Relational fuzzy Galois connections

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***Abstract*—We propose a suitable generalization of the notion of Galois connection whose components are fuzzy relations. We prove that the construction embeds Yao's notion of fuzzy Galois connection as a particular case. Although the natural framework for the proposed notion is that of fuzzy preposets, we also prove that it behaves properly with respect to the formation of quotient with respect to the fuzzy symmetric kernel relation.**

I. Introduction

Since their inception, Galois connections have shown to be an interesting tool both for theory and for applications [1], [2]. One particular research area on application is Formal Concept Analysis (FCA) [3], since the main properties of the concept-forming operators are consequences of their being part of a Galois connection.

Successive generalizations of FCA to the fuzzy case, suggested the introduction of different fuzzified versions of Galois connection: to the best of our knowledge the first one was due to Bělohlávek [4]; later, Georgescu and Popescu introduced non-commutative versions [5], and Yao and Lu, a version based on fuzzy posets [6]; and more abstract approaches were given in [7]–[9].

An interesting problem in all the generalized notions of Galois connection is its actual construction, namely, the problem of constructing the residual (aka right adjoint) mapping of a given $f\colon A \to B$. The straightforward answer is to apply a suitable version of the well-known Freyd's adjoint theorem, which characterizes when such an residual exists when both A and B have the same structure.

But what if A and B are *differently structured*?

If A has a richer structure than B, firstly the missing structure on B has to be built, and only then the residual could be constructed. This has been one of our preferred research topics in the recent years, in which a number of results have been obtained considering different underlying settings. Namely, in [10] we worked with crisp functions between a poset (resp. preordered set) and an unstructured set; later, in [11] we entered in the fuzzy arena, considering the case in which A is fuzzy preposet; then, in [12], we extended the previous results by allowing fuzzy equivalence relations as an adequate substitute to equality.

Before proceeding to further generalizations, a more adequate notion of Galois connection should be considered since the results in [11], [12] lack of fuzziness precisely on the components of the Galois connection, which turned out to be crisp functions. In [13] we started the search for a more adequate notion involving fuzzy functions as components.

In this work we introduce the notion of relational fuzzy Galois connection, in which the components of the connection are not fuzzy functions but fuzzy relations satisfying certain properties. We prove that the construction embeds Yao's notion of fuzzy Galois connection as a particular case. Although the natural framework for the proposed notion is that of fuzzy preposets, we also prove that it behaves properly with respect to the formation of quotient with respect to the fuzzy symmetric kernel relation, thus leading to a connection between fuzzy posets.

II. Preliminary definitions

Given a complete residuated lattice $\mathbb{L} = (L, \otimes, \Rightarrow)$, an $\mathbb{L}$-fuzzy set is a mapping from the universe set to the membership values structure $X\colon U \to L$ where $X(u)$ means the degree in which u belongs to X. Given X and Y two $\mathbb{L}$-fuzzy sets, X is said to *be included in* Y, denoted as $X \subseteq Y$, if $X(u) \le Y(u)$ for all $u \in U$.

An $\mathbb{L}$-fuzzy binary relation between A and B is an $\mathbb{L}$-fuzzy subset of $A \times B$, i.e. a mapping $\mu\colon A \times B \to L$ and its *domain* and its *image* are defined as follows:

$$\operatorname{dom}(\mu) = \{a \in A \mid \text{ there exists } b \in B \text{ with } \mu(a,b) = \top\}$$
$$\operatorname{im}(\mu) = \{b \in B \mid \text{ there exists } a \in A \text{ with } \mu(a,b) = \top\}$$

Moreover, μ is said to be *total* if $\operatorname{dom}(\mu) = A$ and μ is said to be *surjective* if $\operatorname{im}(\mu) = B$.

From now on, when no confusion arises, we will omit the prefix "$\mathbb{L}$-".

The *identity relation* in a set A is denoted by id_A, i.e. $id_A\colon A \times A \to L$ where $id_A(a,b) = \top$ if $a = b$, and $id_A(a,b) = \bot$ otherwise. The *composition* of two fuzzy binary relations $\mu\colon A \times B \to L$ and $\nu\colon B \times C \to L$ is defined as $\nu \circ \mu\colon A \times C \to L$ where $(\nu \circ \mu)(a,c) = \bigvee_{b \in B}(\mu(a,b) \otimes \nu(b,c))$ for all $a \in A$ and $c \in C$. In addition, the *inverse* of μ is the fuzzy binary relation $\mu^{-1}\colon B \times A \to L$ such that $\mu^{-1}(b,a) = \mu(a,b)$ for all $a \in A$ and $b \in B$. It is trivial that, if μ is total, $id_A \subseteq \mu^{-1} \circ \mu$, and, if μ is surjective, $id_B \subseteq \mu \circ \mu^{-1}$.

A fuzzy binary relation $\mu\colon A \times A \to L$ is said to be:

- *Reflexive* if $id_A \subseteq \mu$.
- $\otimes$-*Transitive* if $\mu \circ \mu \subseteq \mu$.
- *Symmetric* if $\mu = \mu^{-1}$.

