Towards relational fuzzy adjunctions

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***Abstract*—The problem of studying the existence of a right adjoint for a mapping defined between sets with different fuzzy structure naturally leads to the search of new notions of adjunction which fit better with the underlying structure of domain and codomain. In this work, we introduce a version of relational fuzzy adjunction between fuzzy preposets which generalizes previous approaches in that its components are fuzzy relations. We also prove that the construction behaves properly with respect to the formation of quotient with respect to the symmetric kernel relation and, thus, giving rise to a relational fuzzy adjunction between fuzzy posets.**

I. Introduction

This paper focuses on a certain generalization of the notion of adjunction (also called isotone Galois connection), which has proved to be a practical instrument to link different topics and has found a number of applications, both theoretical and practical. The interested reader can find more details for instance in [1], [2].

In the recent years, we have studied the problem of constructing the right adjoint of a mapping $f\colon A \to B$ between *differently structured* environments, in the sense that A has a richer structure than B and, in order to build an adjunction, the necessary structure has to be built on B and, then, the right adjoint should be built. A number of results have been obtained so far on different underlying settings. Namely:

- In [3], our underlying environment was that of crisp functions between a poset (resp. preordered set) and an unstructured set;
- Then, in [4], the paradigm was shifted to the fuzzy case, considering the corresponding problem in which the set A has a fuzzy preposet structure;
- Later, in [5], we considered in addition fuzzy equivalence relations generalizing the equality, both in A and in B;
- More recently, in [6], we started to find a more adequate definition of adjunction in a fuzzy environment, since the fuzzy extensions given in [4], [5] lack of fuzziness precisely on the adjunction, namely, both components of the adjunction are crisp, and we would like to work with a really fuzzy version of the notion of adjunction, in which they are fuzzy functions as well.

In this work we focus on a further generalization in which the components of the fuzzy adjunction are not fuzzy functions but fuzzy relations satisfying less requirements than a fuzzy function, the only requirement being to satisfy the property of totality.

The structure of the paper is the following: in Section II, we introduce the preliminary notions which will be needed thereafter; then, in Section III we introduce the notion of relational fuzzy adjunction; later, Section IV focuses on the behavior of the relational fuzzy adjunctions with respect to forming the quotient over the symmetric kernel relation; finally, in Section V we draw some conclusions and present prospects of future work.

II. Preliminary definitions

Given a complete residuated lattice $\mathbb{L} = (L, \otimes, \Rightarrow)$, an $\mathbb{L}$-fuzzy set is a mapping from the universe set to the membership values structure $X\colon U \to L$ where $X(u)$ means the degree in which u belongs to X.

An $\mathbb{L}$-fuzzy binary relation on U is an $\mathbb{L}$-fuzzy subset of $U \times U$, that is $R_U\colon U \times U \to L$, and it is said to be:

- *Reflexive* if $R_U(a,a) = \top$ for all $a \in U$.
- *$\otimes$-Transitive* if $R_U(a,b) \otimes R_U(b,c) \leq R_U(a,c)$ for all $a,b,c \in U$.
- *Symmetric* if $R_U(a,b) = R_U(b,a)$ for all $a,b \in U$.
- *Antisymmetric* if $R_U(a,b) = R_U(b,a) = \top$ implies $a = b$, for all $a,b \in U$.

From now on, when no confusion arises, we will omit the prefix "$\mathbb{L}$-".

Definition 1: A *fuzzy preposet* is a pair $\mathbb{A} = \langle A, \rho_A \rangle$ in which ρ_A is a reflexive and $\otimes$-transitive fuzzy relation on A. In addition, a *fuzzy poset* is a fuzzy preposet $\mathbb{A} = \langle A, \rho_A \rangle$ in which ρ_A is also antisymmetric.

Definition 2: A fuzzy relation $\approx$ on A is said to be a:

- *Fuzzy equivalence relation* if $\approx$ is a reflexive, $\otimes$-transitive and symmetric fuzzy relation on A.
- *Fuzzy equality* if $\approx$ is a fuzzy equivalence relation satisfying that $\approx(a,b) = \top$ implies $a = b$, for all $a,b \in A$.

