# ON FIXED-POINTS OF MULTI-VALUED FUNCTIONS ON COMPLETE LATTICES AND THEIR APPLICATION TO GENERALIZED LOGIC PROGRAMS

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**Abstract.** Unlike monotone single-valued functions, multi-valued mappings may have none, one or (possibly infinitely) many minimal fixed-points.

The contribution of this work is twofold. At first we overview and investigate about the existence and computation of minimal fixed-points of multi-valued mappings, whose domain is a complete lattice and whose range is its power set. Second, we show how these results are applied to a general form of logic programs, where the truth space is a complete lattice. We show that a multi-valued operator can be defined whose fixed-points are in one-to-one correspondence with the models of the logic program.

**Key words.** Fixed-points; multi-valued functions; complete lattices; logic programming

**AMS subject classifications.** 47H10, 06B23 68N17, 68Q55,

**1. Introduction.** It is well known that fixed-point theorems are useful in many completely disparate and unrelated scientific branches and, thus, in computer science. Among the main fixed-point results is the Tarski Theorem [48] (often called Knaster-Tarski theorem) stating that the set of fixed-points of a monotone *single-valued* function $f\colon L \to L$, over a complete lattice $\langle L, \leq \rangle$ is a complete lattice and therefore has a least fixed-point.

The topic of this work is to overview and to investigate about the fixed-points of *multi-valued* functions $f\colon L \to 2^L$ (multi-valued functions are also called *correspondences*, or *set-valued functions* in the literature). Such functions naturally arise e.g., in the specification of the semantics of non-deterministic programming languages [7, 8, 11, 18, 32, 37, 38, 45], in game theory [6, 34, 46, 54] and disjunctive logic programming [22, 28, 33, 43, 53], where these latter case motivated our work. Informally, $(i)$ in the first case the meaning of a non-deterministic[1] program $P$ may be seen as a function $p : S \to 2^S$, where $S$ is the set of states a program may assume. The image of $p$ is a finite non-empty set, as at a given step of a program execution, due to a non-deterministic statement, more than one successive state is possible. The semantics of a program is then related to the fixed-points of such functions ($s \in p(s)$); $(ii)$ in the second case, a game is represented as a function $g : S \to 2^S$, where $S$ is the strategy space of the involved players and fixed-points ($s \in g(s)$) are related to the so-called Nash equilibria of the game. The image of $g$ is a non-empty (usually finite) set, as at each step of the game, more than one incomparable strategic choice is possible; and $(iii)$ in the third case, models of disjunctive logic programs are related to fixed-points ($I \in T_{\mathcal{P}}(I)$) of a function $T_{\mathcal{P}}\colon \hat{L} \to 2^{\hat{L}}$, where $\hat{L}$ is the set of interpretations of a disjunctive logic program. Here, $T_{\mathcal{P}}$ is a so-called immediate consequence operator, which at each "step" provides a better approximation of the models of a disjunctive logic program. The image of $T_{\mathcal{P}}$ is a *possibly empty, non-finite* set as

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[1] An example of non-deterministic statement is "$\pi_1 \mathbf{or} \pi_2$" with informal semantics "execute either program $\pi_1$ or program $\pi_2$".

at each step of the model approximation computation, none, or potentially infinite incomparable better approximations are possible.

We point out that, in all three cases, fixed-point computations may be seen roughly as a tree, where a node is an element of the domain and the children of it are the alternative (non-deterministic) choices provided by the image of the multi-valued function.

Generally, multi-valued functions present the following fundamental challenge to the ordinary fixed-point approach: unlike monotone single-valued functions, it is possible that none, one or (infinitely) many minimal fixed-points exist.

The contribution of this work is twofold:

- We provide conditions for the existence of fixed-points and minimal fixed-points and show how to recursively obtain them in a slightly more general setting as considered so far (such as the image of a multi-valed function may be empty, see below). A summary of main findings in described in Table 3.1. To the best of our knowledge, we have compared the results obtained with respect to all related work using similar order-theoretic approaches; when a reformulation or easier proof of a known result is presented, then appropriate credit is given.
- The results are then applied to a general form of logic programs, encompassing the disjunctive and many-valued extensions. The rules in such logic programs have the form $g(B_1, \ldots, B_k) \leftarrow f(A_1, \ldots, A_n)$, where $f, g$ are arbitrary computable functions over a complete lattice (which acts as the truth space) and $B_i$ and $A_j$ are atoms. The form of the rules is sufficiently expressive to generalize all approaches we are aware of in (monotone) many-valued logic programming. The main difference of this application to e.g. semantics of non-deterministic programming languages and game theory is that the image of $T_{\mathcal{P}}(I)$ may be empty or of infinite size, while in the former two cases both $p(s)$ and $g(s)$ are non-empty and finite. We show that a multi-valued operator $T_{\mathcal{P}}(I)$ can be defined whose fixed-points are in one-to-one correspondence with the models of the logic program. The obtained relationship is novel, and addresses some fundamental theoretical problems that have been neglected so far in the logic programming literature. We conclude, by showing that our results extend current well-known results for classical disjunctive logic programs, where rules are of the form $B_1 \vee \ldots \vee B_k \leftarrow A_1 \wedge \ldots \wedge A_n$.

## 2. Preliminaries.

We recall some basic definitions and notations:

With $\mathcal{L} = \langle L, \leq \rangle$, where $\leq$ is a partial order ($x \leq y$ may be read as "$x$ approximates $y$") over the non-empty set $L$, we denote a *complete lattice*, with *join* (*meet*) operator $\vee$ ($\wedge$), least (greatest) element $\bot$ ($\top$).

Given $S \subseteq L$, with $\min S$ ($\max S$ ) we denote the set of *minimal* (*maximal*) elements in $S$ and with $\bigwedge S$ ($\bigvee S$) the greatest lower bound (least upper bound) of $S$. [2] A non-empty subset $S$ of $L$ is a *sub-lattice* of $L$ if for any $x, y$ of $S$, both $x \vee y$ and $x \wedge y$ belong to $S$. A non-empty subset $S$ of $L$ is $\wedge$-*closed* ($\vee$-*closed*) if for any subset $U$ of $S$, $\bigwedge_{x \in U} x$ ( $\bigvee_{x \in U} x$) belongs to $S$. Note that $S$ is $\wedge$-closed ($\vee$-closed) iff $S$ is a complete meet semi-lattice (complete join semi-lattice). Furthermore, we say that $S$ is *closed* if $S$ is both $\wedge$-closed and $\vee$-closed, i.e. $S$ is a complete sub-lattice of $L$. Given two elements $a, b \in L$ with $a \leq b$, we denote by $[a, b]$ the *interval* $\{x \in L \mid a \leq x \leq b\}$. Clearly, $\mathcal{L} = \langle [a, b], \leq \rangle$ is a complete lattice as well. Finally, with $\bar{\mathcal{L}} = \langle L, \geq \rangle$ we

[2] We recall that $\bigwedge S = \bigwedge_{s \in S} s$ and $\bigvee S = \bigvee_{s \in S} s$.

denote the *dual* lattice of $\mathcal{L} = \langle L, \leq \rangle$, where $x \geq y$ iff $y \leq x$. Of course, $\bar{\mathcal{L}}$ is a complete lattice as well, where $\geq$ is the reversed order of $\leq$ and $\top$ ($\perp$) is the least (greatest) element of $\bar{\mathcal{L}}$.

Two sets $X$ and $Y$ are *equipollent* iff there is a bijection from $X$ to an $Y$. The *cardinality* $|X|$ of a set $X$ is the least ordinal $\alpha$ such that there is a bijection between $X$ and $\alpha$.

We use the notation $(x_\alpha)_{\alpha \in I}$, to denote a (possibly transfinite) non-empty *sequence* of elements $x_\alpha \in L$, where $I$ is an ordinal. We say that the sequence is *increasing* (*decreasing*) iff $x_\alpha \leq x_{\alpha+1}$ ($x_{\alpha+1} \leq x_\alpha$), for all $\alpha \in I$.

If there is an ordinal $\beta \in I$ such that $x_\beta = x_\alpha$ for all $\beta \leq \alpha \in I$, we say that $(x_\alpha)_{\alpha \in I}$ is *eventually stationary or constant.* A property we will frequently rely on is the well known fact that:

Proposition 2.1. *An increasing (decreasing) sequence $(x_\alpha)_{\alpha \in I}$ of elements $x_\alpha \in L$ with $|I| > |L|$ has the property that there is an ordinal $\beta \in I$ such that $|\beta| \leq |L|$ and $x_\beta = x_\alpha$ for all $\beta \leq \alpha \in I$ ($|S|$ is the* cardinal *of a set $S$).* ∎

For ease of presentation and by abuse of terminology, under the condition of Proposition 2.1, we will say that the sequence $(x_\alpha)_{\alpha \in I}$ *converges* to $x_\beta$.

A function $f\colon L \to L$ is *monotone* iff for all $x, y \in L$, $x \leq y$ implies $f(x) \leq f(y)$. $f$ is *inflationary* iff for all $x \in L$, $x \leq f(x)$. A *fixed-point* of $f$ is an element $x \in L$ such that $f(x) = x$. With $Fix(f)$ we denote the set of fixed-points of $f$. $f$ is $\bigvee$*-preserving* ($\bigwedge$*-preserving*) iff for all increasing (decreasing) sequences $(x_\alpha)_{\alpha \in I}$, $f(\bigvee_\alpha x_\alpha) = \bigvee_\alpha f(x_\alpha)$ ($f(\bigwedge_\alpha x_\alpha) = \bigwedge_\alpha f(x_\alpha)$). $f$ is *limit preserving* iff it is both $\bigvee$- and $\bigwedge$-preserving. It is easy to prove that $\bigvee$- or $\bigwedge$-preserving functions are monotone. However, a limit preserving (in particular a monotone) function needs not be inflationary.

Example 1. *Consider $f : \{0, 1\} \to \{0, 1\}$ with $f(x) = 0, \forall x \in \{0, 1\}$, then $f$ is limit preserving and, thus, monotone, but $1 \not\leq f(1)$ and, thus, $f$ is not inflationary.* ∎

The Tarski theorem [48] establishes that a monotone function $f\colon L \to L$ has a fixed-point, the set of fixed-points of $f$ is a complete lattice and, thus, $f$ has a least and a greatest fixed-point. The least (greatest) fixed-point can be obtained by transfinite iteration of $f$ over $\perp$ ($\top$). Furthermore, let $\Phi(f) = \{x \in L\colon f(x) \leq x\}$, $\Psi(f) = \{x \in L\colon x \leq f(x)\}$, and, thus, $\top \in \Phi(f)$, while $\perp \in \Psi(f)$. Then the least fixed-point is $\bigwedge \Phi(f)$, while the greatest fixed-point is $\bigvee \Psi(f)$. If $f$ is inflationary then $f$ has a fixed-point (e.g., obtained by transfinite iteration of $f$ over $\perp$, also $\top \leq f(\top) = \top$), and $x \in \Phi(f)$ iff $x$ fixed-point of $f$. However, inflationary functions may not have a least fixed-point.

Example 2. *Consider $L = [0, 1]$ and function $f$ with $f(0) = 1$ and for $x > 0$, $f(x) = x$. Then $f$ is not monotone, is inflationary, all $x > 0$ are fixed-points, $\Phi(f) = \{x : x > 0\}$, $\bigwedge \Phi(f) = 0 \notin \Phi(f)$, and $0$ is not a fixed-point of $f$.* ∎

**3. Multi-valued functions.** Given $\mathcal{L} = \langle L, \leq \rangle$, a *multi-valued function* is a function $f\colon L \to 2^L$ (if for all $x \in L$, $|f(x)| = 1$ then $f$ is single-valued). Note that we do not require $f(x) \neq \varnothing$ for all $x \in L$. We say that $x \in L$ is a *fixed-point* of $f$ iff $x \in f(x)$. For instance,

Example 3. *Let $L = \{0, 1, 2\}$. Consider $f\colon L \to 2^L$ defined as $f(0) = \{0, 1, 2\}$, $f(1) = \{0, 1\}$ and $f(2) = \{0\}$. Then $0$ and $1$ are fixed-points, whereas $2$ is not a* *fixed-point.* ∎

Furthermore, we say that $f$ is *non-empty* (resp. $\wedge$*-closed*, $\vee$*-closed*) iff for all $x \in L$ we have that $f(x) \neq \varnothing$ ($f(x)$ is resp. $\wedge$*-closed*, $\vee$*-closed*).

In order to define the notion of (multi-valued) monotone function, as $f(x)$ is now a set of elements, we need to extend the partial order $\leq$ to sets of elements. There are mainly three well-known *pre-orders* (reflexive, transitive but not antisymmetric), namely the *Smyth ordering*, the *Hoare ordering* and the *Egli-Milner ordering*, which have been proposed in the context of non-deterministic programming languages (see, e.g. [1, 25]) [3]:

$$X \preceq_S Y \quad \text{iff} \quad \forall y \in Y\ \exists x \in X\ s.t.\ x \leq y \quad \text{(Smyth ordering)} \tag{3.1}$$

$$X \preceq_H Y \quad \text{iff} \quad \forall x \in X\ \exists y \in Y\ s.t.\ x \leq y \quad \text{(Hoare ordering)} \tag{3.2}$$

$$X \preceq_{EM} Y \quad \text{iff} \quad X \preceq_S Y \text{ and } X \preceq_H Y \quad \text{(Egli-Milner ordering)}\ . \tag{3.3}$$

These orderings may be read as follows: $(i)$ $X \preceq_S Y$ iff all $y \in Y$ are approximated by some $x \in X$, $(ii)$ $X \preceq_H Y$ iff all $x \in X$ approximate some $y \in Y$; and $(iii)$ $X \preceq_{EM} Y$ iff all $y \in Y$ are approximated by some $x \in X$ and all $x \in X$ approximate some $y \in Y$.

Clearly the Hoare order is equivalent to the Smyth order in the dual underlying lattice. Indeed it is straightforward to show that:

PROPOSITION 3.1. *Let $X, Y$ be two subsets of $L$. Then $X \preceq_S Y$ in $\mathcal{L}$ iff $Y \preceq_H X$ in $\bar{\mathcal{L}}$.* ∎

As a consequence, many properties we state with respect to the Smyth-ordering in $\mathcal{L}$, have their dual with respect to the Hoare ordering in $\bar{\mathcal{L}}$.

$f$ is *Smyth-monotone*, or simply *S-monotone*, iff for all $x, y \in L$, if $x \leq y$ then $f(x) \preceq_S f(y)$ holds. The notions of *Hoare-monotone*, or simply *H-monotone*, and *Egli-Milner-monotone*, or simply *EM-monotone*, are defined similarly. By using Proposition 3.1, it is straightforward to prove that

PROPOSITION 3.2. *Let $f\colon L \to 2^L$ be a multi-valued function. Then $f$ is S-monotone in $\mathcal{L}$ iff $f$ is H-monotone in $\bar{\mathcal{L}}$.* ∎

We say that $f$ is *inflationary* iff for all $x$, $\{x\} \preceq_S f(x)$, i.e. all elements in $f(x)$ are greater or equal than $x$. Dually, we say that $f$ is *deflationary* iff $\forall x \in L$, $f(x) \preceq_H \{x\}$, i.e. all elements in $f(x)$ are smaller or equal than $x$. Of course, a deflationary function is an inflationary function in the dual lattice $\bar{\mathcal{L}}$.

PROPOSITION 3.3. *Let $f\colon L \to 2^L$ be a multi-valued function. Then $f$ is deflationary in $\mathcal{L}$ iff $f$ is inflationary in $\bar{\mathcal{L}}$.* ∎

We next generalise the notion of limit preserving function to the multi-valued case. A multi-valued function $f\colon L \to 2^L$ is $\bigvee$*-preserving* iff for all increasing sequences $(x_\alpha)_{\alpha \in I}$,

$$f(\bigvee_\alpha x_\alpha) = \{y \mid \text{ there is } (y_\alpha)_{\alpha \in I}\ s.t.\ y_\alpha \in f(x_\alpha) \text{ and } y = \bigvee_\alpha y_\alpha\}\ . \tag{3.4}$$

[3] [37] describes another order, called the *Plotkin order*, which extends the Egli-Milner ordering. However, we will not address it here.