- *Antisymmetric* if $\mu(a,b)=\mu(b,a)=\top$ implies $a=b$, for all $a,b\in A$.

Definition 1: A *fuzzy preposet* is a pair $\mathbb{A}=\langle A,\rho_A\rangle$ in which ρ_A is a reflexive and $\otimes$-transitive fuzzy relation on A. In addition, a *fuzzy poset* is a fuzzy preposet $\mathbb{A}=\langle A,\rho_A\rangle$ in which ρ_A is also antisymmetric.

Definition 2: A fuzzy relation $\approx$ on A is said to be a:

- *Fuzzy equivalence relation* if $\approx$ is a reflexive, $\otimes$-transitive and symmetric fuzzy relation on A.
- *Fuzzy equality* if $\approx$ is a fuzzy equivalence relation satisfying that $\approx(a,b)=\top$ implies $a=b$, for all $a,b\in A$.

We will use the infix notation for a fuzzy equivalence relation, that is: for $\approx\colon A\times A\to L$ a fuzzy equivalence relation, we denote $a_1\approx a_2$ to refer to $\approx(a_1,a_2)$.

Definition 3: Let $\approx_A$ and $\approx_B$ be fuzzy equivalence relations on A and B respectively. A fuzzy relation $\mu\colon A\times B\to L$ is said to be *extensional* if $\mu\circ\approx_A\subseteq\mu$ and $\approx_B\circ\,\mu\subseteq\mu$.

That is, μ is extensional if $(a_1\approx_A a_2)\otimes\mu(a_2,b)\le\mu(a_1,b)$ and $\mu(a,b_1)\otimes(b_1\approx_B b_2)\le\mu(a,b_2)$, for all $a,a_1,a_2\in A$ and $b,b_1,b_2\in B$.

III. Relational fuzzy Galois connections

In order to introduce the notion of relational fuzzy Galois connection, it will be convenient to adopt, whenever possible, a relational notation and, specifically, the properties will be stated in terms of compositions of relations. For instance, the usual notion of μ being *antitone* if $\mu(a_1,b_1)\otimes\rho_A(a_1,a_2)\otimes\mu(a_2,b_2)\le\rho_B(b_2,b_1)$ for all $a_1,a_2\in A$ and $b_1,b_2\in B$; and ν being *inflationary* if $\nu(a_1,a_2)\le\rho_A(a_1,a_2)$ for all $a_1,a_2\in A$, will be rephrased as follows:

Definition 4: Let $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$ be fuzzy preposets.

- $\mu\colon A\times B\to L$ is said to be *antitone* if $\mu\circ\rho_A\circ\mu^{-1}\subseteq\rho_B^{-1}$.
- $\nu\colon A\times A\to L$ is said to be *inflationary* if $\nu\subseteq\rho_A$.

The definition of relational fuzzy Galois connection requires that the two involved relations are linked together in a certain manner. This is formally introduced below:

Definition 5: Let $\mu\colon A\times B\to\mathbb{L}$ and $\nu\colon B\times A\to\mathbb{L}$ be fuzzy relations. The pair (μ,ν) is said to be *coupled* if for all $a_1\in A$ and $b_1\in B$ there exist $a_2\in A$ and $b_2\in B$ such that $\mu(a_1,b_1)\otimes\nu(b_1,a_2)=\mu(a_1,b_1)$ and $\nu(b_1,a_1)\otimes\mu(a_1,b_2)=\nu(b_1,a_1)$.

As a direct consequence of the previous definition, if the pair (μ,ν) is coupled, then $\mathrm{im}(\mu)\subseteq\mathrm{dom}(\nu)$ and $\mathrm{im}(\nu)\subseteq\mathrm{dom}(\mu)$. In addition, it is straightforward that any pair of total fuzzy relations $\mu\colon A\times B\to L$ and $\nu\colon B\times A\to L$ are coupled.

Now we have all the notions required in order to introduce the definition of relational fuzzy Galois connection in which the role of left and right adjoints is played by fuzzy relations.

Definition 6: Let $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$ be fuzzy preposets and $\mu\colon A\times B\to\mathbb{L}$ and $\nu\colon B\times A\to\mathbb{L}$ be fuzzy relations. The pair (μ,ν) is said to be a *relational fuzzy Galois connection* between $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$ if μ and ν are antitone, the compositions $\mu\circ\nu$ and $\nu\circ\mu$ are inflationary and the pair (μ,ν) is coupled.

The following theorem extends the classical characterization of Galois connections.

Theorem 1: Let $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$ be fuzzy preposets and $\mu\colon A\times B\to\mathbb{L}$ and $\nu\colon B\times A\to\mathbb{L}$ be fuzzy relations. The pair (μ,ν) is a relational fuzzy Galois connection between $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$ if and only if the pair (μ,ν) is coupled, $\mu\circ\rho_A^{-1}\circ\nu\subseteq\rho_B$ and $\nu\circ\rho_B^{-1}\circ\mu\subseteq\rho_A$.