We will use the infix notation for a fuzzy equivalence relation, that is: for $\approx\colon A \times A \to L$ a fuzzy equivalence relation, we denote $a_1 \approx a_2$ to refer to $\approx(a_1, a_2)$.

Definition 3: For a fuzzy equivalence relation $\approx\colon A \times A \to L$, the equivalence class of an element $a \in A$ is a fuzzy set $\overline{a}\colon A \to L$ defined by $\overline{a}(u) = (a \approx u)$ for all $u \in A$.
Note that $\overline{a} = \overline{b}$ if and only if $(a \approx b) = \top$: on the one hand, if $\overline{a} = \overline{b}$, then $(a \approx b) = \overline{a}(b) = \overline{b}(b) = \top$, by reflexive property; conversely, if $(a \approx b) = \top$, then $\overline{a}(u) = (a \approx u) = (b \approx a) \otimes (a \approx u) \leq (b \approx u) = \overline{b}(u)$, for all $u \in A$.

Definition 4: Let $\approx_A$ and $\approx_B$ be fuzzy equivalence relations on A and B respectively. A fuzzy relation $\mu\colon A \times B \to L$ is said to be *extensional* if the following conditions hold:

(Ext1) $\mu(a_1, b) \otimes (a_1 \approx_A a_2) \leq \mu(a_2, b)$ for all $a_1, a_2 \in A$ and $b \in B$.

(Ext2) $\mu(a, b_1) \otimes (b_1 \approx_B b_2) \leq \mu(a, b_2)$ for all $a \in A$ and $b_1, b_2 \in B$.

Moreover, μ is said to be *total* whenever the following condition holds:

(Tot) For all $a \in A$ there exists $b \in B$ satisfying that $\mu(a, b) = \top$.

Hereinafter, all the fuzzy relations considered will be total.

III. Fuzzy relations and relational fuzzy adjunctions

In this section, we introduce a novel definition of fuzzy adjunction in which the role of left and right adjoints is played by total fuzzy relations.

Definition 5: Let $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ be fuzzy preposets and $\mu\colon A \times B \to \mathbb{L}$ and $\nu\colon B \times A \to \mathbb{L}$ be total fuzzy relations. The pair (μ, ν) is said to be a *relational fuzzy adjunction* between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ if the following conditions hold for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$:

i) $\rho_A(a_1, a_2) \otimes \mu(a_1, b_1) \otimes \nu(b_2, a_2) \leq \rho_B(b_1, b_2)$.

ii) $\rho_B(b_1, b_2) \otimes \mu(a_1, b_1) \otimes \nu(b_2, a_2) \leq \rho_A(a_1, a_2)$.

This definition can be given in terms of composition of fuzzy relations as follows:

$$\nu^{-1} \circ \rho_A \circ \mu^{-1} \leq \rho_B \quad \text{and} \quad \nu \circ \rho_B \circ \mu \leq \rho_A$$

where the composition of two fuzzy relations is defined as usual.

In order to study the properties of relational fuzzy adjunctions, we need to adapt the well-known notions of isotone, inflationary, deflationary mapping.

Definition 6: Let $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ be fuzzy preposets. A fuzzy relation $\mu\colon A \times B \to \mathbb{L}$ is said to be *isotone* if $\rho_A(a_1, a_2) \otimes \mu(a_1, b_1) \otimes \mu(a_2, b_2) \leq \rho_B(b_1, b_2)$ for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Definition 7: Let $\langle A, \rho_A \rangle$ be a fuzzy preposet. A fuzzy relation $\mu\colon A \times A \to \mathbb{L}$ is said to be:

- *inflationary* if $\mu(a_1, a_2) \leq \rho_A(a_1, a_2)$ for all $a_1, a_2 \in A$.
- *deflationary* if $\mu(a_1, a_2) \leq \rho_A(a_2, a_1)$ for all $a_1, a_2 \in A$.

The following theorem gives a first characterization of relational fuzzy adjunctions, which resembles the classical behavior of crisp adjunctions.

Theorem 1 (See [6]): Let $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ be fuzzy preposets and $\mu\colon A \times B \to \mathbb{L}$ and $\nu\colon B \times A \to \mathbb{L}$ be total fuzzy relations. Then, (μ, ν) is a relational fuzzy adjunction if and only if μ and ν are isotone, $\nu \circ \mu$ is inflationary and $\mu \circ \nu$ is deflationary.