Dually, we say that $f\colon L \to 2^L$ is $\bigwedge$-*preserving*, iff for all decreasing sequences $(x_\alpha)_{\alpha\in I}$,

$$f(\bigwedge_\alpha x_\alpha) = \{y \mid \text{ there is } (y_\alpha)_{\alpha\in I} \textit{ s.t. } y_\alpha \in f(x_\alpha) \text{ and } y = \bigwedge_\alpha y_\alpha\} \ . \tag{3.5}$$

$f$ is *limit preserving* iff it is both $\bigvee$- and $\bigwedge$-preserving. For ease of presentation, sometimes we use the notation $\bigvee_\alpha f(x_\alpha)$ (resp. $\bigwedge_\alpha f(x_\alpha)$) to denote the right hand side of Equation (3.4) (resp. Equation (3.5)). Note that if for all $x \in L$, $|f(x)| = 1$ then the definition reduces to the usual one for single-valued functions.
Of course,

Proposition 3.4. *Let $f\colon L \to 2^L$ be a multi-valued function. Then $f$ is $\bigwedge$-preserving in $\mathcal{L}$ iff $f$ is $\bigvee$-preserving in $\bar{\mathcal{L}}$.* ▮

We can prove that:

Proposition 3.5. *Consider a multi-valued function $f\colon L \to 2^L$.*
1. *If $f$ is $\bigvee$-preserving then $f$ is S-monotone;*
2. *If $f$ is $\bigwedge$-preserving then $f$ is H-monotone;*
3. *If $f$ is limit preserving, then $f$ is EM-monotone.*

*Proof.* Case 1. Let $x_1 \le x_2$ and $f$ $\bigvee$-preserving. Then for the increasing sequence $x_1 \le x_2$, $f(x_2) = f(x_1 \vee x_2) = \{y\colon$ there are $y_i \in f(x_i)$ *s.t.* $y = y_1 \vee y_2\} = X$. If $f(x_2) = \varnothing$ then trivially $f(x_1) \preceq_S f(x_2) = \varnothing$. If $f(x_1) = \varnothing$ then by definition $X = \varnothing$ and, thus, $f(x_2) = \varnothing$. Therefore, $\varnothing = f(x_1) \preceq_S f(x_2) = \varnothing$. Otherwise assume $f(x_1)$ and $f(x_2)$ non-empty. Therefore, as $f$ is $\bigvee$-preserving, for $y \in f(x_2) = X$ there are $y_i \in f(x_i)$ $(i = 1, 2)$ such that $y = y_1 \vee y_2$. In particular, $y_1 \le y$. Therefore, $f(x_1) \preceq_S f(x_2)$ and, thus, $f$ is S-monotone.

Case 2. The proof is dual to case 1 (see appendix, Proposition A.1).

Case 3. Straightforward, by case 1. and case 2. □

Note that a $\bigwedge$-preserving function needs not be S-monotone and, similarly, a $\bigvee$-preserving function needs not be H-monotone and, thus, a EM-monotone function needs not be limit preserving.

Example 4. *Consider $L = \{0, 1\}$ with $0 \le 1$. Then the multi-valued function $f : L \to 2^L$, $f(0) = \varnothing, f(1) = \{1\}$ is $\bigwedge$-preserving, but not S-monotone, as $0 \le 1$ and $f(0) = \varnothing \not\preceq_S f(1) = \{1\}$. Similarly, the multi-valued function $g : L \to 2^L$, $g(0) = \{0\}, g(1) = \varnothing$ is $\bigvee$-preserving, but not H-monotone, as $0 \le 1$ and $g(0) = \{0\} \not\preceq_H g(1) = \varnothing$.* ▮

But, we can easily show that:

Proposition 3.6. *Consider a multi-valued function $f\colon L \to 2^L$ and $x_1 \le x_2$ with $f(x_1) \neq \varnothing \neq f(x_2)$.*
1. *If $f$ is $\bigwedge$-preserving then $f(x_1) \preceq_S f(x_2)$;*
2. *If $f$ is $\bigvee$-preserving then $f(x_1) \preceq_H f(x_2)$;*

*Proof.* Case 1. For the decreasing sequence $x_2 \ge x_1$, as $f$ is $\bigwedge$-preserving, $f(x_1) = f(x_2 \wedge x_1) = \{y\colon$ there are $y_i \in f(x_i)$ *s.t.* $y = y_2 \wedge y_1\} = X$. Now, for $y \in f(x_2)$ choose a $y' \in f(x_1) \neq \varnothing$ and consider $y'' = y \wedge y'$. Therefore, $y'' \in X = f(x_1)$, $y'' \le y$ and, thus, $f(x_1) \preceq_S f(x_2)$.

Case 2. Like for case 1 (see appendix, Proposition A.2). □

Example 1 can be adapted to multi-valued functions and prove that a limit preserving (in particular a S-monotone) function needs not be inflationary.

EXAMPLE 5. *Consider* $f : \{0,1\} \to 2^{\{0,1\}}$ *such that for all* $x \in \{0,1\}$, $f(x) = \{0\}$, *then* $f$ *is limit preserving and, thus, S-monotone, but* $\{1\} \not\preceq_S f(1) = \{0\}$. ∎

We next want to investigate about the existence of (minimal) fixed-points of multi-valued functions. Similarly to the single-valued case, for $f \colon L \to 2^L$, let us define

$$\begin{aligned}\Phi(f) &= \{x \in L \colon f(x) \preceq_S \{x\}\} \\ \Psi(f) &= \{x \in L \colon \{x\} \preceq_H f(x)\}\ .\end{aligned}$$

Note that, unlike the single-valued case, not necessarily $\top \in \Phi(f)$ (i.e., if $f(\top) = \varnothing$). Similarly, $\bot \in \Psi(f)$ iff $f(\bot) \neq \varnothing$. Also, if $f(x) = \varnothing$ then $x \notin \Phi(f)$, i.e. if $x \in \Phi(f)$ then $f(x) \neq \varnothing$. Finally, note that if $f(\top) \neq \varnothing$ then $\top \in \Phi(f)$ (we will use these straightforward facts often in the paper). Furthermore, note that $\Phi(f)$ is related to the $\preceq_S$ order, while $\Psi(f)$ is related to $\preceq_H$. One might wonder why we did not consider, for instance, $\Phi_H(f) = \{x \mid f(x) \preceq_H \{x\}\}$. As we will see later, $\bigwedge \Phi(f)$ relates to the least fixed-point of $f$ (if it exists), while $\bigvee \Psi(f)$ relates to the greatest fixed-point of $f$. Example 6 shows that $\bigwedge \Phi_H(f)$ is not related to the least fixed-point of $f$.

EXAMPLE 6. *Consider* $L = \{0,1\}$ *and the multi-valued function* $f \colon L \to 2^L$, $f(0) = \{0,1\}, f(1) = \{1\}$. *Then* $f$ *is EM-monotone,* $Fix(f) = \{0,1\}$, *but* $\Phi_H(f) = \{x \mid f(x) \preceq_H \{x\}\} = \{1\}$ *and* $1 = \bigwedge \Phi_H(f)$ *is not the least fixed-point of* $f$. ∎

We can show that:

PROPOSITION 3.7. *Let* $f \colon L \to 2^L$ *be a multi-valued function.*
1. *If* $f$ *is inflationary then* $x \in \Phi(f)$ *iff* $x$ *fixed-point of* $f$.
2. *If* $f$ *is deflationary then* $x \in \Psi(f)$ *iff* $x$ *fixed-point of* $f$.

*Proof.* Case 1. Let $x \in \Phi(f)$. As $f$ is inflationary, $\{x\} \preceq_S f(x) \preceq_S \{x\}$ and, thus, for $x \in \{x\}$ there is $y \in f(x)$ such that $x \leq y \leq x$, i.e. $x = y \in f(x)$. Vice-versa, if $x \in f(x)$ then $f(x) \preceq_S \{x\}$ and, thus, $x \in \Phi(f)$.

Case 2. Similar to case 1 (see appendix, Proposition A.3). □

Note that Proposition 3.7 does not hold if a function is e.g. S-monotone, but not inflationary.

EXAMPLE 7. *In Example 3,* $f$ *is S-monotone, not inflationary with* $2 \in \Phi(f)$, *but* $2 \notin f(2)$. ∎

The following examples show that a multi-valued S-monotone function $f \colon L \to 2^L$ may have several minimal fixed-points or even no minimal fixed-point at all.

EXAMPLE 8. *Consider Belnap's truth space* $\mathcal{FOUR}$ *[3],* $L = \{\bot, f, t, \top\}$ *with* $f, t$ *incomparable. Here, besides* $f$ *for 'false' and* $t$ *for 'true',* $\bot$ *stands for 'unknown', whereas* $\top$ *stands for inconsistency.* $\leq$ *is the so-called knowledge order. Consider the multi-valued function* $g \colon L \to 2^L$ *defined as* $g(\bot) = \{f, t, \top\}, g(f) = \{f, \top\}, g(t) = \{t, \top\}$ *and* $g(\top) = \{\top\}$. *Then* $g$ *is EM-monotone, inflationary and* $\bigvee$*-preserving. Furthermore,* $f \in g(f), t \in g(t)$ *and* $\top \in g(\top)$, *but* $\bot \notin g(\bot)$ *and, thus,* $f, t$ *and* $\top$ *are fixed-points of* $g$, *while* $\bot$ *is not. The minimal fixed-points are* $f$ *and* $t$. *Note that not for all* $x$, $g(x)$ *has least element (e.g.,* $g(\bot)$*). Additionally, note that* $\Phi(g) = \{f, t, \top\}$, $\bigwedge \Phi(g) = \bot \notin \Phi(g)$ *and* $\min \Phi(g) = \{f, t\}$. *Therefore, unlike the single-valued case,* $\bigwedge \Phi(g)$ *is not a fixed-point of* $g$. ∎

The four-element Belnap's truth space $\mathcal{FOUR}$ was introduced as a very suitable setting for computerized reasoning; it has a bilattice structure, since two orderings can be naturally defined and, as a result, it can be viewed as a class of truth values that can accommodate incomplete and inconsistent information and in certain cases default information.

EXAMPLE 9. *Let $L = [0,1]$. Consider the multi-valued function $f\colon L \to 2^L$ defined as $f(x) = \{y \mid y > 0, y \geq x)\}$. Then $f$ is non-empty, $\bigvee$-preserving and inflationary. Furthermore, for all $x > 0$, $x \in f(x)$, but $0 \notin f(0)$ and, thus, all $x > 0$ are fixed-points of $f$, while $0$ is not. Therefore, $f$ has no minimal fixed-point. Also, note that $\Phi(f) = \{x \mid x > 0\}$, $\bigwedge \Phi(f) = 0 \notin \Phi(f)$ and $\min \Phi(f) = \varnothing$. Like for Example 8, $0 = \bigwedge \Phi(f)$ is not a fixed-point of $f$, but now $\min \Phi(f) = \varnothing$. Also note, that $f(0)$ has not least element.*

*Similarly, for $g(x) = \{y \mid y < 1, y \leq x\}$. Then $g$ is non-empty, $\bigwedge$-preserving and deflationary. $\Psi(g) = \{x \mid x < 1\}$, $\bigvee \Psi(g) = 1 \notin \Psi(g)$, $\max \Psi(g) = \varnothing$ and $1 \notin g(1)$. Hence, $g$ has no greatest fixed-point.*

*Likewise, $h(x) = \{y \mid 0 < y < 1\}$. Then $h$ is non-empty and EM-monotone. $\Phi(h) = \{x \mid x > 0\}$, $\Psi(h) = \{x \mid x < 1\}$, and $h$ has neither a least nor a greatest fixed-point.* ∎

Like the single-valued case, a multi-valued inflationary function may not have a minimal fixed-point, even if $f(x)$ has least element for all $x \in L$.

EXAMPLE 10. *Consider $f\colon [0,1] \to 2^{[0,1]}$, where $f(0) = \{1\}$ and for $x > 0$, $f(x) = \{x\}$. Then $f$ is not S-monotone, but is inflationary. Also, $f(x)$ has least element for all $x \in L$. All $x > 0$ are fixed-points as $x \in f(x)$, $\Phi(f) = \{x \mid x > 0\}$ (in accordance with Proposition 3.7), and $\bigwedge \Phi(f) = 0$, but $0 \notin f(0)$. Note that $\min \Phi(f) = \varnothing$.* ∎

However, we will show later in Proposition 3.10 that a multi-valued S-monotone function such that $f(x)$ has least element for all $x \in L$, has indeed a least fixed-point.

We next show that if $\Phi(f)$ has minimals then a S-monotone or inflationary function $f$ has minimal fixed-points.

PROPOSITION 3.8. *Let $f\colon L \to 2^L$ be a multi-valued function.*

1. *If $f$ is a S-monotone or inflationary multi-valued function, and $\Phi(f)$ has minimals then all $y \in \min \Phi(f)$ are minimal fixed-points of $f$. In particular, if $x = \bigwedge \Phi(f) \in \Phi(f)$ then $x$ is the least fixed-point of $f$;*
2. *If $f$ is a H-monotone or deflationary multi-valued function, and $\Psi(f)$ has maximals then all $y \in \max \Psi(f)$ are maximal fixed-points of $f$. In particular, if $x = \bigvee \Psi(f) \in \Psi(f)$ then $x$ is the greatest fixed-point of $f$.*

*Proof.* Case 1. To begin with, let us show that any $y \in \min \Phi(f)$ is a fixed-point of $f$. As $\Phi(f)$ has minimals, $\min \Phi(f) \neq \varnothing$. So, let $y \in \min \Phi(f)$. As $\varnothing \neq f(y) \preceq_S \{y\}$, thus, there is $y' \in f(y)$ such that $y' \leq y$. If $f$ S-monotone, then $f(y') \preceq_S f(y)$ and, thus, for $y' \in f(y)$ there is $y'' \in f(y')$ such that $y'' \leq y'$. Therefore, $f(y') \preceq_S \{y'\}$ and, thus, $y' \in \Phi(f)$. But $y \in \min \Phi(f)$, so it cannot be $y' < y$. Therefore, $y = y' \in f(y)$, i.e. $y$ is a fixed-point of $f$. If $f$ is inflationary, by Proposition 3.7, $y$ is a fixed-point of $f$.

Now, let us show that any $y \in \min \Phi(f)$ is also a minimal fixed-point of $f$. So, consider $y \in \min \Phi(f)$ and, thus, $y$ is a fixed-point of $f$. Now, consider another fixed-point $x \in f(x)$. Therefore, $f(x) \preceq_S \{x\}$ and, thus, $x \in \Phi(f)$. But $y \in \min \Phi(f)$ so it cannot be $x < y$ and, thus, $y$ is a minimal fixed-point of $f$.

Finally, consider $x = \bigwedge \Phi(f)$. By hypothesis, $x \in \Phi(f)$ and $x$ is least element of $\Phi(f)$. Hence, we know that $x \in f(x)$. Let $y \in f(y)$. Hence $y \in \Phi(f)$, and, thus, $x \leq y$. As a consequence, $x$ is the least fixed-point of $f$.

Case 2. Similar to case 1 (see appendix, Proposition A.4). □

Note that $\Phi(f)$ in Examples 3 and 8 has minimals, while $\Phi(f)$ in Example 9 does not.

The following proposition establishes a condition on a S-monotone function $f$ under which $\Phi(f)$ has minimals and, thus, minimal fixed-points.

PROPOSITION 3.9. *Let $f\colon L \to 2^L$ be a multi-valued function.*

1. *If $f$ is a $\bigwedge$-preserving multi-valued function with $\Phi(f) \neq \varnothing$ then $\Phi(f)$ has minimals and, thus, minimal fixed-points;*
2. *If $f$ is a $\bigvee$-preserving multi-valued function with $\Psi(f) \neq \varnothing$ then $\Psi(f)$ has maximals and, thus, maximal fixed-points.*

*Proof.* Case 1. By hypothesis $\Phi(f) \neq \varnothing$. Let $(x_\alpha)_{\alpha\in I}$ be a decreasing sequence of $x_\alpha \in \Phi(f)$ and let $\bar{x} = \bigwedge_\alpha x_\alpha$. As $f$ is $\bigwedge$-preserving, by definition $f(\bar{x}) = \{y\colon$ there is $(y_\alpha)_{\alpha\in I}$ *s.t.* $y_\alpha \in f(x_\alpha)$ and $y = \bigwedge_\alpha y_\alpha\}$. Now, for any $\alpha$, $x_{\alpha+1} \leq x_\alpha$, by Proposition 3.6 and, as $x_\alpha \in \Phi(f)$, $f(x_{\alpha+1}) \preceq_S f(x_\alpha) \preceq_S \{x_\alpha\}$. Therefore, for any $x_\alpha$ there is $y_\alpha \in f(x_\alpha)$ and $y_{\alpha+1} \in f(x_{\alpha+1})$ such that $y_{\alpha+1} \leq y_\alpha \leq x_\alpha$. Note that if $\alpha$ is a limit ordinal then, as $x_\alpha \leq x_\beta$ for all $\beta < \alpha$, it follows that $f(x_\alpha) \preceq_S f(x_\beta) \preceq_S \{x_\beta\}$ and, thus, $y_\alpha \leq y_\beta \leq x_\beta$ for all $\beta < \alpha$. Therefore, there is a decreasing sequence $(y_\alpha)_{\alpha\in I}$ of elements $y_\alpha \in f(x_\alpha)$ such that $\bar{y} = \bigwedge_\alpha y_\alpha \leq \bigwedge_\alpha x_\alpha = \bar{x}$. By definition of $f(\bar{x})$, $\bar{y} \in f(\bar{x})$ and, thus, $f(\bar{x}) \preceq_S \{\bar{x}\}$. Therefore $\bar{x} \in \Phi(f)$ and, thus, every decreasing sequence has a lower bound in $\Phi(f)$. So, by Zorn's lemma, $\Phi(f)$ has minimals, which by Proposition 3.8 are also minimal fixed-points.