Proof: Assume that (μ,ν) is a relational fuzzy Galois connection between $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$. By definition, the pair (μ,ν) is coupled. We prove just $\nu\circ\rho_B^{-1}\circ\mu\subseteq\rho_A$ since the other inclusion is proved similarly. Specifically, we prove that for all $a_1,a_2\in A$ and $b_1,b_2\in B$:

$$\mu(a_1,b_1)\otimes\rho_B(b_2,b_1)\otimes\nu(b_2,a_2)\le\rho_A(a_1,a_2)$$

By using the facts that the pair (μ,ν) is coupled, $\nu\circ\mu$ is inflationary and ν is antitone, and transitivity of ρ_A, one has that there exists $a_3\in A$ such that:

$$\begin{aligned}
&\mu(a_1,b_1)\otimes\rho_B(b_2,b_1)\otimes\nu(b_2,a_2)\\
&=\mu(a_1,b_1)\otimes\nu(b_1,a_3)\otimes\nu(b_1,a_3)\otimes\rho_B(b_2,b_1)\otimes\nu(b_2,a_2)\\
&\le(\nu\circ\mu)(a_1,a_3)\otimes(\nu\circ\rho_B\circ\nu^{-1})(a_2,a_3)\\
&\le\rho_A(a_1,a_3)\otimes\rho_A(a_3,a_2)\le\rho_A(a_1,a_2)
\end{aligned}$$

Conversely, assume that the pair (μ,ν) is coupled, $\mu\circ\rho_A^{-1}\circ\nu\subseteq\rho_B$ and $\nu\circ\rho_B^{-1}\circ\mu\subseteq\rho_A$.

- The following sequence proves that $\nu\circ\mu$ is inflationary by using reflexivity of ρ_B and the second inclusion:
$$\nu\circ\mu=\nu\circ id_B\circ\mu\subseteq\nu\circ\rho_B^{-1}\circ\mu\subseteq\rho_A$$
- In order to prove the antitonicity of μ, first we prove that, for all $a_1,a_2\in A$ and $b\in B$, there exists $a_3\in A$ with
$$\rho_A(a_1,a_2)\otimes\mu(a_2,b)\le\nu(b,a_3)\otimes\rho_A^{-1}(a_3,a_1)\qquad(1)$$
Since (μ,ν) is coupled, $\nu\circ\mu$ is inflationary and ρ_A is transitive, one has:
$$\begin{aligned}
&\rho_A(a_1,a_2)\otimes\mu(a_2,b)\\
&\quad=\rho_A(a_1,a_2)\otimes\mu(a_2,b)\otimes\nu(b,a_3)\otimes\nu(b,a_3)\\
&\quad\le\rho_A(a_1,a_2)\otimes\rho_A(a_2,a_3)\otimes\nu(b,a_3)\\
&\quad\le\rho_A(a_1,a_3)\otimes\nu(b,a_3)=\nu(b,a_3)\otimes\rho_A^{-1}(a_3,a_1)
\end{aligned}$$
Now, by using (1) and $\mu\circ\rho_A^{-1}\circ\nu\subseteq\rho_B$, one has:
$$\begin{aligned}
&\rho_A(a_1,a_2)\otimes\mu(a_1,b_1)\otimes\mu(a_2,b_2)\\
&\quad\le\nu(b_2,a_3)\otimes\rho_A^{-1}(a_3,a_1)\otimes\mu(a_1,b_1)\\
&\quad\le(\mu\circ\rho_A^{-1}\circ\nu)(b_2,b_1)\le\rho_B(b_2,b_1)
\end{aligned}$$
- Analogously, it is proved that $\mu\circ\nu$ is inflationary and ν is antitone.

■

Definition 7: Let $\langle A,\rho_A\rangle$ be a fuzzy preposet. The *fuzzy symmetric kernel relation* $\approx_A\colon A\times A\to L$ is defined by

$$(a_1\approx_A a_2)=\rho_A(a_1,a_2)\wedge\rho_A(a_2,a_1)$$

which turns out to be a fuzzy equivalence relation.

It is straightforward that ρ_A is extensional w.r.t. $\approx_A$. The following lemma goes beyond extensionality.

Lemma 1: Let $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$ be fuzzy preposets and $\mu\colon A\times B\to L$.

1) $\rho_A\circ\approx_A=\approx_A\circ\rho_A=\rho_A$ and $\rho_B\circ\approx_B=\approx_B\circ\rho_B=\rho_B$.
2) If μ is antitone, then $\mu\circ\mu^{-1}\subseteq\approx_B$.

Proof: From the definition of $\approx_A$, and the facts that ρ_A is reflexive and $\otimes$-transitive, and $\approx_A$ is reflexive, one has: $\rho_A \subseteq \rho_A \circ id_A \subseteq \rho_A \circ \approx_A \subseteq \rho_A \circ \rho_A \subseteq \rho_A$. The rest of the equalities in item 1) are proved similarly.