Proposition 1: Let (μ, ν) be a relational fuzzy adjunction between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$. The following conditions hold:

1) $\mu(a, b_1) \otimes (\mu \circ \nu \circ \mu)(a, b_2) \leq \rho_B(b_1, b_2) \wedge \rho_B(b_2, b_1)$ for all $a \in A$ and $b_1, b_2 \in B$.
2) $\nu(b, a_1) \otimes (\nu \circ \mu \circ \nu)(b, a_2) \leq \rho_A(a_1, a_2) \wedge \rho_A(a_2, a_1)$ for all $a_1, a_2 \in A$ and $b \in B$.
3) For all $a_1 \in A$ and $b \in B$ there exists $a_2 \in A$ such that $\mu(a_1, b) \leq \rho_A(a_1, a_2)$.
4) For all $a \in A$ and $b_1 \in B$ there exists $b_2 \in B$ such that $\nu(b_1, a) \leq \rho_B(b_2, b_1)$.

Proof: We only prove the items 1) and 3) because 2) and 4) are analogous.

1) On the one hand, since $\nu \circ \mu$ is inflationary, $\otimes$ is distributive wrt $\vee$, and μ is isotone, one has:

$$\begin{aligned}
&\mu(a, b_1) \otimes (\mu \circ \nu \circ \mu)(a, b_2) \\
&\quad = \mu(a, b_1) \otimes \bigvee_{x \in A} \bigvee_{y \in B} \big(\mu(a, y) \otimes \nu(y, x) \otimes \mu(x, b_2)\big) \\
&\quad \leq \mu(a, b_1) \otimes \bigvee_{x \in A} \big(\rho_A(a, x) \otimes \mu(x, b_2)\big) \\
&\quad \leq \bigvee_{x \in A} \big(\rho_A(a, x) \otimes \mu(a, b_1) \otimes \mu(x, b_2)\big) \\
&\quad \leq \rho_B(b_1, b_2)
\end{aligned}$$

On the other hand, since $\nu \circ \mu$ is deflationary, $\otimes$ is distributive wrt $\vee$, ρ_A is reflexive, μ is isotone, and ρ_B is $\otimes$-transitive, one has:

$$\begin{aligned}
&\mu(a, b_1) \otimes (\mu \circ \nu \circ \mu)(a, b_2) \\
&\quad = \mu(a, b_1) \otimes \bigvee_{x \in A} \bigvee_{y \in B} \big(\mu(a, y) \otimes \nu(y, x) \otimes \mu(x, b_2)\big) \\
&\quad \leq \mu(a, b_1) \otimes \bigvee_{y \in B} \big(\mu(a, y) \otimes \rho_B(b_2, y)\big) \\
&\quad \leq \bigvee_{y \in B} \big(\mu(a, b_1) \otimes \mu(a, y) \otimes \rho_B(b_2, y)\big) \\
&\quad \leq \bigvee_{y \in B} \big(\rho_A(a, a) \otimes \mu(a, y) \otimes \mu(a, b_1) \otimes \rho_B(b_2, y)\big) \\
&\quad \leq \bigvee_{y \in B} \big(\rho_B(y, b_1) \otimes \rho_B(b_2, y)\big) \\
&\quad \leq \rho_B(b_2, b_1)
\end{aligned}$$

3) Since ν is total, for all $b \in B$ there exists $a_2 \in A$ such that $\nu(b, a_2) = \top$ and, hence $\mu(a_1, b) = \mu(a_1, b) \otimes \nu(b, a_2)$ and, since $\nu \circ \mu$ is inflationary, one has $\mu(a_1, b) \leq \rho_A(a_1, a_2)$.