Case 2. As for case 1 (see appendix, Proposition A.5). □

The converse of Proposition 3.9 above is not true.

EXAMPLE 11. *Consider $L = \{0, 0.5, 1\}$, where $f\colon L \to 2^L$ with $f(0) = \{0\}$, $f(0.5) = \{0.5\}$, and $f(1) = \{0, 1\}$. Then $\Phi(f) = L$ has minimals, but $f$ is not S-monotone: $0.5 \leq 1$ but $f(0.5) \not\preceq_S f(1)$. Therefore, by Proposition 3.6, $f$ cannot be $\bigwedge$-preserving.* ∎

One might wonder whether a S-monotone $f\colon L \to 2^L$ such that for all $x \in L$, $f(x)$ has minimals implies that $\Phi(f)$ has minimals. This is not true as the following example shows.

EXAMPLE 12. *Consider $Y = \{y_\alpha : \alpha \in \omega\}$, $Y$ antichain, $X = \{x_\alpha : \alpha \in \omega\}$, $x_{\alpha+1} \leq x_\alpha$, $\bar{x} = \bigwedge_\alpha x_\alpha$, $y_\alpha \leq x_\alpha$, each pair $\bar{x}, y_\alpha$ incomparable, $L = \{\bar{x}\} \cup X \cup Y \cup \{\bot, \top\}$, and $f\colon L \to 2^L$ with $f(\bot) = Y$, $f(\bar{x}) = Y$, $f(x_\alpha) = \{x_\alpha\}$, $f(y_\alpha) = \{x_\alpha\}$ and $f(\top) = \{\top\}$. Then $f$ S-monotone, for all $x \in L$, $f(x)$ has minimals, $\Phi(f) = X \cup \{\top\}$ and $(x_\alpha)_{\alpha\in\omega}$ is a decreasing sequence of elements in $\Phi(f)$. As neither $\bar{x}$ nor $\bot$ is in $\Phi(f)$, $\Phi(f)$ does not have minimals.* ∎

However, we can prove that:

PROPOSITION 3.10. *Let $f\colon L \to 2^L$ be a multi-valued function.*

1. *If $f$ is S-monotone and for all $x \in L$, $f(x)$ has least element then $f$ has least fixed-point;*
2. *If $f$ is H-monotone and for all $x \in L$, $f(x)$ has greatest element then $f$ has greatest fixed-point.*

*Proof.* Case 1. As for all $x \in L$, $f(x)$ has least element, by definition $\bigwedge f(x) \in f(x) \neq \varnothing$. Therefore, $\Phi(f) \neq \varnothing$ as $\varnothing \neq f(\top) \preceq_S \{\top\}$. Consider $a = \bigwedge_{c\in\Phi(f)} c$. If

$a \in \Phi(f)$ then by Proposition 3.8, $a$ is the least fixed-point of $f$. So, let us show that $a \in \Phi(f)$. For $c \in \Phi(f)$ there is a $x_c \in f(c)$ such that $x_c \leq c$. As $a \leq c$ and $f$ is S-monotone, $f(a) \preceq_S f(c)$ and, thus, for $x_c \in f(c)$ there is $y_c \in f(a)$ such that $y_c \leq x_c \leq c$. Since $f(a)$ has least element, there is $y \in f(a)$ such that $y \leq \bigwedge_{c \in \Phi(f)} y_c \leq \bigwedge_{c \in \Phi(f)} x_c \leq \bigwedge_{c \in \Phi(f)} c = a$. Hence, $f(a) \preceq_S \{a\}$, i.e. $a \in \Phi(f)$.

Case 2. As for case 1 (see appendix, Proposition A.6). □

Note that if e.g. $f(x)$ has a least element for all $x \in L$ then this does not imply necessarily that $f$ is $\bigwedge$-preserving or $\bigvee$-preserving.

Example 13. *Consider Belnap's truth space $\mathcal{FOUR}$, $L = \{f, t, \bot, \top\}$. Let $h(\top) = \{f\}$, $h(t) = \{\bot, f\}$, $h(f) = \{\bot, t\}$, $h(\bot) = \{\bot\}$. Then for all $x \in L$, $h(x)$ has least element. Consider the decreasing sequence $(\top, f)$. Then $h(\top \wedge f) = h(f) = \{\bot, t\}$, while $h(\top) \wedge h(f) = \{\bot\}$ and, thus, $h$ is not $\bigwedge$-preserving. Consider the increasing sequence $(f, \top)$. Then $h(f \vee \top) = h(\top) = \{f\}$, while $h(f) \vee h(\top) = \{f, \top\}$ and, thus, $h$ is not $\bigvee$-preserving.* ∎

The following example shows that e.g. a H-monotone function such that for all $x \in L$, $f(x)$ has least element, does not imply that $f$ has least fixed-point.

Example 14. *Consider the lattice $\mathcal{FOUR}$ as in Example 13. Let $g(\bot) = \{t\}, g(f) = \{f, t, \bot\}, g(t) = \{f, t, \bot\}, g(\top) = \{\top\}$. $g$ is H-monotone, but not S-monotone. Furthermore, for all $x \in L$, $g(x)$ has least element. As $Fix(g) = \{f, t, \top\}$, $g$ has no least fixed-point.* ∎

The following example shows that a H-monotone or S-monotone non-empty function may not have a fixed-point at all.

Example 15. *Consider $L = [0, 1]$ and multi-valued function $f$, with $f(x) = \{(x+1)/2\}$ for $x < 1$ and $f(1) = \{1 - 1/n \mid n = 1, 2, \ldots\}$. Then $f$ is H-monotone without any fixed-point.*

*Similarly, let $g(x) = \{x/2\}$ for $x > 0$ and $g(0) = \{1/n \mid n = 1, 2, \ldots\}$. Then $g$ is S-monotone without any fixed-point.* ∎

Next, we describe properties about the structure of the set of fixed-points. The following example shows that the meet of two fixed-points of a monotone multi-valued function may not be a fixed-point and, thus, the set of fixed-points may not be a sub-lattice.

Example 16. *Consider $L = \{f, t, \bot, \top, c\}$, where $\bot \leq c$, $c \leq f \leq \top$ and $c \leq t \leq \top$. Let $g(\bot) = \{\bot\}, g(c) = \{\bot\}, g(t) = \{t\}, g(f) = \{f\}, g(\top) = \{\top\}$. Then $g$ is EM-monotone, limit preserving, deflationary, but not inflationary, and for all $x \in L$, $g(x)$ is a closed sub-lattice of $L$. However, $Fix(g) = \{\bot, \top, f, t\}$ is not a sub-lattice of $L$, e.g. $f, t \in Fix(f)$, but $c = f \wedge t \notin Fix(f)$ ($Fix(f)$ is even not a meet semi-lattice).* ∎

However, we can show that:

Proposition 3.11. *Let $f \colon L \to 2^L$ be a S-monotone, non-empty and $\wedge$-closed multi-valued function. Then*

1. *$\Phi(f)$ is $\wedge$-closed;*
2. *$f$ has a least fixed-point.*

*Proof.* Note that $\Phi(f) \neq \varnothing$ as $\varnothing \neq f(\top) \preceq_S \{\top\}$.

Point 1. Consider a subset $S$ of $\Phi(f)$ and $a = \bigwedge S$. Let us show that $a \in \Phi(f)$. We know that for each $c \in S$, $f(c) \preceq_S \{c\}$ holds, i.e. there is $x_c \in f(c)$ such that

$x_c \leq c$. But, $f$ is S-monotone and, thus, from $a \leq c$, $f(a) \preceq_S f(c) \preceq_S \{c\}$ follows. That is, there is $y_c \in f(a)$ such that $y_c \leq x_c \leq c$. Let $y = \bigwedge_{c \in S} y_c$. As $f$ is $\wedge$-closed, $y \in f(a)$ follows. Therefore, $y = \bigwedge_{c \in S} y_c \leq \bigwedge_{c \in S} c = a$, $f(a) \preceq_S \{a\}$ and, thus, $a \in \Phi(f)$. Therefore, $\Phi(f)$ is $\wedge$-closed.

Point 2. From point 1, $\Phi(f)$ has least element $a$ and, thus, by Proposition 3.8, $f$ has $a$ as least fixed-point. □

Dually, we have

PROPOSITION 3.12. *Let $f\colon L \to 2^L$ be a H-monotone, non-empty and $\vee$-closed multi-valued function. Then*

1. *$\Psi(f)$ is $\vee$-closed;*
2. *$f$ has a greatest fixed-point.*

*Proof.* Dual of proof of Proposition 3.11 (see appendix, Proposition A.7). □

Clearly, from Proposition 3.11 and 3.12 we have immediately:

PROPOSITION 3.13. *Let $f\colon L \to 2^L$ be a EM-monotone multi-valued function such that for any $x \in L$, $f(x)$ is non-empty closed sub-lattice of $L$. Then $f$ has a least fixed-point and a greatest fixed-point.* ∎

Also, it follows immediately from Proposition 3.7 that

PROPOSITION 3.14. *Let $f\colon L \to 2^L$ be a non-empty multi-valued function. Then*

1. *if $f$ is S-monotone, inflationary and $\wedge$-closed then $Fix(f)$ is non-empty and $\wedge$-closed, and, thus has a least element;*
2. *if $f$ is H-monotone, deflationary and $\vee$-closed then $Fix(f)$ is non-empty and $\vee$-closed and, thus has a greatest element.* ∎

Note that if $f$ is both inflationary and deflationary then for all $x \in L$ such that $f(x) \neq \varnothing$, we can easily show that $f(x) = \{x\}$, i.e. $f$ is a single-valued, constant, limit preserving function and each such $x$ is a fixed-point, and, thus, are not interesting.

We have seen in Proposition 3.14 that under rather strong conditions, we have a rather strong structure on the set of fixed-points (e.g., the conjunction of two fixed-points is a fixed-point). On the other hand, Example 16 shows that e.g. if we omit the inflationary condition then $Fix(f)$ is not $\wedge$-closed (e.g., the conjunction of two fixed-points needs not be a fixed-point) and, thus, $Fix(f)$ cannot be a closed sub-lattice of $L$.

The following proposition, due to [54] establishes that the set of fixed-points is a complete lattice, though not a closed sub-lattice.

PROPOSITION 3.15 (Zhou [54]). *Let $f\colon L \to 2^L$ be a multi-valued function. If $f$ is EM-monotone and for any $x \in L$, $f(x)$ is non-empty closed sub-lattice of $L$, then $Fix(f)$ is a non-empty complete lattice.* ∎

We next look at limit preserving functions and their impact to the set of fixed-points. We first notice that

PROPOSITION 3.16. *Let $f\colon L \to 2^L$ be a multi-valued function. Then*

1. *if $f$ is $\bigwedge$-preserving then $f$ is $\wedge$-closed;*
2. *if $f$ is $\bigvee$-preserving then $f$ is $\vee$-closed;*
3. *if $f$ is limit-preserving then for any $x \in L$, $f(x)$ is a closed sub-lattice of $L$.*

*Proof.* Point 1. Consider $x \in L$. If $f(x)$ is empty then it is also $\wedge$-closed. Otherwise, consider any subset of $f(x)$ in the form of a sequence $(y_\alpha)_{\alpha \in I}$ of elements $y_\alpha \in f(x)$. We show that $f(x)$ is $\wedge$-closed by showing that $y = \bigwedge_{\alpha \in I} y_\alpha \in f(x)$.

So, consider the decreasing sequence $(x_\alpha)_{\alpha\in I}$, where $x = x_\alpha$, for all $\alpha \in I$. By construction, $x = \bigwedge_{\alpha\in I} x_\alpha$. As $f$ is $\bigwedge$-preserving, we have that

$$\begin{array}{rcl} f(x) & = & f(\bigwedge_\alpha x_\alpha) \\ & = & \{z \mid \text{ there is } (z_\alpha)_{\alpha\in I} \ s.t.\ z_\alpha \in f(x_\alpha) \text{ and } z = \bigwedge_\alpha z_\alpha\} \\ & = & \{z \mid \text{ there is } (z_\alpha)_{\alpha\in I} \ s.t.\ z_\alpha \in f(x) \text{ and } z = \bigwedge_\alpha z_\alpha\} . \end{array}$$

Therefore, as for $(y_\alpha)_{\alpha\in I}$ we have $y_\alpha \in f(x)$, it follows that $y = \bigwedge_{\alpha\in I} y_\alpha \in f(\bigwedge_\alpha x_\alpha) = f(x)$, which concludes.

The other points can be shown similarly. □

Note that the converse in Proposition 3.16 does not hold. For instance, in Example 14, the function $g$ is such that for all $x \in L$, $g(x)$ is a closed sub-lattice, but $g$ is not $\bigwedge$-preserving (as $g$ is not S-monotone).

We already know from Proposition 3.9 that if $f$ is $\bigwedge$-preserving and $\Phi(f) \neq \varnothing$ (e.g. $f(\top) \neq \varnothing$) then $f$ has minimal fixed-points and, similarly, from Proposition 3.9 that if $f$ is $\bigvee$-preserving and $\Psi(f) \neq \varnothing$ (e.g. $f(\bot) \neq \varnothing$) then $f$ has maximal fixed-points. By further relying on Proposition 3.14 and 3.16, we have:

Proposition 3.17. *Let $f\colon L \to 2^L$ be a non-empty multi-valued function. Then*

1. *if $f$ is $\bigwedge$-preserving and inflationary then $Fix(f)$ is non-empty, $\wedge$-closed and, thus has a least element;*
2. *if $f$ is $\bigvee$-preserving and deflationary then $Fix(f)$ is non-empty, $\vee$-closed and, thus has a greatest element;*
3. *if $f$ is limit preserving then $Fix(f)$ is a non-empty complete lattice.* ∎

Note that the condition for non-emptiness in the above proposition is mandatory as e.g. a $\bigwedge$-preserving function $f$ may not necessarily imply that $f$ is non-empty, as the example below shows. This example also shows that Proposition 3.17 does neither subsume nor is in contrast with Proposition 3.8.

Example 17. *Consider the lattice $\mathcal{FOUR}$. Let $g$ be multi-valued function on $L$ such that $g(\bot) = \varnothing$, $g(\top) = \{\top\}$, $g(f) = \{f\}$, and $g(t) = \{t\}$. It can be easily verified that $g$ is $\bigwedge$-preserving, deflationary, though $Fix(g) = \{f, t, \top\}$ and, thus, no least fixed-point exists. $g$ has two minimal fixed-points instead.* ∎

As already pointed out, we are more interested in cases in which $f(x)$ may be empty for some $x \in L$. The literature we are aware of does not report results in such cases [6, 22, 34, 46, 54]. The following result (compare to Proposition 3.15), reveals the structure of the set of fixed-points for limit-preserving functions under weaker conditions than those in Proposition 3.17. It says that the set of fixed-points of a limit-preserving function, if not empty, is a complete multilattice. A *complete multilattice* [4, 30, 31] is a partially ordered set $\mathcal{M} = \langle M, \leq\rangle$, such that for every subset $X \subseteq M$, the set of upper (resp. lower) bounds of $X$ has minimal (resp. maximal) elements, which are called *multi-suprema* (resp. *multi-infima*). The sets of multi-suprema and multi-infima of a set $X$ are denoted multsup$(X)$ and multinf$(X)$.

Proposition 3.18. *Let $f\colon L \to 2^L$ be a multi-valued function. If $f$ is limit-preserving and $Fix(f)$ is non-empty then $Fix(f)$ is a complete multilattice.*

*Proof.* The proof is inspired on the one for Proposition 3.15.