Assume now that μ is antitone. Then

$$\mu \circ \mu^{-1} = \mu \circ id_A \circ \mu^{-1} \subseteq \mu \circ \rho_A \circ \mu^{-1} \subseteq {\rho_B}^{-1}$$

On the other hand, it is easy to prove that $(\mu \circ \mu^{-1})^{-1} = \mu \circ \mu^{-1}$ and, thus, $(\mu \circ \mu^{-1})^{-1} = \mu \circ \mu^{-1} \subseteq {\rho_B}^{-1}$. Therefore, $\mu \circ \mu^{-1} \subseteq {\rho_B}^{-1} \wedge \rho_B = \approx_B$. ■

It is well-known that, for classical Galois connections (f, g), one has $f \circ g \circ f = f$ and $g \circ f \circ g = g$. In the following theorem, and its two corollaries, we explore what is the behavior of the analogous compositions in the framework of relational fuzzy Galois connections.

Theorem 2: Let (μ, ν) be a relational fuzzy Galois connection between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$. Then $\mu \circ \nu \circ \mu \circ \mu^{-1} \subseteq \approx_B$ and $\nu \circ \mu \circ \nu \circ \nu^{-1} \subseteq \approx_A$.

Proof: We only prove the first inclusion because the second one is analogous. Firstly, since $\nu \circ \mu$ is inflationary and μ is antitone, one has:

$$\begin{aligned}(\mu \circ \nu \circ \mu \circ \mu^{-1})(b_1, b_2) &\leq (\mu \circ \rho_A \circ \mu^{-1})(b_1, b_2)\\ &\leq \rho_B^{-1}(b_1, b_2) = \rho_B(b_2, b_1)\end{aligned}$$

On the other hand, since $\mu \circ \nu$ is inflationary and Lemma 1, one has:

$$\begin{aligned}(\mu \circ \nu \circ \mu \circ \mu^{-1})(b_1, b_2) &\leq (\rho_B \circ \mu \circ \mu^{-1})(b_1, b_2)\\ &\leq (\rho_B \circ \approx_B)(b_1, b_2)\\ &\leq \rho_B(b_1, b_2)\end{aligned}$$

Therefore, $(\mu \circ \nu \circ \mu \circ \mu^{-1})(b_1, b_2) \leq (b_1 \approx_B b_2)$. ■

Corollary 1: Let (μ, ν) be a relational fuzzy Galois connection between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$. The following conditions hold for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$,

1) $\mu(a, b_1) \otimes (\mu \circ \nu \circ \mu)(a, b_2) \leq (b_1 \approx_B b_2)$.
2) $\nu(b, a_1) \otimes (\nu \circ \mu \circ \nu)(b, a_2) \leq (a_1 \approx_A a_2)$.

Proof: Item 1) is a consequence of the fact that

$$\mu(a, b_1) \otimes (\mu \circ \nu \circ \mu)(a, b_2) \leq (\mu \circ \nu \circ \mu \circ \mu^{-1})(b_1, b_2)$$

Item 2) is analogous. ■

Corollary 2: Let (μ, ν) be a relational fuzzy Galois connection between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$. For all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, the following conditions hold:

1) If $b_1 \in \mathrm{im}(\mu)$ then $(\mu \circ \nu)(b_1, b_2) \leq (b_1 \approx_B b_2)$.
2) If $a_1 \in \mathrm{im}(\nu)$ then $(\nu \circ \mu)(a_1, a_2) \leq (a_1 \approx_A a_2)$.

Proof: We prove item 1) and item 2) is proved analogously. If $b_1 \in \mathrm{im}(\mu)$ then $(\mu \circ \mu^{-1})(b_1, b_1) = \top$ and $(\mu \circ \nu)(b_1, b_2) = (\mu \circ \mu^{-1})(b_1, b_1) \otimes (\mu \circ \nu)(b_1, b_2) \leq (\mu \circ \nu \circ \mu \circ \mu^{-1})(b_1, b_2) \leq (b_1 \approx_B b_2)$. ■

IV. Relational fuzzy Galois connections between fuzzy posets

In this section we will study some additional properties of relational fuzzy Galois connections when the underlying structures are fuzzy posets. Note that, a fuzzy poset is a fuzzy preposet $\langle A, \rho_A \rangle$ in which $(a_1 \approx_A a_2) = \top$ implies $a_1 = a_2$ for all $a_1, a_2 \in A$, i.e. the fuzzy symmetric kernel relation $\approx_A$ is a fuzzy equality.

Proposition 1: Let (μ, ν) be a relational fuzzy Galois connection between the fuzzy posets $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$.

1) For all $a \in A$ there exists at most one $b \in B$ such that $\mu(a, b) = \top$. In addition, $(\mu \circ \nu \circ \mu)(a, y) = \top$ if and only if $y = b$.
2) For all $b \in B$ there exists at most one $a \in A$ such that $\nu(b, a) = \top$. In addition, $(\nu \circ \mu \circ \nu)(b, x) = \top$ if and only if $x = a$.