■

Proposition 2: Let (μ, ν) be a relational fuzzy adjunction between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$. Then,

1) For all $a_1 \in A$ there exists $a_2 \in A$ such that
$$(\nu \circ \mu)(a_1, a_2) = \rho_A(a_1, a_2) = \top$$
2) For all $b_1 \in A$ there exists $b_2 \in A$ such that
$$(\mu \circ \nu)(b_1, b_2) = \rho_B(b_2, b_1) = \top$$

Proof: It is straighforward from the fact that μ and ν are total (and so are both compositions) and that $\nu \circ \mu$ is inflationary, and $\mu \circ \nu$ is deflationary. ■

Given a fuzzy relation μ, we denote $\mu_\top$ its $\top$-cut, ie.,

$$\mu_\top = \{(a, b) \in A \times B \mid \mu(a, b) = \top\}$$

which is a crisp binary relation.

Proposition 3: Let (μ, ν) be a relational fuzzy adjunction between $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$. Then, $(\mu_\top, \nu_\top)$ is a relational adjunction between $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$, ie., for all $(a_1, b_1) \in \mu_\top$ and $(b_2, a_2) \in \nu_\top$, one has

$$\rho_A(a_1, a_2) = \rho_B(b_1, b_2)$$

Proof: If $(a_1, b_1) \in \mu_\top$ then $\mu(a_1, b_1) = \top$ and if $(b_2, a_2) \in \nu_\top$ then $\nu(b_2, a_2) = \top$. Therefore,

$$\rho_A(a_1, a_2) = \mu(a_1, b_1) \otimes \rho_A(a_1, a_2) \otimes \nu(b_2, a_2) \le \rho_B(b_1, b_2)$$

Analogously, $\rho_B(b_1, b_2) \le \rho_A(a_1, a_2)$. ■

Corollary 1: Let (μ, ν) be a relational fuzzy adjunction between $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$. There exist mappings $f \subseteq \mu_\top$ and $g \subseteq \nu_\top$ such that (f, g) is an adjunction between $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$ (in the sense of [4]).

Proof: Since the relation μ is total, for all $a \in A$, there exists $b \in B$ such that $(a, b) \in \mu_\top$ thus by axiom of choice, a mapping $f\colon A \to B$ can be defined with $(a, f(a)) \in \mu_\top$. Similarly, a mapping $g\colon B \to A$ can be obtained from ν. Therefore, $\rho_A(a, g(b)) = \rho_B(b, f(a))$. ■

Proposition 4: Let (μ, ν) be a relational fuzzy adjunction between the fuzzy posets $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$. Then,

1) $(\mu \circ \nu \circ \mu)(a, b) = \top$ if and only if $\mu(a, b) = \top$
2) $(\nu \circ \mu \circ \nu)(b, a) = \top$ if and only if $\nu(b, a) = \top$

for all $a \in A$ and $b \in B$.

Furthermore, for all $a \in A$ there exists a unique $b \in B$ such that $\mu(a, b) = \top$.

Proof: Assume that $\mu(a, b) = \top$, since $\mu \circ \nu \circ \mu$ is also total, there exists $y \in B$ sucht that

$$\begin{aligned}\top &= (\mu \circ \nu \circ \mu)(a, y) \\ &= (\mu \circ \nu \circ \mu)(a, y) \otimes \mu(a, b) \le \rho_B(b, y) \wedge \rho_B(y, b)\end{aligned}$$

by Proposition 1. By antisymmetry of ρ_B, we obtain $b = y$, hence $(\mu\circ\nu\circ\mu)(a, b) = \top$. The other implications are obtained in a similar way. Suppose that there exist $b_1, b_2 \in B$ such that $\top = \mu(a, b_1) = \mu(a, b_2)$, then

$$\top = (\mu \circ \nu \circ \mu)(a, b_1) \otimes \mu(a, b_2) \le \rho_B(b_1, b_2) \wedge \rho_B(b_2, b_1)$$

By antisymmetry of ρ_B, we have $b_1 = b_2$. ■

In the case of fuzzy posets, the mappings f and g in the previous corollary are unique, and define an adjunction between fuzzy posets.

Corollary 2: Let (μ, ν) be a relational fuzzy adjunction between the fuzzy posets $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$. There exist two unique mappings $f = \mu_\top$ and $g = \nu_\top$ such that (f, g) is a fuzzy adjunction between $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$ (in the sense of Yao [12]).

The following result ensures that the converse result also holds.

