approx_B p$, $o \leq_B q$ and $p \leq_B q$, and the mapping $g\colon B \to A$ defined as $g(o) = g(p) = c$ and $g(q) = d$.



To begin with, we have that $S = gf(A) = \{c, d\}$ is a subset of $\bigcup\limits_{x \in A} \text{p-max}[x]_{\cong_A}$ and, then, check the two conditions in Theorem 4.

It is not difficult to check that $\text{p-min}(UB[x]_{\cong_A} \cap S) \neq \varnothing$ for all $x \in A$. Specifically, we have

$$\text{p-min}(UB[a]_{\cong_A} \cap S) = \text{p-min}(UB[b]_{\cong_A} \cap S) = \text{p-min}(UB[c]_{\cong_A} \cap S) = \{c, d\}$$

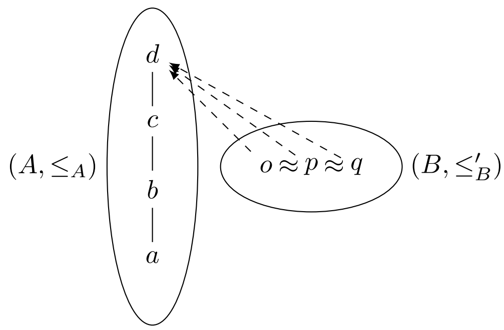
and

$$\text{p-min}(UB[d]_{\cong_A} \cap S) = \{d\}$$

Finally, with the previous computation, it is straightforward to check that if $a_1 \leq_A a_2$ then $\text{p-min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq \text{p-min}(UB[a_2]_{\cong_A} \cap S)$.

As a result, the pair (f, g) is an adjunction between $\mathbb{A}$ and $\mathbb{B}$. □

Example 2. Now, consider $\mathbb{B}' = (B, \leq'_B)$ preordered by $o \approx'_B p$ and $p \approx'_B q$, and the mapping $g' \colon B \to A$ defined as $g'(o) = g'(p) = g'(q) = d$.



Again we will check the conditions in Theorem 4.

In this case, $S = g'f(A) = \{d\}$ which is a subset of $\bigcup_{x \in A} \text{p-max}[x]_{\cong_A} = \{b, c, d\}$. The first condition holds since $\text{p-min}(UB[a]_{\cong_A} \cap S) = \text{p-min}(UB[b]_{\cong_A} \cap S) = \text{p-min}(UB[c]_{\cong_A} \cap S) = \text{p-min}(UB[d]_{\cong_A} \cap S) = \{d\}$. As all the previous sets coincide, the second condition follows trivially.

As a result, the pair (f, g') is an adjunction between the preorders $\mathbb{A}$ and $\mathbb{B}'$.

□

5 Conclusions

Given a mapping $f \colon A \to B$ from a preordered set A into an unstructured set B, we have obtained necessary and sufficient conditions which allow us for defining a suitable preordering relation on B such that there exists mapping $g \colon B \to A$ such that the pair of mappings (f, g) forms an adjunction between preordered sets.

Whereas the results in the partially ordered case followed more or less the intuition of what should be expected (Theorem 3), the description of the conditions on the preordered case is much more involved (Theorem 4). A first piece of future work should be to consider alternative approaches to this problem in order to obtain, if possible, a simpler alternative characterization.

Concerning potential applications of the present work, let us recall that the Galois connections used in FCA are given between the Boole algebras of the powersets of objects and the powerset of attributes. There exist several generalizations in FCA which weaken the structure on which a Galois connection is defined: for instance, in fuzzy FCA the residuated structure of the powerset of fuzzy sets is used. In [16], a general approach called *pattern structures* was proposed, which allows for extending FCA techniques to arbitrary partially ordered data descriptions. Using pattern structures, one can compute taxonomies, ontologies, implications, implication bases, association rules, concept-based (or JSM-) hypotheses in the same way it is done with standard concept lattices [26].

In this generalization, instead of associating each object with the set of attributes it satisfies, a pattern is given, which can be either a graph, or a sequence or an interval, and the semantics of these patterns can be different in each case. For instance, [15] represents scenarios of conflict between human agents, or [22] use gene expression data. These sets of patterns are provided with a partial ordering relation such as "*being a subgraph of*" or "*being a subchain of*".

The results obtained in this work are aimed at not only extending these results to sets in which there is a preordering previously defined but, more specifically, to the problem of knowledge discovery on the existing structure between the patterns. The scenario in which this work could be applied is as follows: we start from a set of objects each one related to the set of patterns it satisfies, ignoring whether there exists some (pre-)ordering relation between patterns, but assuming that the semantics of the problem guarantees the existence of a Galois connection between them, the goal would be to obtain as much information as possible about the relation existing in the set of patterns.

To finish with the future work, it is remarkable the number of papers on fuzzy Galois connections have been written since its introduction in [2]; consider for instance [4, 14, 19, 25] for some recent ones. As future work in the short term, we would like to extend the results in this work to the fuzzy case, for instance to the framework of fuzzy posets and fuzzy preorders, and study the potential relationship to other approaches based on generalized structures.

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