Proof: Suppose that there exist $b_1, b_2 \in B$ such that $\top = \mu(a, b_1) = \mu(a, b_2)$. From $\mu(a, b_1) = \top$, since (μ, ν) is coupled, we have that there exists $y \in B$ such that $(\mu \circ \nu \circ \mu)(a, y) = \top$ and, by Corollary 1,

$$\begin{aligned}\top &= (\mu \circ \nu \circ \mu)(a, y)\\ &= (\mu \circ \nu \circ \mu)(a, y) \otimes \mu(a, b_1) \leq (y \approx_B b_1)\end{aligned}$$

and, by antisymmetry of ρ_B, we obtain $y = b_1$. The same reasoning leads to

$$\top = (\mu \circ \nu \circ \mu)(a, b_1) \otimes \mu(a, b_2) \leq (b_1 \approx_B b_2)$$

By antisymmetry of ρ_B, we have $b_1 = b_2$.

The proof of item 2) is similar. ■

Proposition 2: Let (μ, ν) be a relational fuzzy Galois connection between the fuzzy posets $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$. For all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, the following conditions hold:

1) If $b_1 \in \mathrm{im}(\mu)$, then $(\mu \circ \nu)(b_1, b_2) = \top$ iff $b_1 = b_2$.
2) If $a_1 \in \mathrm{im}(\nu)$, then $(\nu \circ \mu)(a_1, a_2) = \top$ iff $a_1 = a_2$.

Proof: Assume $b_1 \in \mathrm{im}(\mu)$ and $(\mu \circ \nu)(b_1, b_2) = \top$. Then by Corollary 2, one has $(b_1 \approx_B b_2) = \top$ and hence $b_1 = b_2$.

Conversely, if $b_1 \in \mathrm{im}(\mu)$, there exists $a_1 \in A$ such that $\mu(a_1, b_1) = \top$. Since (μ, ν) is coupled, there exist $a_2 \in A$ and $y \in B$ such that $\nu(b_1, a_2) = \mu(a_2, y) = \top$. Therefore, $(\mu \circ \nu \circ \mu)(a_1, y) = \top$. Now, by Proposition 1, $b_1 = y$. Finally, $\top = \nu(b_1, a_2) \otimes \mu(a_2, b_1) \leq (\mu \circ \nu)(b_1, b_1)$. ■

Let us see now that our definition embeds that given by Yao [6]. For this, we have to recall some notions:

Given a fuzzy relation μ, we denote $\mu_\top$ its $\top$-cut, ie.,

$$\mu_\top = \{(a, b) \in A \times B \mid \mu(a, b) = \top\}$$

which is a crisp binary relation. Thus, as a consequence of Proposition 1, if (μ, ν) is a relational fuzzy Galois connection between the fuzzy posets $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$, then:

- $(a, b_1), (a, b_2) \in \mu_\top$ implies $b_1 = b_2$.
- $(b, a_1), (b, a_2) \in \nu_\top$ implies $a_1 = a_2$.

Then, $\mu_\top$ and $\nu_\top$ are *partial* functions whose domains are $\mathrm{dom}(\mu)$ and $\mathrm{dom}(\nu)$, respectively. Moreover, since the pair (μ, ν) is coupled, we have also $\mathrm{im}(\mu) \subseteq \mathrm{dom}(\nu)$ and $\mathrm{im}(\nu) \subseteq \mathrm{dom}(\mu)$ and, then, both partial functions can be composed.

Obviously, if μ and ν are total fuzzy relations, then $\mu_\top$ and $\nu_\top$ are functions.

Hereafter, we will use prefix notation when dealing with $\mu_\top$ and $\nu_\top$.

The following result recovers the definition of fuzzy Galois connection given by Yao [6] in terms of relational fuzzy Galois connections.

Proposition 3: Let (μ, ν) be a relational fuzzy Galois connection between the fuzzy posets $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$. Then, for all $a \in \mathrm{dom}(\mu)$ and $b \in \mathrm{dom}(\nu)$, one has

$$\rho_A(a, \nu_\top(b)) = \rho_B(b, \mu_\top(a))$$

Proof: Consider $a \in \mathrm{dom}(\mu)$ and $b \in \mathrm{dom}(\nu)$, i.e. $\mu(a, \mu_\top(a)) = \top$ and $\nu(b, \nu_\top(b)) = \top$. Then, by Theorem 1,

$$\begin{aligned}\rho_A(a, \nu_\top(b)) &= \nu(b, \nu_\top(b)) \otimes \rho_A^{-1}(\nu_\top(b), a) \otimes \mu(a, \mu_\top(a)) \\ &\le \rho_B(b, \mu_\top(a)) \\ \rho_B(b, \mu_\top(a)) &= \mu(a, \mu_\top(a)) \otimes \rho_B^{-1}(\mu_\top(a), b) \otimes \nu(b, \nu_\top(b)) \\ &\le \rho_A(a, \nu_\top(b))\end{aligned}$$

Thus $\rho_A(a, \nu_\top(b)) = \rho_B(b, \mu_\top(a))$. ■

It is worth noting that if μ and ν are total, then $(\mu_\top, \nu_\top)$ is a fuzzy Galois connection between the fuzzy posets $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ in the sense of Yao. Conversely, any fuzzy Galois connection (in Yao's sense) define a relational fuzzy Galois connection.