Proposition 5: Let (f, g) be a fuzzy adjunction between two fuzzy posets $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$ in Yao's sense. Consider $\mu\colon A \times B \to L$ and $\nu\colon A \times B \to L$ defined as $\mu(a, b) = \rho_B(f(a), b) \wedge \rho_B(b, f(a))$ and $\nu(b, a) = \rho_A(g(b), a) \wedge \rho_A(a, g(b))$ for all $a \in A$ and $b \in B$. Then, the pair (μ, ν) is a relational fuzzy adjunction.

Proof: Straightforward. ■

The following example shows that the relation between fuzzy adjunctions in Yao's sense and relational fuzzy functions is not a bijection. Specifically, it is possible to find different relational fuzzy adjuntions between fuzzy posets with the same $\top$-cuts.

Example 1: Consider the underlying truth-values set $\mathbb{L}$ to be the real unit interval with its residuated lattice structure induced by the Łukasiewicz t-norm.

Consider the following fuzzy posets $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$ where $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$ and the fuzzy relations ρ_A and ρ_B given below:

ρ_A	a_1	a_2	a_3
a_1	1	1	1
a_2	0.5	1	1
a_3	0	0.5	1

ρ_B	b_1	b_2	b_3
b_1	1	1	1
b_2	0.5	1	0.5
b_3	0	1	1

Consider also the mappings $f\colon A \to B$ defined by $f(a_1) = f(a_2) = b_1$ and $f(a_3) = b_2$ and $g\colon B \to A$ given by $g(b_1) = g(b_3) = a_2$ and $g(b_2) = a_3$. The pair (f, g) is an adjunction in Yao's sense, i.e.

$$\rho_A(a, g(b)) = \rho_B(f(a), b), \text{ for all } a \in A, b \in B$$

Moreover, we can obtain a relational fuzzy adjunction by considering the construction given in Proposition 5. That is, considering the pair of fuzzy relations $\mu\colon A \times B \to L$ and $\nu\colon B \times A \to L$ defined by the following tables:

μ	b_1	b_2	b_3
a_1	1	0.5	0
a_2	1	0.5	0
a_3	0.5	1	0.5

ν	a_1	a_2	a_3
b_1	0.5	1	0.5
b_2	0	0.5	1
b_3	0.5	1	0.5

It is just a matter of computation to check that (μ, ν) is a relational fuzzy adjunction between $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$

Nevertheless, it is not the only relational fuzzy adjunction whose $\top$-cuts are f and g. Thus, for instance, consider the fuzzy relation $\mu'\colon A \times B \to L$ given by the following table:

μ'	b_1	b_2	b_3
a_1	1	0.1	0
a_2	1	0.5	0
a_3	0.5	1	0.5

The pair (μ', ν) is also a relational fuzzy adjunction between $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$ and $\mu_\top = \mu'_\top$.

IV. From adjunctions between fuzzy preposets to adjunctions between fuzzy posets

In this section we will study the behaviour of the relational fuzzy adjunction with respect to the construction of the quotient set modulo a certain fuzzy equivalence relation, the fuzzy symmetric kernel relation defined below.

Specifically, given a fuzzy preposet $\langle A, \rho_A\rangle$ we will consider the relation $\approx_A\colon A \times A \to L$ defined by

$$(a_1 \approx_A a_2) = \rho_A(a_1, a_2) \wedge \rho_A(a_2, a_1)$$

which turns out to be a fuzzy equivalence relation.

The corresponding quotient set is $\bar{A} = \{\bar{a} \mid a \in A\}$. Now, we will introduce a fuzzy preorder relation $\bar{\rho}_{\bar{A}}\colon \bar{A} \times \bar{A} \to L$

in order to induce a fuzzy poset structure on $\bar{A}$. The definition is straightforward:

$$\bar{\rho}_{\bar{A}}(\overline{a_1}, \overline{a_2}) = \rho_A(a_1, a_2)$$

Proposition 6: Let $\langle A, \rho_A\rangle$ be a fuzzy preposet and $\overline{A}$ and $\overline{\rho_A}$ be defined as above. Then $\langle \overline{A}, \overline{\rho_A}\rangle$ is a fuzzy poset.

Proof: Let us see that ρ_A is extensional with respect to $\approx_A$, ie. the following inequalities hold for all $x, y, z \in A$:

1) $\rho_A(x, y) \otimes (y \approx z) \leq \rho_A(x, z)$,
2) $\rho_A(x, y) \otimes (x \approx z) \leq \rho_A(z, y)$.