Let us show that $\langle Fix(f), \leq\rangle$ is a complete multilattice. By assumption, $Fix(f)$ is non-empty, by Proposition 3.5, $f$ is EM-monotone, and by Proposition 3.16, for any $x \in L$, $f(x)$ is a closed sub-lattice of $L$. Let $S \subseteq Fix(f)$. Let us show that the

TABLE 3.1
*Main results about $Fix(f)$.*

| Prop. | $\bigwedge$-pr. | $\bigvee$-pr. | S-mo. | H-mo. | $f(x)$ | infl. | defl. | $\Phi(f)$ | $\Psi(f)$ | $Fix(f)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3.7 | | | | | | • | | $\neq \varnothing$ | | $\neq \varnothing$ |
| 3.7 | | | | | | | • | | $\neq \varnothing$ | $\neq \varnothing$ |
| 3.8 | | | • | | | | | min | | min |
| 3.8 | | | • | | | | | $\bigwedge$ | | $\bigwedge$ |
| 3.8 | | | | • | | | | | max | max |
| 3.8 | | | | • | | | | | $\bigvee$ | $\bigvee$ |
| 3.8 | | | | | | • | | min | | min |
| 3.8 | | | | | | • | | $\bigwedge$ | | $\bigwedge$ |
| 3.8 | | | | | | | • | | max | max |
| 3.8 | | | | | | | • | | $\bigvee$ | $\bigvee$ |
| 3.9 | • | | | | | | | $\neq \varnothing$ | | min |
| 3.9 | | • | | | | | | | $\neq \varnothing$ | max |
| 3.10 | | | • | | $\bigwedge$ | | | | | $\bigwedge$ |
| 3.10 | | | | • | | $\bigvee$ | | | | $\bigvee$ |
| 3.11 | | | • | | $\neq \varnothing$, $\wedge$-cl. | | | | | $\bigwedge$ |
| 3.12 | | | | • | $\neq \varnothing$, $\vee$-cl. | | | | | $\bigvee$ |
| 3.14 | | | • | | $\neq \varnothing$, $\wedge$-cl. | • | | | | $\neq \varnothing$, $\wedge$-cl. |
| 3.14 | | | | • | $\neq \varnothing$, $\vee$-cl. | | • | | | $\neq \varnothing$, $\vee$-cl. |
| 3.15 | | | • | • | $\neq \varnothing$, sub-latt. | | | | | $\neq \varnothing$, compl. latt. |
| 3.17 | • | | | | $\neq \varnothing$ | • | | | | $\neq \varnothing$, $\wedge$-cl. |
| 3.17 | | • | | | $\neq \varnothing$ | | • | | | $\neq \varnothing$, $\vee$-cl. |
| 3.17 | • | • | | | $\neq \varnothing$ | | | | | $\neq \varnothing$, compl. latt. |
| 3.18 | • | • | | | | | | | | compl. multilatt. |

set multsup$(S)$ is non-empty in $\langle Fix(f), \leq \rangle$. So, consider $a = \bigvee S = \bigvee_{c \in S} c$ and the complete lattice $\mathcal{B} = \langle [a, \top], \leq \rangle$. Let $g$ be the multi-valued function from $[a, \top]$ to $2^{[a,\top]}$ defined by $g(s) = f(s) \cap [a, \top]$ for all $s \in [a, \top]$. Since both $f$ and $h$, which assigns to each $s \in [a, \top]$ the constant interval $[a, \top]$, are $\bigwedge$-preserving on $S$, it is not difficult to check that $g = f \cap h$ is $\bigwedge$-preserving on $[a, \top]$.

Now, let's show that $\Phi(g) \neq \varnothing$. For $c \in S$, as $c \leq a$ and $f$ is H-monotone, $f(c) \preceq_H f(a)$ follows. Hence, for $c \in f(c)$ there is $x_c \in f(a)$ such that $c \leq x_c$. Consider $b = \bigvee_{c \in S} x_c$. Therefore, $a = \bigvee_{c \in S} c \leq \bigvee_{c \in S} x_c = b$. We show now that $b \in f(a)$. Consider the sequence $(a, a, \ldots, a)$ of length $|S|$. As $f$ is limit-preserving and all $x_c \in f(a)$, we have that $b = \bigvee_{c \in S} x_c \in f(a \vee a \vee \ldots \vee a) = f(a)$, i.e. $b \in f(a)$. Now, consider $s \in [a, \top]$. As $a \leq s$ and $f$ is H-monotone, $f(a) \preceq_H f(s)$ follows, i.e. for $b \in f(a)$ there is $s_b \in f(s)$ such that $a \leq b \leq s_b$. It follows that $g(s) = f(s) \cap [a, \top] \neq \varnothing$ for all $s \in [a, \top]$. In particular, $g(\top) \neq \varnothing$ and, thus, $g(\top) \preceq_S \{\top\}$, i.e. $\top \in \Phi(g) \neq \varnothing$.

As a consequence, by Proposition 3.9, $g$ has minimal fixed-points $S'$. Obviously, as $Fix(g) = Fix(f) \cap [a, \top]$, any $a' \in S'$ is also a fixed-point of $f$ with $a \leq a'$. In fact, $a'$ is a minimal fixed-point of $f$ which is an upper bound of all elements of $S$; in other words, $a' \in$ multsup$(S)$ and $a' \in Fix(f)$, which concludes.

Similarly, it can be shown that multinf$(S)$ is non-empty in $\langle Fix(f), \leq \rangle$ and, thus, we can conclude that $\langle Fix(f), \leq \rangle$ is a complete multilattice. □

Note that by Proposition 3.9, in Proposition 3.18 above, $\Phi(f) \neq \varnothing$ guarantees that $Fix(f)$ is non-empty.

For convenience, Table 3.1 reports a summary of main results about $Fix(f)$ reported in this section. In the table min (max) means that the set contains minimals (maximals), while $\bigwedge$ ($\bigvee$) means that the set contains least (greatest) element.
For completeness, Table 3.2 summarizes instead the impact of the multi-valued functions in the examples on the set of fixed-points.

**3.1. Orbits.** We next describe how to obtain minimal fixed-points (if they exist) of multi-valued functions $f \colon L \to 2^L$. An *orbit* [4] of $f$ is a (possibly transfinite)

[4]The definition is a generalization of the usual iteration of $f$ over $\bot$ for single-valued functions.

TABLE 3.2
*Impact of multi-valued functions in the examples on $Fix(f)$.*

| Ex. | $\bigwedge$-pr. | $\bigvee$-pr. | S-mo. | H-mo. | $f(x)$ | infl. | defl. | $\Phi(f)$ | $\Psi(f)$ | $Fix(f)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | | | | | $\bigwedge, \bigvee$ | • | | $\neq \varnothing$, $\nexists \bigwedge$ | $\bigvee$ | $\bigvee$, $\nexists$ min |
| 8 | | • | • | • | $\bigvee$ | • | | $\exists$ min, $\nexists \bigwedge$ | $\bigvee$ | $\bigvee$, $\exists$ min, $\nexists \bigwedge$ |
| 9 | | • | | | $\bigvee$ | • | | $\bigvee$, $\nexists$ min | $\bigvee$ | $\bigvee$, $\nexists$ min |
| 9 | • | | | | $\bigwedge$ | | • | $\bigwedge$ | $\bigwedge$, $\nexists$ max | $\bigwedge$, $\nexists$ max |
| 9 | | | • | • | $\neq \varnothing$, $\nexists$ min $\nexists$ max | | | $\neq \varnothing$, $\nexists$ min $\nexists$ max | $\neq \varnothing$, $\nexists$ min $\nexists$ max | $\neq \varnothing$, $\nexists$ min $\nexists$ max |
| 14 | | | . | • | $\bigwedge$ | | | $\exists$ min $\nexists \bigwedge$ | compl. latt. | $\nexists$ min $\bigvee$ |
| 15 | | | | • | $\bigwedge$ | | | $\bigvee$ | $\bigwedge$ | $= \varnothing$ |
| 15 | | | • | | $\bigvee$ | | | $\bigvee$ | $\bigwedge$ | $= \varnothing$ |
| 16 | • | • | | | closed sub.-latt. | | • | compl. latt. | $\bigwedge$ | $\exists \bigwedge$, $\exists \bigvee$, $\neg\wedge$-cl. |
| 17 | • | | | | | | • | $\exists$ min $\bigvee$ | $\exists$ min $\bigvee$ | $\exists$ min, $\bigvee$ |

sequence $(x_\alpha)_{\alpha \in I}$ of elements $x_\alpha \in L$, with $|I| > |L|$ and

$$\begin{array}{lcl} x_0 & = & \bot \\ x_{\alpha+1} & \in & f(x_\alpha) \\ x_\lambda & = & \bigvee_{\alpha<\lambda} x_\alpha, \text{ for limit ordinals } \lambda \ . \end{array}$$

Some comments are in order:

- due to the non-deterministic choice of $x_{\alpha+1}$, $f$ may have many possible orbits;
- for the sake of this paper we consider the starting point of the orbit $x_0 = \bot$. However, this can be made more flexible by considering any $x_0 = a \in L$ as starting point. We consider $x_0 = \bot$ as we are interested in how to obtain minimal fixed-points. Of course, a special and interesting alternative case is $x_0 = \top$ (in that case, we postulate that for limit ordinal $\lambda$, $x_\lambda = \bigwedge_{\alpha<\lambda} x_\alpha$), which relates to the computation of maximal fixed-points. We call such sequences $\top$-*orbits*;
- a sequence $x_0, x_1, \ldots, x_\alpha$, where $x_{\beta+1} \in f(x_\beta)$ for $\beta < \alpha$ and $f(x_\alpha) = \varnothing$ is *not* an orbit;
- for convenience, we require that the length $|I|$ of an orbit is strictly greater than $|L|$, so that, if the orbit is increasing (decreasing), we may apply Proposition 2.1, which guarantees then that the orbit becomes eventually stationary;
- if an orbit $(x_\alpha)_{\alpha \in I}$ becomes stationary, i.e., there is $\beta \in I$ such that $|\beta| \leq |L|$ and $x_\alpha = x_\beta$ for all $\beta \leq \alpha \in I$, then by construction $x_\beta = x_{\beta+1} \in f(x_\beta)$ and, thus, $x_\beta$ is a fixed-point of $f$;
- as any increasing (decreasing) orbit converges to a fixed-point, it is clear that if we can guarantee that such an orbit exists then also the existence of a fixed-point is shown;
- of course, from an practical point of view, whenever we try to build an orbit, we may stop as soon as we have $x_\beta = x_{\beta+1}$.

EXAMPLE 18. *Consider the lattice $\mathcal{FOUR}$. Let $g$ be multi-valued function such that $g(\bot) = \{f, t\}$, $g(f) = \{f, t\}$, $g(t) = \{f, t\}$, $g(\top) = \{\top\}$. It can easily be verified that $g$ is S-monotone and $Fix(g) = \{f, t, \top\}$. Then, for instance, we may have the* *following orbits:*

$$\begin{aligned} o_1 &= (\bot, f, f, f, f) \\ o_2 &= (\bot, t, t, t, t, t) \\ o_3 &= (\bot, f, t, t, t) \\ o_4 &= (\bot, t, f, f, t) \\ o_5 &= (\bot, f, t, f, t, f, t) \ . \end{aligned}$$

∎

As already pointed out, unlike the single-valued case, Examples 9 and 18 show that e.g. S-monotonicity does not guarantee the existence of a minimal fixed-point. Also, S-monotonicity does not guarantee that an orbit $(x_\alpha)_{\alpha \in I}$ becomes eventually stationary (consider Example 3 and the orbit $(0, 2, 0, 2, \ldots))$ or in Example 18 orbit $o_5$. Note also that in Example 18 no orbit converges to the fixed-point $\top$.

Our main contribution in this context is the following:

PROPOSITION 3.19. *For a multi-valued function $f$,*
1. *if $f$ is inflationary then each orbit is increasing;*
2. *each increasing orbit converges to a fixed-point of $f$ (if no fixed-point exists then there is no orbit);*
3. *if $f$ is S-monotone and inflationary then for any minimal fixed-point of $f$ there is an orbit converging to it.*

*Proof.* Let $(x_\alpha)_{\alpha \in I}$ be an orbit of $f$. Recall that for ordinal $\alpha$ then $x_{\alpha+1} \in f(x_\alpha) \neq \varnothing$. As $f$ is inflationary, $\{x_\alpha\} \preceq_S f(x_\alpha)$. But, by definition of $\preceq_S$, for $x_{\alpha+1} \in f(x_\alpha)$, $x_\alpha \leq x_{\alpha+1}$. For a limit ordinal $\lambda$, $x_\lambda = \bigvee_{\alpha<\lambda} x_\alpha$, $\{x_\lambda\} \preceq_S f(x_\lambda) \neq \varnothing$ and, thus, there is $x_{\lambda+1} \in f(x_\lambda)$ such that $x_\lambda \leq x_{\lambda+1}$.

For the second point, as $(x_\alpha)_{\alpha \in I}$ is an increasing sequence and $|I| > |L|$, by Proposition 2.1 there is an ordinal $\alpha$ such that $x_\alpha = x_{\alpha+1} \in f(x_\alpha)$. That is, $x_\alpha$ is a fixed-point of $f$.

Finally, for the third point, assume $\bar{x} \in f(\bar{x})$ is a minimal fixed-point of $f$. Now, let us show on (transfinite) induction on $\alpha$ that there is an increasing orbit $(x_\alpha)_{\alpha \in I}$ of $f$ s.t. $x_\alpha \leq \bar{x}$ for all $\alpha$.

**$\alpha = 0$.** $x_0 = \bot \leq \bar{x}$.

**$\alpha$ successor ordinal.** By induction, $x_\alpha \leq \bar{x}$. As $f$ is S-monotone and inflationary, $\{x_\alpha\} \preceq_S f(x_\alpha) \preceq_S f(\bar{x})$. But, $\bar{x} \in f(\bar{x})$, so we can choose $x_{\alpha+1} \in f(x_\alpha)$ s.t. $x_\alpha \leq x_{\alpha+1} \leq \bar{x}$.

**$\alpha$ limit ordinal.** By induction, $x_\beta \leq \bar{x}$ holds for all $\beta < \alpha$, which implies that $x_\alpha = \bigvee_{\beta<\alpha} x_\beta \leq \bar{x}$.

The sequence $(x_\alpha)_{\alpha \in I}$ is increasing and, thus, by Proposition 2.1 there is an ordinal $\alpha$ such that $x_\alpha = x_{\alpha+1} \in f(x_\alpha)$. So, $x_\alpha$ is a fixed-point of $f$ with $x_\alpha \leq \bar{x}$. As $\bar{x}$ is a minimal, $x_\alpha = \bar{x}$. □

EXAMPLE 19. *Consider the lattice $\mathcal{FOUR}$. Let $g$ be a multi-valued function such that $g(\bot) = \{f, t\}$, $g(f) = \{f\}$, $g(t) = \{t\}$, $g(\top) = \{\top\}$. It can easily be verified that $g$ is S-monotone and inflationary and $Fix(g) = \{f, t, \top\}$. Then, we may have the following orbits:*

$$\begin{aligned} o_1 &= (\bot, f, f, f, f) \\ o_2 &= (\bot, t, t, t, t, t) \ . \end{aligned}$$

*Orbit* $o_1$ *converges to the minimal fixed-point* $f$*, while* $o_2$ *converges to the minimal fixed-point* $t$*.* ∎

Of course, the dual of Proposition 3.19 holds as well.

PROPOSITION 3.20. *For a multi-valued function* $f$*,*

1. *if* $f$ *is deflationary then each* $\top$*-orbit is decreasing;*
2. *each decreasing* $\top$*-orbit converges to a fixed-point of* $f$ *(if no fixed-point exists then there is no orbit);*
3. *if* $f$ *is H-monotone and deflationary then for any maximal fixed-point of* $f$ *there is a* $\top$*-orbit converging to it.*

*Proof.* The proof is dual to Proposition 3.19 (see appendix, Proposition A.8). □

By a straightforward adaptation of the proof of Point 3 in Proposition 3.19, we can show that:

PROPOSITION 3.21. *Let* $f$ *be a H-monotone, non-empty multi-valued function, such that for any increasing sequence* $(y_\alpha)_{\alpha\in I}$ *there is* $y \in f(\bigvee_{\alpha\in I} y_\alpha)$ *such that* $y_\alpha \leq y$ *for all* $\alpha \in I$*. Then, there is an increasing orbit and, thus, a fixed-point of* $f$*.*

*Proof.* Let us show on (transfinite) induction on $\alpha$ that there is an increasing orbit $(x_\alpha)_{\alpha\in I}$ of $f$ and, thus, by Proposition 3.19, point 2., converging to a fixed-point of $f$.

$\alpha = 0$. $x_0 = \bot$.

$\alpha$ **successor ordinal.** By induction, $x_{\alpha-1} \leq x_\alpha$ and $x_\alpha \in f(x_{\alpha-1})$. As $f$ is H-monotone, we have $f(x_{\alpha-1}) \preceq_H f(x_\alpha)$. So, for $x_\alpha \in f(x_{\alpha-1})$, there is $x_{\alpha+1} \in f(x_\alpha)$ s.t. $x_\alpha \leq x_{\alpha+1}$.

$\alpha$ **limit ordinal.** Consider $(x_\beta)_{\beta\in\alpha}$. By hypothesis, there is $x_{\alpha+1} \in f(\bigvee_{\beta\in\alpha} x_\beta)$ with $x_\beta \leq x_{\alpha+1}$ for all $\beta < \alpha$ and, thus, $x_\alpha = \bigvee_{\beta<\alpha} x_\beta \leq x_{\alpha+1}$.