Proposition 4: Let (f, g) be a fuzzy Galois connection between two fuzzy posets $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ in Yao's sense. The pair (μ, ν), where $\mu(a, b) = (f(a) \approx_B b)$ and $\nu(b, a) = (g(b) \approx_A a)$, is a relational fuzzy Galois connection.

Proof: Since f and g are mappings, the relations μ and ν are total and (μ, ν) is coupled. By Theorem 1, we must prove $\mu \circ \rho_A^{-1} \circ \nu \subseteq \rho_B$ and $\nu \circ \rho_B^{-1} \circ \mu \subseteq \rho_A$. From Lemma 1 and the fact that $\rho_A(a, g(b)) = \rho_B(b, f(a))$ for all $a \in A$ and $b \in B$, we have that

$$\begin{aligned}&(\mu \circ \rho_A^{-1} \circ \nu)(b_1, b_2) = \\ &= \bigvee_{a_1, a_2 \in A} \big(\nu(b_1, a_1) \otimes \rho_A(a_2, a_1) \otimes \mu(a_2, b_2)\big) \\ &= \bigvee_{a_1, a_2 \in A} \big((g(b_1) \approx_A a_1) \otimes \rho_A(a_2, a_1) \otimes (f(a_2) \approx_B b_2)\big) \\ &= \bigvee_{a_2 \in A} \big(\rho_A(a_2, g(b_1)) \otimes (f(a_2) \approx_B b_2)\big) \\ &= \bigvee_{a_2 \in A} \big(\rho_B(b_1, f(a_2)) \otimes (f(a_2) \approx_B b_2)\big) \\ &\le (\rho_B \circ \approx_B)(b_1, b_2) = \rho_B(b_1, b_2)\end{aligned}$$

The other inclusion is similarly proved. ■

The following example shows that the relation between fuzzy Galois connections in Yao's sense and relational fuzzy Galois connections is not a bijection. Specifically, it is possible to find different relational fuzzy Galois connections between fuzzy posets with the same $\top$-cuts.

Example 1: Consider the underlying truth-values set $\mathbb{L}$ to be the real unit interval with its residuated lattice structure induced by the Łukasiewicz t-norm.

Consider the following fuzzy posets $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ where $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$ and the fuzzy relations ρ_A and ρ_B given below:

ρ_A	a_1	a_2	a_3
a_1	1	1	1
a_2	0.5	1	1
a_3	0	0.5	1

ρ_B	b_1	b_2	b_3
b_1	1	0.5	0
b_2	1	1	1
b_3	1	0.5	1

The fuzzy equivalence relations induced by ρ_A and ρ_B are given below:

$\approx_A$	a_1	a_2	a_3
a_1	1	0.5	0
a_2	0.5	1	0.5
a_3	0	0.5	1

$\approx_B$	b_1	b_2	b_3
b_1	1	0.5	0
b_2	0.5	1	0.5
b_3	0	0.5	1

Consider also the mappings $f\colon A \to B$ defined by $f(a_1) = f(a_2) = b_1$ and $f(a_3) = b_2$ and $g\colon B \to A$ given by $g(b_1) = g(b_3) = a_2$ and $g(b_2) = a_3$. The pair (f, g) is a fuzzy Galois connection in Yao's sense, i.e.

$$\rho_A(a, g(b)) = \rho_B(b, f(a)), \text{ for all } a \in A, b \in B$$

Moreover, we can obtain a relational fuzzy Galois connection by considering the construction given in Proposition 4. That is, considering the pair of fuzzy relations $\mu\colon A \times B \to L$ and $\nu\colon B \times A \to L$ defined by the following tables:

μ	b_1	b_2	b_3
a_1	1	0.5	0
a_2	1	0.5	0
a_3	0.5	1	0.5

ν	a_1	a_2	a_3
b_1	0.5	1	0.5
b_2	0	0.5	1
b_3	0.5	1	0.5

It is just a matter of computation to check that (μ, ν) is a relational fuzzy Galois connection between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$.

Nevertheless, it is not the only relational fuzzy Galois connection whose $\top$-cuts are f and g. Thus, for instance, consider the fuzzy relation $\mu'\colon A \times B \to L$ given by the following table:

μ'	b_1	b_2	b_3
a_1	1	0.1	0
a_2	1	0.5	0
a_3	0.5	1	0.5

The pair (μ', ν) is also a relational fuzzy Galois connection between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ and $\mu_\top = \mu'_\top$.

V. Relational fuzzy Galois connections by means of quotient construction

In this section, we study the behavior of the relational fuzzy Galois connections with respect to the construction of the quotient set modulo the fuzzy symmetric kernel relation. Firstly, let us recall some previous notions.

For a fuzzy equivalence relation $\approx\colon A \times A \to L$, the corresponding quotient set is $\bar{A} = \{\bar{a} \mid a \in A\}$ where $\bar{a}$ is the equivalence class of a, i.e the fuzzy set $\bar{a}\colon A \to L$ defined by $\bar{a}(u) = (a \approx u)$ for all $u \in A$.