In effect,

$$\begin{aligned}\rho_A(x, y) \otimes (y \approx z) &= \rho_A(x, y) \otimes (\rho_A(y, z) \wedge \rho_A(z, y)) \\ &\leq \rho_A(x, y) \otimes \rho_A(y, z) \\ &\leq \rho_A(x, z).\end{aligned}$$

The other inequality follows similarly.

We can now prove that $\bar{\rho}$ is well-defined. That is, assuming $\overline{a_i} = \overline{x_i}$, let us show that $\rho_A(a_1, a_2) = \rho_A(x_1, x_2)$:

$$\begin{aligned}\rho_A(a_1, a_2) &= (a_1 \approx_A x_1) \otimes \rho_A(a_1, a_2) \otimes (a_2 \approx_A x_2) \\ &\leq \rho_A(x_1, a_2) \otimes (a_2 \approx_A x_2) \\ &\leq \rho_A(x_1, x_2)\end{aligned}$$

The other inequality is similar.

Finally, it is straightforward that $\overline{\rho_A}$ is reflexive, $\otimes$-transitive and antisymmetric. ■

Now, given a fuzzy relation $\mu \colon A \times B \to L$ between fuzzy preposets $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$, we define $\overline{\mu} \colon \bar{A} \times \bar{B} \to L$ as follows

$$\overline{\mu}(\overline{a}, \overline{b}) = \bigvee_{x\in A} \bigvee_{y \in B} \mu(x, y) \otimes (x \approx_A a) \otimes (y \approx_B b)$$

It is straightforward to check that $\overline{\mu}$ is well-defined. Similarly, we can see that totality and isotonicity is inherited by $\overline{\mu}$.

Lemma 1: If μ is a total fuzzy relation, then so is $\overline{\mu}$.

Proof: Given $\bar{a} \in \bar{A}$, consider $a \in A$ and the fact that μ is total, then there exists $b \in B$ such that $\mu(a, b) = \top$. Now, it is not difficult to check $\overline{\mu}(\overline{a}, \overline{b}) = \top$, therefore $\overline{\mu}$ is total. ■

Lemma 2: If μ is isotone, then $\overline{\mu}$ is isotone as well.

Proof: For all $a_1, a_2, x_1, x_2 \in A$ and $b_1, b_2, y_1, y_2 \in B$, the following inequalities hold:

$$\begin{aligned}&\mu(x_1, y_1) \otimes (y_1 \approx_A b_1) \otimes (x_1 \approx_A a_1) \otimes \rho_A(a_1, a_2) \otimes \\ &\qquad \otimes \mu(x_2, y_2) \otimes (y_2 \approx_B b_2) \otimes (x_2 \approx_B a_2) \\ &\leq \mu(x_1, y_1) \otimes (y_1 \approx_A b_1) \otimes \rho_A(x_1, a_2) \otimes \\ &\qquad \otimes \mu(x_2, y_2) \otimes (y_2 \approx_B b_2) \otimes (x_2 \approx_B a_2) \\ &\leq \mu(x_1, y_1) \otimes (y_1 \approx_A b_1) \otimes \\ &\qquad \otimes \rho_A(x_1, x_2) \otimes \mu(x_2, y_2) \otimes (y_2 \approx_B b_2) \\ &\leq \rho_B(y_1, y_2) \otimes (y_1 \approx_A b_1) \otimes (y_2 \approx_B b_2) \\ &\leq \rho_B(b_1, b_2)\end{aligned}$$

Therefore, by the definition of $\overline{\mu}$ and $\bar{\rho}$, one has

$$\begin{aligned}&\overline{\mu}(\overline{a_1}, \overline{b_1}) \otimes \bar{\rho}_{\bar{A}}(\overline{a_1}, \overline{a_2}) \otimes \overline{\mu}(\overline{a_2}, \overline{b_2}) = \\ &\bigvee_{\substack{x_1 \in A \\ y_1 \in B}} \big(\mu(x_1, y_1) \otimes (y_1 \approx_A b_1) \otimes (x_1 \approx_A a_1)\big) \otimes \\ &\otimes \rho_A(a_1, a_2) \otimes \bigvee_{\substack{x_2 \in A \\ y_2 \in B}} \big(\mu(x_2, y_2) \otimes (y_2 \approx_B b_2) \otimes (x_2 \approx_B a_2)\big) \\ &\leq \rho_B(b_1, b_2) = \bar{\rho}_{\bar{B}}(\overline{b_1}, \overline{b_2}) \,.\end{aligned}$$