□

Note that the condition on the limit is essential as Example 15 shows: $(0, 0.5, 0.75, \ldots)$ is the increasing sequence that can be built, which converges to 1. But, there is no $x \in f(1)$ such that $1 \leq x$. The dual of Proposition 3.21 is:

PROPOSITION 3.22 (Khamsi [22]). *Let* $f$ *be a S-monotone, non-empty multi-valued function such that for any decreasing sequence* $(y_\alpha)_{\alpha\in I}$ *there is* $y \in f(\bigwedge_{\alpha\in I} y_\alpha)$ *such that* $y \leq y_\alpha$ *for all* $\alpha \in I$*. Then there is an decreasing* $\top$*-orbit and, thus, a fixed-point of* $f$*.* ∎

We recall that Proposition 3.22 is the main result described in [22] (see also [16]).

A closer look to the induction step in the previous proof of Point 3 of Proposition 3.19 reveals a useful practical case. Indeed, rather than choosing an arbitrary $x_{\alpha+1} \in f(x_\alpha)$ s.t. $x_{\alpha+1} \leq \bar{x}$ with $x_\alpha \leq x_{\alpha+1}$, if $\min f(x_\alpha)$ is non-empty we may choose an appropriate $x_{\alpha+1} \in \min f(x_\alpha)$.

In the following, let $(x_\alpha)_{\alpha\in I}$ be an orbit ($\top$-orbit) of $f$. We say that $(x_\alpha)_{\alpha\in I}$ is an *orbit ($\top$-orbit) of minimals (maximals)* of $f$ iff $x_{\alpha+1} \in \min f(x_\alpha)$ if $\min f(x_\alpha) \neq \varnothing$ ($x_{\alpha+1} \in \max f(x_\alpha)$ if $\max f(x_\alpha) \neq \varnothing$). Hence,

PROPOSITION 3.23. *Consider a multi-valued function* $f\colon L \to 2^L$*.*

1. *If* $f$ *is inflationary and S-monotone, then for any minimal fixed-point of* $f$ *there is an orbit* $(x_\alpha)_{\alpha\in I}$ *of minimals converging to it;*
2. *If* $f$ *is deflationary and H-monotone, then for any maximal fixed-point of* $f$ *there is an* $\top$*-orbit*$(x_\alpha)_{\alpha\in I}$ *of maximals, converging to it.* ∎

Similarly,

PROPOSITION 3.24. *Consider a multi-valued function $f\colon L \to 2^L$.*

1. *If $f\colon L \to 2^L$ is S-monotone and for all $x \in L$, $f(x)$ has least element then there is an orbit $(x_\alpha)_{\alpha\in I}$ of least elements, i.e. $x_{\alpha+1} = \bigwedge f(x_\alpha)$, converging to the least fixed-point of $f$;*
2. *If $f\colon L \to 2^L$ is H-monotone and for all $x \in L$, $f(x)$ has greatest element then there is a $\top$-orbit $(x_\alpha)_{\alpha\in I}$ of greatest elements, i.e. $x_{\alpha+1} = \bigvee f(x_\alpha)$, converging to the greatest fixed-point of $f$.*

*Proof.* Point 1. From Proposition 3.10, we know that $f$ has least fixed-point $\bar{x}$. Now, we proceed similarly as for Proposition 3.19, point 3. Let us show on (transfinite) induction on $\alpha$ that there is an increasing orbit $(x_\alpha)_{\alpha\in I}$ of $f$ s.t. $x_{\alpha+1} = \bigwedge f(x_\alpha)$ (if $\alpha$ ordinal), and $x_\alpha \leq \bar{x}$ for all $\alpha$.

$\alpha = 0$. $x_0 = \bot \leq \bar{x}$.

$\alpha$ **successor ordinal.** By induction, $x_{\alpha-1} \leq x_\alpha \leq \bar{x}$ and $x_\alpha = \bigwedge f(x_{\alpha-1})$. As $f$ S-monotone, $f(x_{\alpha-1}) \preceq_S f(x_\alpha) \preceq_S f(\bar{x})$. But, $\bar{x} \in f(\bar{x})$, and, thus, there is $y_1 \in f(x_\alpha)$ such that $y_1 \leq \bar{x}$. Consider $x_{\alpha+1} = \bigwedge f(x_\alpha)$. As $x_{\alpha+1} \in f(x_\alpha)$, $x_{\alpha+1} \leq y_1 \leq \bar{x}$ follows. But then, for $x_{\alpha+1} \in f(x_\alpha)$ there is $y_2 \in f(x_{\alpha-1})$ such that $y_2 \leq x_{\alpha+1}$. Consider $x_\alpha = \bigwedge f(x_{\alpha-1})$. By induction $x_\alpha \in f(x_{\alpha-1})$ and, thus, $x_\alpha \leq y_2 \leq x_{\alpha+1} \leq y_1 \leq \bar{x}$.

$\alpha$ **limit ordinal.** By induction, $x_\beta \leq x_{\beta+1} \leq \bar{x}$ holds for all $\beta < \alpha$, which implies that $x_\alpha = \bigvee_{\beta<\alpha} x_\beta = \bigvee_{\beta<\alpha} x_{\beta+1} \leq \bar{x}$. As $f$ S-monotone, $f(x_\beta) \preceq_S f(x_\alpha) \preceq_S f(\bar{x})$ for $\beta < \alpha$. But, $\bar{x} \in f(\bar{x})$, and, thus, there is $y_1 \in f(x_\alpha)$ such that $y_1 \leq \bar{x}$. Consider $x_{\alpha+1} = \bigwedge f(x_\alpha)$. As $x_{\alpha+1} \in f(x_\alpha)$, $x_{\alpha+1} \leq y_1 \leq \bar{x}$ follows. Similarly, as $f(x_\beta) \preceq_S f(x_\alpha)$, for $x_{\alpha+1} \in f(x_\alpha)$ and $x_{\beta+1} = \bigwedge f(x_\beta)$, we have by induction $x_{\beta+1} \in f(x_\beta)$ and, thus, $x_{\beta+1} \leq x_{\alpha+1}$. Therefore, $x_\beta \leq x_{\beta+1} \leq x_{\alpha+1} \leq \bar{x}$ and, thus, $x_\alpha = \bigvee_{\beta<\alpha} x_\beta = \bigvee_{\beta<\alpha} x_{\beta+1} \leq x_{\alpha+1} \leq \bar{x}$.

The sequence $(x_\alpha)_{\alpha\in I}$ is increasing and, thus, by Proposition 2.1 there is an ordinal $\alpha$ such that $x_\alpha = x_{\alpha+1} \in f(x_\alpha)$. So, $x_\alpha$ is a fixed-point of $f$ with $x_\alpha \leq \bar{x}$. As $\bar{x}$ is the least fixed-point, $x_\alpha = \bar{x}$.

Point 2. can be shown similarly. □

Interestingly, $f$ S-monotone and inflationary does not guarantee that $\Phi(f)$ has minimals and, thus, a minimal fixed point may not exist (Example 9). However, we have:

PROPOSITION 3.25. *Let $f$ be an inflationary, $\bigwedge$-preserving multi-valued function such that $\Phi(f) \neq \varnothing$.*

- *Then $f$ has minimal fixed-points and there are orbits converging to them;*
- *If $f$ is also $\bigvee$-preserving, then $\omega$ steps are sufficient to reach a minimal fixed-point.*

*Proof.* The first item follows immediately from Proposition 3.7, Proposition 3.9 and Proposition 3.19. For the second item, consider an orbit $(x_\alpha)_{\alpha\in I}$ converging to a minimal fixed-point $\bar{x}$ of $f$. Let us show that $x_\omega$ is a fixed-point of $f$. As $f$ is inflationary, the orbit is increasing. Then $x_\omega = \bigvee_{\alpha<\omega} x_\alpha$. As $f$ is $\bigvee$-preserving we have that $f(x_\omega) = f(\bigvee_{\alpha<\omega} x_\alpha) = \{y\colon \text{there is } (y_\alpha)_{\alpha<\omega} \textit{ s.t. } y_\alpha \in f(x_\alpha) \text{ and } y = \bigvee_{\alpha<\omega} y_\alpha\}$. For $0 \leq \alpha < \omega$, let $y_\alpha = x_{\alpha+1}$. Therefore, $y_\alpha \in f(x_\alpha)$ and, thus, $x_\omega = y = \bigvee_{\alpha<\omega} y_\alpha \in f(x_\omega)$. That is, $x_\omega$ is a fixed-point of $f$ and $x_\omega \leq \bar{x}$ and, thus,

$x_\omega = \bar{x}$. □

Clearly, the dual of Proposition 3.25 holds as well.

PROPOSITION 3.26. *If a multi-valued function $f$ is deflationary, $\bigvee$-preserving and $\Psi(f) \neq \varnothing$, then $f$ has maximal fixed-points and there are $\top$-orbits converging to them. If $f$ is also $\bigwedge$-preserving, then $\omega$ steps are sufficient to reach a maximal fixed-point.* ∎

We conclude this part showing a strict relationship between S-monotone and inflationary operators. For a multi-valued function $f\colon L \to 2^L$, let us define

$$g(x) = x \oplus f(x) = \{x \vee y \colon y \in f(x)\} \ . \tag{3.6}$$

Note that if $f(x) = \varnothing$ then $g(x) = \varnothing$.

PROPOSITION 3.27. *For $f\colon L \to 2^L$, $g(x) = x \oplus f(x)$ is inflationary. Furthermore, if $f$ is S-monotone, then*

1. *$g$ is S-monotone;*
2. *$x \in f(x)$ implies $x \in g(x)$;*
3. *$x \in g(x)$ implies $f(x) \preceq_S \{x\}$;*
4. *if $x$ is a minimal fixed point of $g$ then $x$ is a minimal fixed point of $f$.*
5. *if $x$ is a minimal fixed point of $f$ and $f$ is also inflationary then $x$ is a minimal fixed point of $g$.*

*Proof.* Consider $f$ and $g$. If $f(x) = \varnothing$ then $\{x\} \preceq_S g(x) = \varnothing$. Otherwise, for $y \in g(x)$, $x \leq y$. Therefore, $\{x\} \preceq_S g(x)$ and, thus, $g$ is inflationary. Now, suppose $f$ S-monotone. Point 1. Easy as $g$ is a combination of S-monotone functions. Point 2. If $x \in f(x)$ then by definition of $g$, $x = x \vee x \in g(x)$. Point 3. If $x \in g(x)$ then for some $y \in f(x)$, $x = x \vee y$. Therefore, $y \leq x$ and, thus, $f(x) \preceq_S \{x\}$. Point 4. Assume $x$ is a minimal fixed-point of $g$, i.e. $x \in g(x) = x \oplus f(x)$. Therefore, there is $y \in f(x)$ such that $y \leq x$. As $f$ is S-monotone, $f(y) \preceq_S f(x)$. That is, there is $z \in f(y)$ such that $z \leq y$ and, thus, $y = y \vee z$. Therefore, $y \in g(y)$. As $x$ is minimal and $y \leq x$, $y = x$ follows and, thus, $x \in f(x)$. To prove that $x$ is a minimal fixed-point of $f$, assume there is $y \leq x$ such that $y \in f(y)$. By Point 2., $y \in g(y)$ and, thus, as $x$ is a minimal fixed-point of $g$, $y = x$ follows; Point 5. Assume $x$ is a minimal fixed-point of $f$. By Point 2. $x \in g(x)$. To prove that $x$ is a minimal fixed-point of $g$, assume there is $y \leq x$ such that $y \in g(y)$. Then by Point 3. $f(y) \preceq_S \{y\}$ and, thus, $y \in \Phi(f)$. By Proposition 3.7, $y \in f(y)$, and, thus, as $x$ is a minimal fixed-point of $f$, $y = x$ follows. □

We note that the inflationary condition in Point 5. in Proposition 3.27 is necessary.

EXAMPLE 20. *Consider $L = \{0\} \cup \{1/n : n = 1, 2, \dots\}$ and the multi-valued mapping $f : L \to 2^L$ defined as follows:*

$$\begin{aligned} f(0) &= \{1/n : n = 1, 2, \dots\} \\ f(1/n) &= \{1\} \cup \{1/(n+k) : k = 1, 2, \dots\} \end{aligned}$$

*$f$ is S-monotone, but not inflationary ($\{1/n\} \not\preceq_S f(1/n)$), and that 1 is its only fixed-point. However, the function $g(x) = x \oplus f(x)$ has the following definition*

$$\begin{aligned} g(0) &= \{1/n : n = 1, 2, \dots\} \\ g(1/n) &= \{1, 1/n\} \ , \end{aligned}$$

*which has infinitely many fixed points and no one is minimal.* ∎

Of course, Proposition 3.27 has its dual as well. Let

$$h(x) = x \otimes f(x) = \{x \wedge y \colon y \in f(x)\} \ . \tag{3.7}$$

PROPOSITION 3.28. *For $f \colon L \to 2^L$, $h(x) = x \otimes f(x)$ is deflationary. Furthermore, if $f$ is H-monotone, then*

1. *$h$ is H-monotone;*
2. *$x \in f(x)$ implies $x \in h(x)$;*
3. *$x \in h(x)$ implies $\{x\} \preceq_H f(x)$;*
4. *if $x$ is a maximal fixed point of $h$ then $x$ is a maximal fixed point of $f$.*
5. *if $x$ is a maximal fixed point of $f$ and $f$ is also deflationary then $x$ is a maximal fixed point of $h$.*

*Proof.* The proof is dual to Proposition 3.27 (see appendix, Proposition A.9). □

We report here some other related results known in the literature. For instance, [46] (which relies on [34]) gives a condition for the existence of a least fixed-point.

PROPOSITION 3.29 (Stouti [46]). *Let $f \colon L \to 2^L$ be a multi-valued function, where $\mathcal{L} = \langle L, \preceq \rangle$ is a* complete partial order *(cpo) with $\bot$, i.e. any non-empty chain in L has a supremum in L, and $\bot \in L$. Assume that for any $x \in L$, $f(x)$ is non-empty and that for any $x, y \in L$ with $x < y$, then for every $a \in f(x)$ and $b \in f(y)$, we have that $a \leq b$.* [5]

1. *Then $f$ has a least fixed-point;*
2. *If there is $a \in L$ such that for all $b \in f(a)$ we have $a \leq b$ then $f$ has a least fixed-point in the subset $\{a \in L \mid a \leq x\}$.* ∎

For completeness, we recall that [34] states that:

PROPOSITION 3.30 (Orey [34]). *Let $f \colon L \to 2^L$ be a multi-valued function, where $\mathcal{L} = \langle L, \preceq \rangle$ is a* complete partial order *(cpo) with $\bot$, i.e. any non-empty chain in L has a supremum in L, and $\bot \in L$. Assume that for any $x \in L$, $f(x)$ is non-empty and that for any $x, y \in L$ with $x < y$, then for every $a \in f(x)$ and $b \in f(y)$, we have that $a \leq b$. If there is $a \in L$ such that $\{a\} \preceq_S f(a)$ then $f$ has a fixed-point.* ∎

The above proposition relies on the fact that under its condition we have that $\{a\} \preceq_S f(a) \preceq_S f^2(a) \preceq_S \ldots$, which allows us to build an increasing and, thus, eventually stationary orbit.

We conclude this section by extending $\leq$ to $L^n$ point-wise: for $(x_1, \ldots, x_n) \in L^n$ and $(y_1, \ldots, y_n) \in L^n$, we say that $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ iff for all $i$, $x_i \leq y_i$. For $\mathbf{x}, \mathbf{y} \in L^n$, $\mathbf{x} \wedge \mathbf{y}$ and $\mathbf{x} \vee \mathbf{y}$ are defined point-wise, i.e. $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$ and $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, \ldots, x_n \vee y_n)$. Since $\mathcal{L} = \langle L, \leq \rangle$ is a complete lattice, so is $\mathcal{L}^n = \langle L^n, \leq \rangle$. All definitions and properties of single-valued functions and multi-valued functions over the domain $L$ of $\mathcal{L}$ can be extended to $\mathcal{L}^n$ as well.

**4. Generalized logic programs.** We apply now the results developed so far to a general form of logic programs. Consider a complete lattice $\mathcal{L} = \langle L, \leq \rangle$, which will act as our truth-value set. Formulae will have a degree of truth in $L$. Let $\mathcal{F}$ be a family of computable $n$-ary functions $f \colon L^n \to L$, called (logical) *connectors* [6].

[5] Hence, this is a strictly stronger monotonicity condition than the EM-monotonicity.

[6] With computable we mean that the result of $f$ is computable in a finite amount of time.

Connectors will be used to build logical formulae from logical atoms. For instance, the join (disjunction function) $\vee$ and the meet (conjunction function) $\wedge$ are connectors. $f(x,y) = \max(0, x+y-1)$ is also a connector over $[0,1]^2$. Connectors have not necessarily to be monotone functions. Let $\mathcal{V}$ be a set of variable symbols and $\mathcal{A}$ be a set of atomic formulae $P(t_1,\ldots,t_m)$, where $P$ is an $m$-ary predicate symbol and all $t_i$ are terms. A *term* is defined inductively, as usual, as being either a variable, a constant or the application of a logical function symbol to terms [27].