Note that $\bar{a} = \bar{b}$ if and only if $(a \approx b) = \top$: on the one hand, if $\bar{a} = \bar{b}$, then $(a \approx b) = \bar{a}(b) = \bar{b}(b) = \top$, by reflexive property; on the other hand, if $(a \approx b) = \top$, then $\bar{a}(u) = (a \approx u) = (b \approx a) \otimes (a \approx u) \le (b \approx u) = \bar{b}(u)$, for all $u \in A$.

In addition, given two fuzzy equivalence relations $\approx_A$ and $\approx_B$ in A and B respectively, any fuzzy relation $\mu\colon A\times B\to L$ induces a fuzzy relation in their quotient sets $\bar\mu\colon \bar A\times\bar B\to L$ defined as $\bar\mu(\bar a,\bar b)=(\approx_B\circ\mu\circ\approx_A)(a,b)$. It is straightforward to check that $\overline{\mu}$ is well-defined, i.e. $\bar a_1=\bar a_2$ and $\bar b_1=\bar b_2$ imply $\bar\mu(\bar a_1,\bar b_1)=\bar\mu(\bar a_2,\bar b_2)$.

The following proposition shows how a fuzzy preorder relation $\rho_A\colon A\times A\to L$ induces a fuzzy poset structure on the quotient set $\bar A$.

Proposition 5: Let $\langle A,\rho_A\rangle$ be a fuzzy preposet, $\approx_A$ its symmetric kernel relation, and $\overline{A}$ and $\overline{\rho_A}$ as defined above. Then

1) $\bar\rho_{\bar A}(\overline{a_1},\overline{a_2})=\rho_A(a_1,a_2)$ for all $a_1,a_2\in A$.
2) $\langle\overline{A},\overline{\rho_A}\rangle$ is a fuzzy poset.

Proof: By Lemma 1, one has

$$\bar\rho_{\bar A}(\overline{a_1},\overline{a_2})=(\approx_A\circ\rho_A\circ\approx_A)(a_1,a_2)=\rho_A(a_1,a_2)$$

and, therefore, $\bar\rho_{\bar A}$ is reflexive and $\otimes$-transitive. Finally, it is trivial that $\overline{\rho_A}$ is antisymmetric because $\rho_A(a_1,a_2)=\rho_A(a_2,a_1)=\top$ implies $(a_1\approx_A a_2)=\top$ and hence $\bar a_1=\bar a_2$. ■

Now, we can see that totality and antitonicity is inherited by $\overline{\mu}$.

Lemma 2: Let $\mu\colon A\times B\to L$ be a fuzzy relation between fuzzy preposets $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$. Then,

1) If μ is a total fuzzy relation, then so is $\overline{\mu}$.
2) If μ is antitone, then $\overline{\mu}$ is antitone as well.

Proof: The first item is trivial because $\mu(a,b)=\top$ implies $\overline{\mu}(\overline{a},\overline{b})=\top$.

Assume now that μ is antitone: $\mu\circ\rho_A\circ\mu^{-1}\subseteq\rho_B^{-1}$. Then, by Proposition 5 and Lemma 1, one has

$$\begin{aligned}\overline{\mu}\circ\overline{\rho_A}\circ\overline{\mu}^{-1}&=\approx_B\circ\mu\circ\approx_A\circ\rho_A\circ\approx_A\circ\mu^{-1}\circ\approx_B\\&=\approx_B\circ\mu\circ\rho_A\circ\mu^{-1}\circ\approx_B\\&\subseteq\approx_B\circ\rho_B^{-1}\circ\approx_B=\overline{\rho_B}^{-1}\end{aligned}$$

Therefore, $\overline{\mu}$ is antitone. ■

We conclude this section with the following theorem that describes how the quotient construction transforms any relational fuzzy Galois connection between fuzzy preposets into another one between fuzzy posets. Then, an example illustrates this construction.

Theorem 3: Let (μ,ν) be a relational fuzzy Galois connection between fuzzy preposets $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$. If μ and ν are total, then $(\overline{\mu},\overline{\nu})$ is a relational fuzzy Galois connection between the fuzzy posets $\langle\overline{A},\overline{\rho_A}\rangle$ and $\langle\overline{B},\overline{\rho_B}\rangle$.

Proof: If μ and ν are total, Lemma 2 ensures that $\overline{\mu}$ and $\overline{\nu}$ are total as well, and therefore $(\overline{\mu},\overline{\nu})$ is coupled. Thus, by Theorem 1, it is sufficient to prove $\overline{\mu}\circ\overline{\rho_A}^{-1}\circ\overline{\nu}\subseteq\overline{\rho_B}$ and $\overline{\nu}\circ\overline{\rho_B}^{-1}\circ\overline{\mu}\subseteq\overline{\rho_A}$.

$$\begin{aligned}\overline{\mu}\circ\overline{\rho_A}^{-1}\circ\overline{\nu}&=\approx_B\circ\mu\circ\approx_A\circ\rho_A^{-1}\circ\approx_A\circ\nu\circ\approx_B\\&=\approx_B\circ\mu\circ\rho_A^{-1}\circ\nu\circ\approx_B\\&\subseteq\approx_B\circ\rho_B\circ\approx_B=\overline{\rho_B}\end{aligned}$$

The proof of $\overline{\nu}\circ\overline{\rho_B}^{-1}\circ\overline{\mu}\subseteq\overline{\rho_A}$ is analogous. ■

Example 2: Consider the underlying truth-values set $\mathbb{L}$ to be the real unit interval with its residuated lattice structure induced by the Łukasiewicz t-norm.