■

Now, we can state and prove the following result:

Theorem 2: Let (μ, ν) be a relational fuzzy adjunction between fuzzy preposets $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$, then $(\overline{\mu}, \overline{\nu})$ is a relational fuzzy adjunction between the fuzzy posets $\langle \overline{A}, \overline{\rho_A}\rangle$ and $\langle \overline{B}, \overline{\rho_B}\rangle$.

Proof: We have to prove the following inequality

$$\bar{\rho}_{\bar{A}}(\overline{a_1}, \overline{a_2}) \otimes \overline{\mu}(\overline{a_1}, \overline{b_1}) \otimes \overline{\nu}(\overline{b_2}, \overline{a_2}) \leq \bar{\rho}_{\bar{B}}(\overline{b_1}, \overline{b_2})$$

Unfolding the definitions we prove that, for all $a_1, a_2, x_1, x_2 \in A$ and $b_1, b_2, y_1, y_2 \in B$:

$$\begin{aligned}&\rho_A(a_1, a_2) \otimes (a_1 \approx_A x_1) \otimes (y_1 \approx_B b_1) \otimes \mu(x_1, y_1) \otimes \\ &\qquad \otimes (b_2 \approx_B y_2) \otimes (x_2 \approx_A a_2) \otimes \nu(y_2, x_2) \\ &\leq \rho_A(x_1, x_2) \otimes \mu(x_1, y_1) \otimes \nu(y_2, x_2) \otimes \\ &\qquad \otimes (y_1 \approx_B b_1) \otimes (b_2 \approx_B y_2) \\ &\leq \rho_B(y_1, y_2) \otimes (y_1 \approx_B b_1) \otimes (b_2 \approx_B y_2) \\ &\leq \rho_B(b_1, b_2)\end{aligned}$$

Therefore, by the definition of $\overline{\mu}$ and $\bar{\rho}$, one has

$$\begin{aligned}&\bar{\rho}_{\bar{A}}(\overline{a_1}, \overline{a_2}) \otimes \overline{\mu}(\overline{a_1}, \overline{b_1}) \otimes \overline{\nu}(\overline{b_2}, \overline{a_2}) = \\ &\rho_A(a_1, a_2) \otimes \bigvee_{\substack{x_1 \in A \\ y_1 \in B}} \big((a_1 \approx_A x_1) \otimes (y_1 \approx_B b_1) \otimes \mu(x_1, y_1)\big) \otimes \\ &\otimes \bigvee_{\substack{x_2 \in A \\ y_2 \in B}} \big((b_2 \approx_B y_2) \otimes (x_2 \approx_A a_2) \otimes \nu(y_2, x_2)\big) \\ &\leq \rho_B(b_1, b_2) = \bar{\rho}_{\bar{B}}(\overline{b_1}, \overline{b_2}) \,.\end{aligned}$$

■

Example 2: Consider the underlying truth-values set $\mathbb{L}$ to be the real unit interval with its residuated lattice structure induced by the Łukasiewicz t-norm.

Consider the following fuzzy preposets $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$ where $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_1, b_2, b_3, b_4\}$ and the fuzzy relations ρ_A and ρ_B given below:

ρ_A	a_1	a_2	a_3	a_4
a_1	1	1	1	1
a_2	0.2	1	1	1
a_3	0.2	1	1	1
a_4	0	0.1	0.1	1

ρ_B	b_1	b_2	b_3	b_4
b_1	1	1	1	1
b_2	1	1	1	1
b_3	1	1	1	1
b_4	0.1	0.1	0.1	1

The pair of fuzzy relations $\mu \colon A \times B \to L$ and $\nu \colon B \times A \to L$ given by the following tables constitutes a relational fuzzy adjunction between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$.