A *formula* is either an atom $A$ or an expression of the form $f(A_1,\ldots,A_n)$, where $f$ is an $n$-ary connector and each $A_i$ is an atom. For ease of presentation, the connectors $\wedge$ and $\vee$ are used in infix notation. The intuition behind a formula $f(A_1,\ldots,A_n)$ is that the truth degree of the formula is given by evaluating the truth degree of each $A_i$ and then to apply $f$ to these degrees to obtain the final degree. Of course, the function $f$ may well be the composition of functions, $f_1 \circ \ldots \circ f_n$. For instance, over $[0,1]$, $\min(A(x,y), B(y,z)) \cdot \max(\neg R(z), 0.7) + G(x)$ is a formula. In this case, the truth of the formula is determined from the truth of the atoms $A(x,y), B(y,z), R(z)$ and $G(x)$ by applying the specified arithmetic functions. Truth degrees in $L$ may appear in formulae (like 0.7 above).

A *logic program* $\mathcal{P}$ is a set of rules $\psi \leftarrow \varphi$, where $\psi$ and $\varphi$ are formulae, i.e. rules are of the form

$$g(B_1,\ldots,B_k) \leftarrow f(A_1,\ldots,A_n) \ ,$$

where $f, g$ are connectors and $B_i$ and $A_j$ are atoms. Free variables in a rule are understood to be *universally quantified*. For instance, over $[0,1]$

$$\max(A(x), B(x)) \leftarrow 0.7 \cdot \max(0, A(x,y) + B(y,z) - 1)$$

is a rule. The intuition is that the truth of either $A(x)$ or $B(x)$ is at least the truth degree of the body. We point out that the form of the rules is sufficiently expressive to encompass all approaches we are aware of to monotone many-valued logic programming [7]. So far, in many-valued logic programming rules are either of the "deterministic" form $B \leftarrow f(A_1,\ldots,A_n)$ or of the form $B_1 \vee \ldots \vee B_k \leftarrow A_1 \wedge \ldots \wedge A_n$ (see, e.g. [47]).

In the following, with $\mathcal{P}^*$ we denote the ground instantiation of $\mathcal{P}$. If there is no constant in $\mathcal{P}$ then we consider some constant, say $c$ to form ground terms. Note that $|\mathcal{P}^*|$ may be not finite, but is countable. If we restrict a term to be either a variable or a constant then $|\mathcal{P}^*|$ is finite.

We next consider the usual notion of interpretation and generalise the notion of satisfiability (see, e.g. [47]) to our setting. An *interpretation* is a mapping $I$ from ground atoms to members of $L$. For a ground atom $A$, $I(A)$ indicates the degree of truth to which $A$ is true under $I$. An interpretation $I$ is extended from atoms to non-atomic formulae in the usual way:

1. for $b \in L$, $I(b) = b$;
2. $I(f(A_1,\ldots,A_n)) = f(I(A_1),\ldots,I(A_n))$ .

An interpretation $I$ *satisfies* (is a *model* of) a ground rule $\psi \leftarrow \varphi \in \mathcal{P}^*$, denoted $I \models \psi \leftarrow \varphi$ iff $I(\varphi) \leq I(\psi)$. Essentially, we postulate that the consequent $\psi$ of rule (implication) is at least as true as the antecedent $\varphi$. We further say that $I$ *satisfies*

[7] Also note that any classical first order clause $A_1 \vee \ldots \vee A_k \vee \neg B_1 \vee \ldots \neg B_n$ (with $k+n>0$) is a rule of the form $A_1 \vee \ldots \vee A_k \leftarrow B_1 \wedge \ldots \wedge B_n$. If $k=0$ we use $\bot$ in the left hand side, while if $n=0$ we use $\top$ in the right hand side.

(is a *model* of) a logic program $\mathcal{P}$, denoted $I \models \mathcal{P}$, iff $I$ satisfies all ground rules in $\mathcal{P}^*$. Given an interpretation $I$, with $\mathcal{P}[I]$ we denote the set of ground rules of $\mathcal{P}^*$ in which the body has been evaluated by means of $I$, i.e.

$$\mathcal{P}[I] = \{\psi \leftarrow I(\varphi) \colon \psi \leftarrow \varphi \in \mathcal{P}^*\} \ .$$

It is easily verified that $I \models \mathcal{P}$ iff $I \models \mathcal{P}[I]$.

Given two interpretations $I, J$, we define $I \leq J$ point-wise, i.e. $I \leq J$ iff for all ground atoms $I(A) \leq J(A)$. It is easily verified that the set of interpretations, denoted $\hat{L}$, forms a complete lattice as well, i.e. $\langle \hat{L}, \leq \rangle$ is a complete lattice, with least element $I_\perp$ (mapping all atoms to $\perp$) and greatest element $I_\top$ (mapping all atoms to $\top$). If $L$ is countable then so is $\hat{L}$. If $L$ is finite and a term is either a variable or a constant, then $\hat{L}$ is finite as well.

It is worth noting that $I \leq J$ does not necessarily imply that $I(\psi) \leq J(\psi)$ for a formula $\psi$. However, as one may expect, if the functions involved in $\psi$ are monotone then from $I \leq J$, $I(\psi) \leq J(\psi)$ follows.

Proposition 4.1. *Let $I, J$ be two interpretations such that $I \leq J$. If $\psi$ is a formula involving monotone functions $f \in \mathcal{F}$ then $I(\psi) \leq J(\psi)$.*

*Proof.* The proof is on the structure of $\psi$. Assume $\psi$ is an atomic formula $A$. Then by definition of $I \leq J$, $I(A) \leq J(A)$. If $\psi = f(A_1, \ldots, A_n)$ then using induction on $A_i$ and the fact that $f$ is monotone we have that

$$\begin{aligned} I(f(A_1, \ldots, A_n)) &= f(I(A_1), \ldots, I(A_n)) \\ &\leq f(J(A_1), \ldots, J(A_n)) \\ &= J(f(A_1, \ldots, A_n)) \ , \end{aligned}$$

which concludes. □

Note that the connectors $\wedge, \vee$ are monotone. More generally, let us define the *evaluation* function

$$e(I, \psi) = I(\psi) \ .$$

Then the above proposition establishes that the function $e(I, \psi)$ is monotone in $I$ if all the connectors in $\psi$ are monotone., i.e. if $I \leq J$ then $e(I, \psi) \leq e(J, \psi)$. Similarly, we can show that if all the connectors in $\psi$ are $\bigvee$-preserving ($\bigwedge$-preserving) then $e(I, \psi)$ is $\bigvee$-preserving ($\bigwedge$-preserving) in $I$.

Proposition 4.2. *If all the connectors in $\psi$ are $\bigvee$-preserving ($\bigwedge$-preserving) then $e(I, \psi)$ is $\bigvee$- preserving ($\bigwedge$-preserving) in $I$.*

*Proof.* Let us prove the case $\bigwedge$-preserving. The other case is similar. Consider a decreasing sequence of interpretations $(I_\alpha)_{\alpha \in I}$. We have to show that $e(\bigwedge_\alpha I_\alpha, \psi) = \bigwedge_\alpha e(I_\alpha, \psi)$. That is, $(\bigwedge_\alpha I_\alpha)(\psi) = \bigwedge_\alpha I_\alpha(\psi)$. Let $\bar{I}$ be the interpretation $\bar{I} = \bigwedge_\alpha I_\alpha$. The proof is on the structure of $\psi$. Assume $\psi$ is an atomic formula $A$. Then by definition, $e(\bar{I}, A) = \bar{I}(A) = (\bigwedge_\alpha I_\alpha)(A) = \bigwedge_\alpha I_\alpha(A) = \bigwedge_\alpha e(I_\alpha, A)$. If $\psi = f(A_1, \ldots, A_n)$ then using induction on $A_i$ and the fact that $f$ $\bigwedge$-preserving we have that

$$
\begin{aligned}
e(\bar{I}, f(A_1,\dots,A_n)) &= \bar{I}(f(A_1,\dots,A_n)) \\
&= f(\bar{I}(A_1),\dots,\bar{I}(A_n)) \\
&= f(e(\bar{I},A_1),\dots,e(\bar{I},A_n)) \\
&= f(\bigwedge_\alpha e(I_\alpha,A_1),\dots,\bigwedge_\alpha e(I_\alpha,A_n)) \\
&= \bigwedge_\alpha f(e(I_\alpha,A_1),\dots,e(I_\alpha,A_n)) \\
&= \bigwedge_\alpha f(I_\alpha(A_1),\dots,I_\alpha(A_n)) \\
&= \bigwedge_\alpha I_\alpha(f(A_1,\dots,A_n)) \\
&= \bigwedge_\alpha e(I_\alpha, f(A_1,\dots,A_n)) \ ,
\end{aligned}
$$

which concludes. □

Useful to note is that:

PROPOSITION 4.3. *$\vee$ ($\wedge$) is $\bigvee$-preserving ($\bigwedge$-preserving).*

*Proof.* Let us show that $\vee$ is $\bigvee$-preserving. Indeed, for all increasing sequences $(\langle x_\alpha, y_\alpha\rangle)_{\alpha\in I}$, we have that

$$
\begin{aligned}
\vee(\bigvee_\alpha \langle x_\alpha, y_\alpha\rangle) &= \vee(\langle \bigvee_\alpha x_\alpha, \bigvee_\alpha y_\alpha\rangle) \\
&= (\bigvee_\alpha x_\alpha) \vee (\bigvee_\alpha y_\alpha) = \bigvee_\alpha (x_\alpha \vee y_\alpha) \\
&= \bigvee_\alpha \vee(x_\alpha, y_\alpha) \ .
\end{aligned}
$$

In a similar way, $\wedge$ is $\bigwedge$-preserving. □

In general, $\vee$ ($\wedge$) is not $\bigwedge$- ($\bigvee$-) preserving.

EXAMPLE 21 ([5]). *Let us show that the meet function is not $\bigvee$-preserving in general. Consider the complete lattice obtained from the set of closed subsets of the unit disk, the meet defined as set-intersection and the join defined as the topological closure of set-union (closure is needed here because arbitrary union of closed sets need not be closed). This definition provides a complete distributive lattice structure. Now, for all $n \in \mathbb{N}$, define $x_{n,1} = a =$ the unit circle, i.e. the points $\langle x, y\rangle$ satisfying $x^2+y^2 = 1$, and define $x_{n,2} =$ the disk of radius $1-1/n$, that is, the points $\langle x, y\rangle$ satisfying $x^2 + y^2 \le 1 - 1/n$. The sequence $(\langle x_{n,1}, x_{n,2}\rangle)_{n\in\mathbb{N}}$ is an increasing sequence. $\bigvee_n x_{n,2}$ turns out to be the whole unit disk, therefore $(\bigvee_n x_{n,1}) \wedge (\bigvee_n x_{n,2}) = a \wedge (\bigvee_n x_{n,2})$ is the unit circle. On the other hand, $x_{n,1} \wedge x_{n,2} = a \wedge x_{n,2}$ is the empty set (which is a closed subset), hence $\bigvee_n (x_{n,1} \wedge x_{n,2}) = \bigvee_n (a \wedge x_{n,2})$ is the empty set. As a consequence, $(\bigvee_n x_{n,1}) \wedge (\bigvee_n x_{n,2}) \neq \bigvee_n (x_{n,1} \wedge x_{n,2})$ and, thus, the meet function $\wedge$ is not $\bigvee$-preserving.* ∎

However, it can easily be shown that $\vee$ ($\wedge$) is $\bigwedge$- ($\bigvee$-) preserving if $\mathcal{L} = \langle L, \le\rangle$ is *finite*, i.e. $|L| \in \mathbb{N}$. From a practical point of view this is a limitation we can live with, especially taking into account that computers have finite resources. In particular, this includes also the case of the rational numbers in $[0,1]$ under a given fixed decimal precision $p$ (e.g. $p = 2$) and the Boolean lattice over $\{0,1\}$.

PROPOSITION 4.4. *If $\mathcal{L} = \langle L, \leq \rangle$ is* finite*, then $\vee$ and $\wedge$ are limit preserving.* ∎

Note that Proposition 4.4 can be extended to any *finite* $n$-ary meet (join) function. Furthermore, Proposition 4.4 holds also for any *infinite* $n$-ary meet (join) function, as for a finite lattice, an infinite meet (join) is equivalent to a finite meet (join). Indeed, only finitely many values can appear in the infinite meet (join). Another useful and special case is when $\mathcal{L} = \langle [0,1], \leq \rangle$, as it is used in fuzzy logic programming (see, e.g. [49]).

PROPOSITION 4.5. *$\vee$ and $\wedge$ are limit preserving on* $[0,1] \times [0,1]$. ∎

**4.1. Fixed-point characterization of logic programs.** The aim of this section is to extend the usual fixed-point characterization of classical logic programs [27] to the case of generalized logic programs. So, let $\mathcal{P}$ be a logic program. Consider $\mathcal{L} = \langle L, \leq \rangle$ and the related complete lattice of interpretations $\langle \hat{L}, \leq \rangle$. We next define a multi-valued function over $\hat{L}$ whose set of fixed-points coincides with the set of models of $\mathcal{P}$.

The *multi-valued immediate consequence* operator mapping interpretations into sets of interpretations, $T_{\mathcal{P}} \colon \hat{L} \to 2^{\hat{L}}$, is defined as

$$T_{\mathcal{P}}(I) = \{J \colon J \models \mathcal{P}[I], I \leq J\} \ .$$

Note that either $T_{\mathcal{P}}(I_\top) = \varnothing$ or $T_{\mathcal{P}}(I_\top) = \{I_\top\}$. Also note that unlike the single-valued case, not necessarily $T_{\mathcal{P}}(I) \neq \varnothing$.

EXAMPLE 22. *For any interpretation $I$ and for $\mathcal{P} = \{A \vee B \leftarrow \top, \ \bot \leftarrow A, \ \bot \leftarrow B\ \}$, $T_{\mathcal{P}}(I) = \varnothing$ holds.* ∎

However, note that for the specific case of rules of the form (no $A_i, B_j$ is neither $\top$ nor $\bot$ and $k \geq 1$)

$$A_1 \vee \ldots \vee A_k \leftarrow f(B_1, \ldots, B_n) \ ,$$

it is easily verified that for any $I$, $I_\top \in T_{\mathcal{P}}(I) \neq \varnothing$, in particular $T_{\mathcal{P}}(I_\top) = \{I_\top\}$.

Also, note that $T_{\mathcal{P}}(I)$ may not be countable.

EXAMPLE 23. *Consider $L = [0,1]$ and $\mathcal{P}$ with rule $A \leftarrow 0$. Then for any interpretation $I \neq I_\top$, $T_{\mathcal{P}}(I) = \{J \mid I \leq J \text{ and } J(A) \geq 0.3\}$ holds. Hence, $T_{\mathcal{P}}(I)$ is not countable.* ∎

The $T_{\mathcal{P}}$ function has the desired property that models of logic programs are fixed-points and vice-versa.

PROPOSITION 4.6. *$I \models \mathcal{P}$ iff $I \in T_{\mathcal{P}}(I)$.*
*Proof.* $I \models \mathcal{P}$ iff $I \models \mathcal{P}[I]$ iff $I \in T_{\mathcal{P}}(I)$. □

EXAMPLE 24. *Over $\mathcal{L} = \langle \{0,1\}, \leq \rangle$, consider $\mathcal{P} = \{A \leftarrow 1 - B\}$, and $I(A) = 0, I(B) = 1$. Then*

$$\begin{aligned} T_{\mathcal{P}}(I) &= \{J \mid J \models \mathcal{P}[I], I \leq J\} \\ &= \{J \mid J \models A \leftarrow 0, I \leq J\} \\ &= \{J \mid I \leq J\} \\ &= \{I, I'\} \ . \end{aligned}$$

*where* $I'(A) = I'(B) = 1$*. Note that* $I \in T_{\mathcal{P}}(I)$ *and* $I$ *is a model of* $\mathcal{P}$*. Note also that the truth combination function* $f(x) = 1 - x$ *in rule* $A \leftarrow 1 - B$ *is not monotone.* ∎

Hence determining models of a logic program is equivalent to investigate the fixed-points of the multi-valued function $T_{\mathcal{P}}$.