Consider the following fuzzy preposets $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$ where $A=\{a_1,a_2,a_3,a_4\}$, $B=\{b_1,b_2,b_3,b_4\}$ and the fuzzy relations ρ_A and ρ_B given below:

ρ_A	a_1	a_2	a_3	a_4
a_1	1	1	1	1
a_2	0.2	1	1	1
a_3	0.2	1	1	1
a_4	0	0.1	0.1	1

ρ_B	b_1	b_2	b_3	b_4
b_1	1	1	1	0.1
b_2	1	1	1	0.1
b_3	1	1	1	0.1
b_4	1	1	1	1

The pair of fuzzy relations $\mu\colon A\times B\to L$ and $\nu\colon B\times A\to L$ given by the following tables constitutes a relational fuzzy Galois connection between $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$.

μ	b_1	b_2	b_3	b_4
a_1	1	0.1	0.1	0
a_2	1	0.1	0.1	0
a_3	0.1	1	0.1	0
a_4	0	0	0	1

ν	a_1	a_2	a_3	a_4
b_1	0	1	0.4	0
b_2	0	0.1	1	0.1
b_3	0	1	0.1	0.1
b_4	0	0	0	1

The fuzzy equivalence relations induced by ρ_A and ρ_B are given below:

$\approx_A$	a_1	a_2	a_3	a_4
a_1	1	0.2	0.2	0
a_2	0.2	1	1	0.1
a_3	0.2	1	1	0.1
a_4	0	0.1	0.1	1

$\approx_B$	b_1	b_2	b_3	b_4
b_1	1	1	1	0.1
b_2	1	1	1	0.1
b_3	1	1	1	0.1
b_4	0.1	0.1	0.1	1

Then, the equivalence classes are the following:

$$\begin{aligned}\overline{a_1}&=\{a_1/1,a_2/0.2,a_3/0.2\}\\\overline{a_2}=\overline{a_3}&=\{a_1/0.2,a_2/1,a_3/1,a_4/0.1\}\\\overline{a_4}&=\{a_2/0.1,a_3/0.1,a_4/1\}\\\overline{b_1}=\overline{b_2}=\overline{b_3}&=\{b_1/1,b_2/1,b_3/1,b_4/0.1\}\\\overline{b_4}&=\{b_1/0.1,b_2/0.1,b_3/0.1,b_4/1\}\end{aligned}$$

The quotient relations are:

$\overline{\rho_A}$	$\overline{a_1}$	$\overline{a_2}$	$\overline{a_4}$
$\overline{a_1}$	1	1	1
$\overline{a_2}$	0.2	1	1
$\overline{a_4}$	0	0.1	1

$\overline{\rho_B}$	$\overline{b_1}$	$\overline{b_4}$
$\overline{b_1}$	1	0.1
$\overline{b_4}$	1	1

The pair of fuzzy relations $\overline{\mu}\colon\overline{A}\times\overline{B}\to L$ and $\overline{\nu}\colon\overline{B}\times\overline{A}\to L$ given by the following tables constitutes a relational fuzzy Galois connection between the fuzzy posets $\langle\overline{A},\overline{\rho_A}\rangle$ and $\langle\overline{B},\overline{\rho_B}\rangle$.

$\overline{\mu}$	$\overline{b_1}$	$\overline{b_4}$
$\overline{a_1}$	1	0
$\overline{a_2}$	1	0
$\overline{a_4}$	0	1

$\overline{\nu}$	$\overline{a_1}$	$\overline{a_2}$	$\overline{a_4}$
$\overline{b_1}$	0	1	0
$\overline{b_4}$	0	0	1

VI. Conclusions and further work

We have introduced the notion of relational fuzzy Galois connection, in which the components of the connection are not fuzzy functions but fuzzy relations satisfying certain properties, among which the novelty is that the pair of relations forming the connection should be coupled. We have also proven that the provided construction naturally embeds Yao's notion of fuzzy Galois connection as a particular case.

Although the basic framework for the proposed notion of relational fuzzy Galois connection is that of fuzzy preposets, we have shown that it behaves properly with respect to the formation of quotient over the fuzzy symmetric kernel relation, thus, leading to connections between fuzzy posets.

As future work, we are planning to continue the line initiated in [11], [12] and attempt the construction of the residual, in the sense of relational fuzzy Galois connections, to a given mapping between differently structured domain and codomain, as stated in the introduction of this work.

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