μ	b_1	b_2	b_3	b_4
a_1	1	0.1	0.1	0
a_2	1	0.1	0.1	0
a_3	0.1	1	0.1	0
a_4	0	0	0	1

ν	a_1	a_2	a_3	a_4
b_1	0	1	0.4	0
b_2	0	0.1	1	0.1
b_3	0	1	0.1	0.1
b_4	0	0	0	1

The fuzzy equivalence relations induced by ρ_A and ρ_B are given below:

$\approx_A$	a_1	a_2	a_3	a_4
a_1	1	0.2	0.2	0
a_2	0.2	1	1	0.1
a_3	0.2	1	1	0.1
a_4	0	0.1	0.1	1

$\approx_B$	b_1	b_2	b_3	b_4
b_1	1	1	1	0.1
b_2	1	1	1	0.1
b_3	1	1	1	0.1
b_4	0.1	0.1	0.1	1

Then, the equivalence classes are the following:

$$\begin{aligned} \overline{a_1} &= \{a_1/_1, a_2/_{0.2}, a_3/_{0.2}\} \\ \overline{a_2} = \overline{a_3} &= \{a_1/_{0.2}, a_2/_1, a_3/_1, a_4/_{0.1}\} \\ \overline{a_4} &= \{a_2/_{0.1}, a_3/_{0.1}, a_4/_1\} \\ \overline{b_1} = \overline{b_2} = \overline{b_3} &= \{b_1/_1, b_2/_1, b_3/_1, b_4/_{0.1}\} \\ \overline{b_4} &= \{b_1/_{0.1}, b_2/_{0.1}, b_3/_{0.1}, b_4/_1\} \end{aligned}$$

The quotient relations are:

$\overline{\rho_A}$	$\overline{a_1}$	$\overline{a_2}$	$\overline{a_4}$
$\overline{a_1}$	1	1	1
$\overline{a_2}$	0.2	1	1
$\overline{a_4}$	0	0.1	1

$\overline{\rho_B}$	$\overline{b_1}$	$\overline{b_4}$
$\overline{b_1}$	1	1
$\overline{b_4}$	0.1	1

The pair of fuzzy relations $\overline{\mu}\colon \overline{A} \times \overline{B} \to L$ and $\overline{\nu}\colon \overline{B} \times \overline{A} \to L$ given by the following tables constitutes a relational fuzzy adjunction between the fuzzy posets $\langle \overline{A}, \overline{\rho_A} \rangle$ and $\langle \overline{B}, \overline{\rho_B} \rangle$.

$\overline{\mu}$	$\overline{b_1}$	$\overline{b_4}$
$\overline{a_1}$	1	0
$\overline{a_2}$	1	0
$\overline{a_4}$	0	1

$\overline{\nu}$	$\overline{a_1}$	$\overline{a_2}$	$\overline{a_4}$
$\overline{b_1}$	0	1	0
$\overline{b_4}$	0	0	1

V. Conclusions and further work

This work is a natural continuation of [6] where we revisited the problem of studying when a given fuzzy relation can be characterized in terms of a fuzzy function and provided a notion of fuzzy adjunction as a pair of completely functional fuzzy relations fulfilling certain properties which generalizes naturally the notion used in previous approaches (see [4], [5]). In this paper we have introduced a generalized notion of fuzzy adjunction (between two fuzzy preposets) whose components are total fuzzy relations.

Concerning the generalization to the notion of adjunction to the fuzzy case, to the best of our knowledge, the first approach was due to Bělohlávek [7]. Later, a number of publications have introduced different approaches to either fuzzy adjunctions or fuzzy Galois connections, see [8]–[12]. The latter, introduced by Yao in [12], was used in our previous work [4], where we were interested in constructing a right adjoint associated to a given mapping $f\colon \langle A, \rho_A \rangle \to B$ from a fuzzy preposet $\langle A, \rho_A \rangle$ into an unstructured set B. The fact the mappings in this approach are crisp rather than fuzzy motivated the search for the use of fuzzy functions or even fuzzy relations, and lead to the notion introduced in this work.

As future work, we are planning the characterization of existence of this type of fuzzy adjunctions in different fuzzy environments (preordered or partially ordered sets with or without corresponding fuzzy equivalence relations).

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