In the following, we will determine which properties of Section 3 about multi-valued functions apply to $T_{\mathcal{P}}$ and which are specific of $T_{\mathcal{P}}$ only. To start with, as by definition $J \in T_{\mathcal{P}}(I)$ implies $I \leq J$ we have immediately:

PROPOSITION 4.7. *$T_{\mathcal{P}}$ is inflationary.* ∎

Furthermore, we also can show that:

PROPOSITION 4.8. *If all connector functions in the body $\varphi$ of rules $\psi \leftarrow \varphi \in \mathcal{P}$ are $\bigvee$-preserving then $T_{\mathcal{P}}$ is $\bigvee$-preserving and, thus, S-monotone.*

*Proof.* Let $(I_\alpha)_{\alpha \in I}$ be an increasing sequence of interpretations. Let $\bar{I} = \bigvee_\alpha I_\alpha$. We have to show that $T_{\mathcal{P}}(\bar{I}) = \{J\colon$ there is $(J_\alpha)_{\alpha \in I}$ *s.t.* $J_\alpha \in T_{\mathcal{P}}(I_\alpha)$ and $J = \bigvee_\alpha J_\alpha\}$ $(= \bigvee_\alpha T_{\mathcal{P}}(I_\alpha))$. So, let $J \in T_{\mathcal{P}}(\bar{I})$. Then $J \models \mathcal{P}[\bar{I}]$ and $\bar{I} \leq J$ and, thus, $I_\alpha \leq J$. Then, using Proposition 4.2, for all ground rules $\psi \leftarrow \varphi \in \mathcal{P}^*$, $I_\alpha(\varphi) \leq \bigvee_\alpha I_\alpha(\varphi) = \bar{I}(\varphi) \leq J(\psi)$. Therefore, $J \models \mathcal{P}[I_\alpha]$ and, thus, $J \in T_{\mathcal{P}}(I_\alpha)$. Hence, $J \in \bigvee_\alpha T_{\mathcal{P}}(I_\alpha)$. Vice-versa, let $J \in \bigvee_\alpha T_{\mathcal{P}}(I_\alpha)$. Thus $J = \bigvee_\alpha J_\alpha$ with $J_\alpha \in T_{\mathcal{P}}(I_\alpha)$. It follows that $I_\alpha \leq J_\alpha \leq J$ and $J_\alpha \models P[I_\alpha]$. Then, using Proposition 4.2, for all ground rules $\psi \leftarrow \varphi \in \mathcal{P}^*$, $\bar{I}(\varphi) = \bigvee_\alpha I_\alpha(\varphi) \leq \bigvee_\alpha J_\alpha(\psi) = J(\psi)$ and, thus, $J \models \mathcal{P}[\bar{I}]$. As $\bar{I} = \bigvee_\alpha I_\alpha \leq \bigvee_\alpha J_\alpha = J$, $J \in T_{\mathcal{P}}(\bar{I})$ follows. S-monotonicity follows from Proposition 3.5. □

The analogue of Proposition 4.8 does not hold for $\bigwedge$-preserving connector functions.

EXAMPLE 25. *Consider $L = [0,1]$, $a \geq 1$, the function $f(x) = 1/(a+1-x)$ and the logic program $\mathcal{P} = \{\frac{1}{a+1} \leftarrow f(A)\}$. Consider a decreasing sequence of interpretations $I_n(A) = 1/n$, $n \in \mathbb{N}$. Then $\bar{I}(A) = \bigwedge_\alpha I_\alpha(A) = I_\perp(A) = 0$. The function $f$ is monotone, more precisely, $\bigwedge$-preserving, with maximum value $\frac{1}{a}$ and minimum value $\frac{1}{a+1}$. Furthermore, $f(I_1(A)) = \frac{1}{a}$, while $f(\bar{I}(A)) = \frac{1}{a+1}$ and $f(I_n(A)) = \frac{1}{a+1-1/n} > \frac{1}{a+1}$. Therefore, $T_{\mathcal{P}}(\bar{I}) = \{J\colon J$ interpretation$\}$. On the other hand, $T_{\mathcal{P}}(I_n) = \varnothing$ and, thus* [8]*, $\bigwedge_n T_{\mathcal{P}}(I_n) = \varnothing$. Therefore, $T_{\mathcal{P}}(\bigwedge_n I_n) \not\subseteq \bigwedge_n T_{\mathcal{P}}(I_n)$, i.e. $T_{\mathcal{P}}$ is not $\bigwedge$-preserving.* ∎

Let us define

$$G_{\mathcal{P}}(I) = \{J \vee I\colon J \models \mathcal{P}[I]\} \ . \tag{4.1}$$

Then it is easily verified that $T_{\mathcal{P}}(I) \subseteq G_{\mathcal{P}}(I)$ (from $I \leq J$, $J \vee I = J$). On the other hand, for $J \in G_{\mathcal{P}}(I)$, $J = J' \vee I$, $J' \models \mathcal{P}[I]$, $J' \leq J$ and $I \leq J$. If all connector functions in the head of rules in $\mathcal{P}$ are monotone then for all ground rules $\psi \leftarrow \varphi \in \mathcal{P}^*$, (using Proposition 4.1) $I(\varphi) \leq J'(\psi) \leq J(\psi)$. Therefore, $J \in T_{\mathcal{P}}(I)$, i.e. $G_{\mathcal{P}}(I) \subseteq T_{\mathcal{P}}(I)$. Therefore:

PROPOSITION 4.9. *For any interpretation $I$, $T_{\mathcal{P}}(I) \subseteq G_{\mathcal{P}}(I)$. If all connector functions in the head of rules in $\mathcal{P}$ are monotone, then $T_{\mathcal{P}}(I) = G_{\mathcal{P}}(I)$.* ∎

Monotonicity is a necessary condition to guarantee equivalence among $T_{\mathcal{P}}$ and $G_{\mathcal{P}}$.

EXAMPLE 26. *Over $\mathcal{L} = \langle\{0,1\}, \leq\rangle$, consider the logic program $\mathcal{P} = \{\neg A \leftarrow A\}$.*

[8] Recall that $\bigwedge_n T_{\mathcal{P}}(I_n)$ is a shorthand for the right hand side of Equation (3.5).

*The negation function* $\neg x = 1 - x$ *is obviously not monotone. Consider* $I(A) = 1$ *and* $J'(A) = 0$*. Then,* $J' \models \mathcal{P}[I]$ *and, thus,* $J = I \vee J' = I_\top \in G_\mathcal{P}(I)$*, but* $J \notin T_\mathcal{P}(I)$*.* ∎

A closer analysis shows that we can write $G_\mathcal{P}$ similarly to Equation (3.6). Indeed, let $F_\mathcal{P}$ be the multi-valued function

$$F_\mathcal{P}(I) = \{J \colon J \models \mathcal{P}[I]\} \ .$$

Then, it can easily verified that

$$G_\mathcal{P}(I) = I \oplus F_\mathcal{P}(I) \ .$$

We can show that:

PROPOSITION 4.10. *If all connector functions in the body* $\varphi$ *of rules* $\psi \leftarrow \varphi \in \mathcal{P}$ *are monotone then* $F_\mathcal{P}$ *is a multi-valued S-monotone operator.*

*Proof.* Consider interpretations $I, J$ s.t. $I \leq J$. Let us show that $F_\mathcal{P}(I) \preceq_S F_\mathcal{P}(J)$. If $F_\mathcal{P}(J) = \varnothing$ then obviously $F_\mathcal{P}(I) \preceq_S F_\mathcal{P}(J)$. Otherwise, assume $F_\mathcal{P}(J) \neq \varnothing$. Let $J' \in F_\mathcal{P}(J)$ and, thus, by definition $J' \models \mathcal{P}[J]$, i.e for all ground rules $\psi \leftarrow \varphi \in \mathcal{P}^*$, $J(\varphi) \leq J'(\psi)$. But, $I \leq J$ and, using Proposition 4.1, $I(\varphi) \leq J(\varphi) \leq J'(\psi)$. Therefore, $J' \models \mathcal{P}[I]$ and, thus, $J' \in F_\mathcal{P}(I)$, which concludes. □

Note that the proof of the proposition above shows in fact that if $I \leq J$ then $F_\mathcal{P}(J) \subseteq F_\mathcal{P}(I)$ and, thus, $F_\mathcal{P}(I) \preceq_S F_\mathcal{P}(J)$.

Now, taking into account Propositions 3.27, 4.7, 4.9 the following analogue of Proposition 3.27 can be obtained:

PROPOSITION 4.11. $G_\mathcal{P}$ *is inflationary. Furthermore, if all connector functions in* $\mathcal{P}$ *are monotone then* $(i)$ $T_\mathcal{P} = G_\mathcal{P}$*;* $(ii)$ $T_\mathcal{P}$ *is S-monotone;* $(iii)$ $I \in F_\mathcal{P}(I)$ *implies* $I \in T_\mathcal{P}(I)$*;* $(iv)$ $I \in T_\mathcal{P}(I)$ *implies* $F_\mathcal{P}(I) \leq \{I\}$*;* $(v)$ *for any interpretation* $I$*,* $I$ *minimal fixed point of* $F_\mathcal{P}$ *iff* $I$ *minimal fixed point of* $T_\mathcal{P}$*.* ∎

By relying on Proposition 4.6, 3.7 and Proposition 3.19 we have that

PROPOSITION 4.12. *Let* $\mathcal{P}$ *be a logic program. Then*

1. $\Phi(T_\mathcal{P}) \neq \varnothing$ *iff* $\mathcal{P}$ *has a model.*
2. *Each orbit of* $T_\mathcal{P}$ *is increasing and converges to a model of* $\mathcal{P}$*.*
3. *If* $I$ *is a minimal model of* $\mathcal{P}$ *and all connector functions in* $\mathcal{P}$ *are monotone, then there is an orbit converging to* $I$*.* ∎

Unlike the general case, for $T_\mathcal{P}$ we can even be more precise and reach any model.

PROPOSITION 4.13. *If* $I$ *is a model of* $\mathcal{P}$ *and all connector functions in* $\mathcal{P}$ *are monotone, then there is an orbit converging to* $I$*.*

*Proof.* We show that if $I \models \mathcal{P}$ then there is an orbit converging to $I$. By Proposition 4.6, $I \in T_\mathcal{P}(I)$. The proof is similar as for Point 3. in Proposition 3.19. We know that each orbit of $T_\mathcal{P}$ converges to a model of $\mathcal{P}$. As in Proposition 3.19, we can show on induction on $\alpha$ that there is an orbit $(I_\alpha)_{\alpha \in I}$ of elements $I_{\alpha+1} \in T_\mathcal{P}(I_\alpha)$ with $I_0 = I_\perp$, such that $I_\alpha \leq I$ for all $\alpha$. Therefore, the orbit converges to a model $I_{\bar{\alpha}}$ of $\mathcal{P}$, where $I_{\bar{\alpha}} = I_{\bar{\alpha}+1}$, $I_{\bar{\alpha}} \leq I$. By Proposition 4.6, $I_{\bar{\alpha}} \in T_\mathcal{P}(I_{\bar{\alpha}})$. Now, let us show that $I \in T_\mathcal{P}(I_{\bar{\alpha}})$. Indeed, from $I_{\bar{\alpha}} \models \mathcal{P}$ and $I \models \mathcal{P}$, for all $\psi \leftarrow \varphi \in \mathcal{P}^*$, from $I_{\bar{\alpha}} \leq I$, using Proposition 4.1, we have $I_{\bar{\alpha}}(\varphi) \leq I(\varphi) \leq I(\psi)$. Therefore, $I \in T_\mathcal{P}(I_{\bar{\alpha}})$ and, thus, the sequence $I_0 = \perp, \ldots, I_{\bar{\alpha}}, I, I, \ldots$ is an orbit converging to $I$. □

EXAMPLE 27. *Consider the logic program over the Boolean lattice on* $\{0, 1\}$*,* $\mathcal{P} = \{(a \vee b \leftarrow 1), (c \leftarrow a), (a \wedge c \wedge d \leftarrow b)\}$*. The unique minimal model is* $\bar{I}(a, b, c, d) =$

$\langle 1,0,1,0\rangle$. *The following are two orbits* $p_1, p_2$ *of* $T_{\mathcal{P}}$*:*

$$p_1 = \langle 0,0,0,0\rangle \to \langle 1,0,0,0\rangle \to \langle 1,0,1,0\rangle \to \langle 1,0,1,0\rangle \ .$$
$$p_2 = \langle 0,0,0,0\rangle \to \langle 0,1,0,0\rangle \to \langle 1,1,1,1\rangle \to \langle 1,1,1,1\rangle \ .$$

*Both* $\langle 1,0,1,0\rangle$ *and* $\langle 1,1,1,1\rangle$ *are fixed-points, i.e. models and* $p_1$ *reaches the minimal one.* ∎

Note that the previous two propositions allow us also to decide, if the lattice is *finite*, whether a logic program does not have a model. Indeed, it suffices to try to build an orbit, starting with $I_\bot$ and systematically use all alternatives (which are finite) at each step. If no orbit can be built, no model exists.

As for the general case (see Example 9), $T_{\mathcal{P}}$ may not have minimal fixed-points.

EXAMPLE 28. *Consider the logic program* $\mathcal{P} = \{f(A) \leftarrow 1\}$*, where* $f(x) = 1$ *if* $x > 0$ *and* $f(0) = 0$*. Then* $I \models \mathcal{P}$ *iff* $I(A) > 0$*, and no minimal model exists.* ∎

The following example shows that if a connector function is not $\bigwedge$-preserving then there is a decreasing sequence of models not converging to a model.

EXAMPLE 29. *Consider* $L = [0,1]$ *and the connector function* $f$ *such that* $f(0) = 0$ *and for* $x > 0$*,* $f(x) = 1$*. Now, consider the logic program* $\mathcal{P} = \{A \vee f(B) \leftarrow 1\}$*. Then the decreasing sequence* $(I_n)_{n\in\mathbb{N}}$ *of interpretations* $I_n$*, where* $I_n(A) = 0$ *and* $I_n(B) = 1/n$ *is a decreasing sequence of models of* $\mathcal{P}$ *converging to the interpretation* $I(A) = 0, I(B) = 0$*, which however is not a model of* $\mathcal{P}$*. Note:* $f$ *is not* $\bigwedge$*-preserving. Also note that* $\mathcal{P}$ *has a minimal model* $I(A) = 1$ *and* $I(B) = 0$*, despite the fact that the connector function* $f$ *is not* $\bigwedge$*-preserving.* ∎

We next want to establish a proposition like Proposition 3.9, guaranteeing the existence of minimal fixed points.

PROPOSITION 4.14. *If all connector functions in* $\mathcal{P}$ *are* $\bigwedge$*-preserving and* $\mathcal{P}$ *has models then* $\Phi(T_{\mathcal{P}})$ *has minimals.*

*Proof.* As $\mathcal{P}$ has models, models are fixed-points of $T_{\mathcal{P}}$ (Proposition 4.6), and $T_{\mathcal{P}}$ is inflationary, by Proposition 3.7, $\Phi(T_{\mathcal{P}}) \neq \varnothing$. So, let $(I_\alpha)_{\alpha\in I}$ be a decreasing sequence of interpretations in $\Phi(T_{\mathcal{P}})$ and let $I = \bigwedge_\alpha I_\alpha$. Again, by Zorn's Lemma it suffices to show that $I \in \Phi(T_{\mathcal{P}})$.

By Proposition 4.11 and Proposition 3.7, $I_\alpha \in T_{\mathcal{P}}(I_\alpha)$, i.e. $I_\alpha$ are fixed-points. Now, let us show that $I \in T_{\mathcal{P}}(I)$. From $I_\alpha \in T_{\mathcal{P}}(I_\alpha)$, and $\psi \leftarrow \varphi \in \mathcal{P}^*$, $I_\alpha(\varphi) \leq I_\alpha(\psi)$ holds. Therefore, by Proposition 4.2: $I(\varphi) = (\bigwedge_\alpha I_\alpha)(\varphi) = \bigwedge_\alpha I_\alpha(\varphi) \leq \bigwedge_\alpha I_\alpha(\psi) = (\bigwedge_\alpha I_\alpha)(\psi) = I(\psi)$. As a consequence, $I \models \mathcal{P}[I]$ and, thus, $I \in T_{\mathcal{P}}(I)$. Therefore, $I \in \Phi(T_{\mathcal{P}})$, which concludes. □

We note that, by Proposition 3.19, if $T_{\mathcal{P}}(I_\top) \neq \varnothing$ then, as $T_{\mathcal{P}}$ inflationary, $\mathcal{P}$ has a model. Then, by Proposition 3.8 and Proposition 4.6 it follows directly that

PROPOSITION 4.15. *If all connector functions in* $\mathcal{P}$ *are* $\bigwedge$*-preserving and* $\mathcal{P}$ *has models, then* $T_{\mathcal{P}}$ *has minimal fixed-points and, thus,* $\mathcal{P}$ *has minimal models.* ∎

The analogue of Proposition 3.25 is:

PROPOSITION 4.16. *If* $\mathcal{P}$ *has models and all connector functions in* $\mathcal{P}$ *are* $\bigwedge$*-preserving then* $\mathcal{P}$ *has minimal models and there are orbits converging to them. If all connector functions in* $\mathcal{P}$ *are also* $\bigvee$*-preserving, then* $\omega$ *steps are sufficient to reach a minimal model.* ∎

**4.2. The case of classical logic programs.** We conclude this part by applying our results to classical logic programs [27, 28, 33]. As already pointed out, any classical first order clause $A_1 \vee \ldots \vee A_k \vee \neg B_1 \vee \ldots \neg B_n$ (with $k+n>0$) is a rule of the form

$$A_1 \vee \ldots \vee A_k \leftarrow B_1 \wedge \ldots \wedge B_n \ . \tag{4.2}$$

If $k=0$ we use $\bot$ in the left hand side, while if $n=0$ we use $\top$ in the right hand side. The truth space is $L=\{0,1\}$. Note that usually in disjunctive logic programs $k \geq 1$ is assumed and no $A_i, B_j$ is neither $\top$ nor $\bot$. This slight difference has an impact on the set of models of a disjunctive logic program, as we show next.

Example 30. *Consider the truth space is $L=\{0,1\}$ and consider $\mathcal{P}$ with rules*

$$\begin{aligned}\bot &\leftarrow A\\ A &\leftarrow \top \ .\end{aligned}$$

*The former rule states that $A$ should be false, while the latter states that $A$ should be true. Of course, $T_{\mathcal{P}}(I)=\emptyset$, for any interpretation $I$ and, thus, $T_{\mathcal{P}}$ has no fixed-point, thus, $\mathcal{P}$ has no model.* ∎

On the other hand, if we assume that $k \geq 1$ and that no $A_i, B_j$ is neither $\top$ nor $\bot$, as usual for disjunctive logic programs, as $L$ is finite, by Proposition 4.4, $\vee$ and $\wedge$ are limit preserving. Furthermore, it is easily verified that for any $I$, $I_\top \in T_{\mathcal{P}}(I) \neq \varnothing$, in particular $T_{\mathcal{P}}(I_\top)=\{I_\top\}$, $T_{\mathcal{P}}$ is $\bigvee$-preserving (thus, S-monotone), and, as $T_{\mathcal{P}}$ inflationary, $\mathcal{P}$ has a model. By Propositions 4.16 and 3.23 we have immediately the well-known fact [28, 33]:

Proposition 4.17. *Any classical disjunctive logic program $\mathcal{P}$ has minimal models and there are orbits (of length $\omega$) of minimals converging to them.* ∎

Finally, let us further restrict logic programs to the case where the head contains one atom only (i.e., $k=1$). That is, rules are of the usual deterministic form

$$A \leftarrow B_1 \wedge \ldots \wedge B_n \ . \tag{4.3}$$

Then, for any $I$, $T_{\mathcal{P}}(I)$ has least element.

Proposition 4.18. *For any classical deterministic logic program $\mathcal{P}$ and interpretation $I$, $T_{\mathcal{P}}(I)$ has least element.*

*Proof.* Consider $\bar{J}=\bigwedge T_{\mathcal{P}}(I)$. Let us show that $\bar{J} \in T_{\mathcal{P}}(I)$. As for all $J \in T_{\mathcal{P}}(I)$ we have $I \leq J$, it follows that $I \leq \bigwedge_{J \in T_{\mathcal{P}}(I)} J = \bar{J}$. Now, consider $A \leftarrow I(\varphi)$ with $A \leftarrow \varphi \in \mathcal{P}^*$. Then by Proposition 4.2, as for all $J \in T_{\mathcal{P}}(I)$, $I(\varphi) \leq J(A)$ holds,

$$I(\varphi) \ \leq \ \bigwedge_{J \in T_{\mathcal{P}}(I)} J(A) \ = \ \bigwedge_{J \in T_{\mathcal{P}}(I)} e(J,A) \ = \ e(\bigwedge_{J \in T_{\mathcal{P}}(I)}, A) \ = \ e(\bar{J},A) \ = \ \bar{J}(A)$$

and, thus, $\bar{J} \models \mathcal{P}[I]$. As a consequence, $\bar{J} \in T_{\mathcal{P}}(I)$. □

Now, using Propositions 3.10, 3.24 and 4.17 we have immediately the well-known fact [27]:

Proposition 4.19. *Any classical deterministic logic program $\mathcal{P}$ has least model and there is an orbit (of length $\omega$) of least elements converging to it.* ∎

If terms are restricted to be either variables or constants, then for disjunctive logic programs the set of minimal models is finite (as there are finitely many interpretations). For both Proposition 4.17 as well as for Proposition 4.19 the length of the orbits are finite.

**5. Conclusions and related work.** We have provided conditions for the existence of fixed-points, minimal and maximal fixed-points of multi-valued functions over complete lattices, and have shown how to obtain them. Our main contribution establishes that an inflationary, S-monotone multi-valued function with $\Phi(f) \neq \varnothing$ has minimal fixed-points, each orbit converges to a fixed-point and for each minimal fixed-point an orbit converging to it exists. We have also shown that (see Table 3.1) the set of fixed-points of a limit-preserving multi-valued function is a complete multilattice. We also reported the results of related work we are aware of.

We then applied our results to a general form of logic programs, where the truth space is a complete lattice. We have shown that a multi-valued operator can be defined whose fixed-points are in one-to-one correspondence with the models of the logic program.

**Related work.** To the best of our knowledge, the fixed-point theory over complete lattices is mainly single-valued oriented. Nonetheless, [6, 14, 15, 16, 20, 21, 22, 34, 46, 54], establish a version of the Knaster-Tarski Theorem, though requiring $f(x)$ always non-empty and some other conditions. [20, 21, 22, 14, 15, 16, 17, 39] also investigate the case where metric spaces or Banach spaces are considered in place of complete lattices, and then use the well-known contraction principle (see also [24, 42]) or continuity to guarantee the existence of a fixed-point (if $f(x)$ always non-empty, of course). They also apply then some of their results to disjunctive logic programs (with non-monotone negation). Close in spirit, using mainly Banach spaces, topological spaces and metric spaces in place of complete lattices, are works of the mathematical society such as [2, 10, 19, 23, 26, 41, 35, 36, 44, 50, 51]. We point out that these works do not cover our results. As our initial objective was to study generalized many-valued logic programs, our analysis tried to parallel the usual ones made for single-valued functions over complete lattices.

The research area of semantics for non-deterministic programming languages (e.g. [8, 37, 38, 45] instead does not address multi-valued functions directly, but rather "lift" a multi-valued function $f : D \to 2^D$ to a function $g : \mathcal{P}^*(D) \to \mathcal{P}^*(D)$, where $\mathcal{P}^*(D)$ is a rather complicated and appropriately ordered subset of the powerset of $D$ (so-called *power domains* [1, 37, 45]) and then applies usual fixed-point theory. Here, $D$ is a so-called *domain*, i.e. a complete partial ordered set with some additional constraints [1]. As in all order cases, $f(x)$ is assumed to be non-empty and finite. This constraint is related to the application of non-deterministic programming languages (as indeed, at each step of a program execution, there is at least one next state and there are at most finitely many possible non-deterministic alternatives).

Concerning the application of multi-valued functions to logic programming, to the best of our knowledge, no work considers such general rules. Related to our approach are [14, 15, 16, 20, 21, 22] in which classical disjunctive logic programs has been considered with non-monotone negation. We did not consider non-monotonic negation so far, as an appropriate semantics (for generalized non-monotone many-valued logic programs) has still to be developed. We also point to works such as [13, 40, 52, 53] in which disjunctive logic programs are studied from a domain-theoretic (i.e. Smyth powerdomain) point of view. One feature of these works is that using an appropriate domain, as in the case of non-determinsitic programming languages, the concept of multi-valued function is avoided by representing “disjunctive states” [9] (again, the image of a multi-valued function is assumed to be non-empty and finite). On the other hand, we follow a direct approach, which requires less formal and abstract theory and is likely amenable to less formal audience as well.

We envisage several directions for future research. The fixed-point theory of multi-valued functions is interesting per se (there are many options worth to be investigated, like using some other sets in place of complete lattices, like complete partial orders, domains, Banach spaces, metric spaces, topological spaces or some specific sets such as $[0,1]$, etc., which have mainly been considered by mathematicians—see also [12]). On the other hand, related to general logic programs, besides considering special cases for connectors in the head and body, it would be interesting to generalize the stable model semantics for classical disjunctive logic programs [9] to our case. More generally, we would like to bring the theory on fixed-points of multi-valued functions to the attention of the knowledge representation and reasoning community, where multi-valued functions may be applied to several problems and logic-based languages for knowledge representation.

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## Appendix A. Some other proofs.

PROPOSITION A.1. *Consider a multi-valued function $f\colon L \to 2^L$. If $f$ is $\bigwedge$-preserving then $f$ is H-monotone;*

*Proof.* Consider $x_1 \leq x_2$. Then for the decreasing sequence $x_2 \geq x_1$, $f(x_1) = f(x_2 \wedge x_1) = \{y\colon \text{ there are } y_i \in f(x_i) \text{ s.t. } y = y_2 \wedge y_1\} = X$. If $f(x_1) = \varnothing$ then trivially $\varnothing = f(x_1) \preceq_H f(x_2)$. If $f(x_2) = \varnothing$ then by definition $X = \varnothing$ and, thus, $f(x_1) = \varnothing$. Therefore, $\varnothing = f(x_1) \preceq_H f(x_2) = \varnothing$. Otherwise assume $f(x_1)$ and $f(x_2)$ non-empty. Therefore, as $f$ is $\bigwedge$-preserving, for $y \in f(x_1) = X$ there are $y_i \in f(x_i)$ $(i = 1, 2)$ such that $y = y_2 \wedge y_1$. In particular, $y \leq y_2$. Therefore, $f(x_1) \preceq_H f(x_2)$ and, thus, $f$ is H-monotone. □

PROPOSITION A.2. *Consider a multi-valued function $f\colon L \to 2^L$ and $x_1 \leq x_2$ with $f(x_1) \neq \varnothing \neq f(x_2)$. If $f$ is $\bigvee$-preserving then $f(x_1) \preceq_H f(x_2)$.*

*Proof.* For the increasing sequence $x_1 \leq x_2$, as $f$ is $\bigvee$-preserving, $f(x_2) = f(x_1 \vee x_2) = \{y\colon \text{ there are } y_i \in f(x_i) \text{ s.t. } y = y_2 \vee y_1\} = X$. Now, for $y \in f(x_1)$ choose a $y' \in f(x_2) \neq \varnothing$ and consider $y'' = y \vee y'$. Therefore, $y'' \in X = f(x_2)$, $y \leq y''$ and, thus, $f(x_1) \preceq_H f(x_2)$. □

PROPOSITION A.3. *Let $f\colon L \to 2^L$ be a multi-valued function. If $f$ is deflationary then $x \in \Psi(f)$ iff $x$ fixed-point of $f$.*

*Proof.* Case 2. Let $x \in \Psi(f)$. As $f$ is deflationary, $\{x\} \preceq_H f(x) \preceq_H \{x\}$ and, thus, for $x \in \{x\}$ there is $y \in f(x)$ such that $x \leq y \leq x$, i.e. $x = y \in f(x)$. Vice-versa, if $x \in f(x)$ then $\{x\} \preceq_H f(x)$ and, thus, $x \in \Psi(f)$. □

PROPOSITION A.4. *Let $f\colon L \to 2^L$ be a multi-valued function. If $f$ is a H-monotone or deflationary multi-valued function, and $\Psi(f)$ has maximals then all $y \in \max \Psi(f)$ are maximal fixed-points of $f$. In particular, if $x = \bigvee \Psi(f) \in \Psi(f)$ then $x$ is greatest fixed-point of $f$.*

*Proof.* As $\Psi(f)$ has maximals, $\max \Psi(f) \neq \varnothing$. So, let $y \in \max \Psi(f)$. Therefore, $\{y\} \preceq_H f(y) \neq \varnothing$ and, thus, there is $y' \in f(y)$ such that $y \leq y'$. If $f$ H-monotone, then $f(y) \preceq_H f(y')$ and, thus, for $y' \in f(y)$ there is $y'' \in f(y')$ such that $y' \leq y''$. Therefore, $\{y'\} \preceq_H f(y')$ and, thus, $y' \in \Psi(f)$. But $y \in \max \Psi(f)$, so it cannot be $y < y'$. Therefore, $y = y' \in f(y)$, i.e. $y$ is a fixed-point of $f$. If $f$ is deflationary, by Proposition 3.7, $y$ is a fixed-point of $f$. Now, assume $x \in f(x)$. Therefore, $\{x\} \preceq_H f(x)$ and, thus, $x \in \Psi(f)$. But $y \in \max \Psi(f)$ so it cannot be $y < x$ and, thus, $y$ is a maximal fixed-point of $f$. Finally, consider $x = \bigvee \Psi(f)$. By hypothesis, $x \in \Psi(f)$ and $x$ is greatest element of $\Psi(f)$. Hence, we know that $x \in f(x)$. Let $y \in f(y)$. Hence $y \in \Psi(f)$, and, thus, $y \leq x$. As a consequence, $x$ is the greatest fixed-point of $f$. □

Proposition A.5. *Let $f \colon L \to 2^L$ be a multi-valued function. If $f$ is a $\bigvee$-preserving multi-valued function with $\Psi(f) \neq \varnothing$ then $\Psi(f)$ has maximals and, thus, maximal fixed-points.*

*Proof.* By hypothesis $\Psi(f) \neq \varnothing$. Let $(x_\alpha)_{\alpha \in I}$ be a increasing sequence of $x_\alpha \in \Psi(f)$ and let $\bar{x} = \bigvee_\alpha x_\alpha$. As $f$ is $\bigvee$-preserving, by definition $f(\bar{x}) = \{y \colon$ there is $(y_\alpha)_{\alpha \in I}$ *s.t.* $y_\alpha \in f(x_\alpha)$ and $y = \bigvee_\alpha y_\alpha\}$.

Now, for any $\alpha$, $x_\alpha \leq x_{\alpha+1}$, by Proposition 3.6 and, as $x_\alpha \in \Psi(f)$, $\{x_\alpha\} \preceq_H f(x_\alpha) \preceq_H f(x_{\alpha+1})$. Therefore, for any $x_\alpha$ there is $y_\alpha \in f(x_\alpha)$ and $y_{\alpha+1} \in f(x_{\alpha+1})$ such that $x_\alpha \leq y_\alpha \leq y_{\alpha+1}$.

Note that if $\alpha$ is a limit ordinal then, as $x_\beta \leq x_\alpha$ for all $\beta < \alpha$, it follows that $\{x_\beta\} \preceq_H f(x_\beta) \preceq_H f(x_\alpha)$ and, thus, $x_\beta \leq y_\beta \leq y_\alpha$ for all $\beta < \alpha$. Therefore, there is a increasing sequence $(y_\alpha)_{\alpha \in I}$ of elements $y_\alpha \in f(x_\alpha)$ such that $\bar{x} = \bigvee_\alpha x_\alpha \leq \bigvee_\alpha y_\alpha = \bar{y}$. By definition of $f(\bar{x})$, $\bar{y} \in f(\bar{x})$ and, thus, $\{\bar{x}\} \preceq_H f(\bar{x})$. Therefore $\bar{x} \in \Psi(f)$ and, thus, every increasing sequence has an upper bound in $\Psi(f)$. So, by Zorn's lemma, $\Psi(f)$ has maximals, which by Proposition 3.8 are also maximal fixed-points. □

Proposition A.6. *Let $f \colon L \to 2^L$ be a multi-valued function. If $f$ is H-monotone multi-valued function and for all $x \in L$, $f(x)$ has greatest element then $f$ has greatest fixed-point.*

*Proof.* As for all $x \in L$, $f(x)$ has greatest element, by definition $\bigvee f(x) \in f(x) \neq \varnothing$. Therefore, $\Psi(f) \neq \varnothing$ as $\{\bot\} \preceq_H f(\bot)$. Consider $a = \bigvee_{c \in \Psi(f)} c$. If $a \in \Psi(f)$ then by Proposition 3.8, $a$ is the greatest fixed-point of $f$. So, let us show that $a \in \Psi(f)$. For $c \in \Psi(f)$ there is a $x_c \in f(c)$ such that $c \leq x_c$. As $c \leq a$ and $f$ is H-monotone, $f(c) \preceq_H f(a)$ and, thus, for $x_c \in f(c)$ there is $y_c \in f(a)$ such that $c \leq x_c \leq y_c$. Since $f(a)$ has greatest element, there is $y \in f(a)$ such that $a = \bigwedge_{c \in \Psi(f)} c \leq \bigwedge_{c \in \Psi(f)} x_c \leq \bigwedge_{c \in \Psi(f)} y_c \leq y$. Hence, $\{a\} \preceq_H f(a)$, i.e. $a \in \Psi(f)$. □

Proposition A.7. *Let $f \colon L \to 2^L$ be a H-monotone, non-empty and $\vee$-closed multi-valued function. Then*

1. *$\Psi(f)$ is $\vee$-closed;*
2. *$f$ has a greatest fixed-point.*

*Proof.* Note that $\Psi(f) \neq \varnothing$ as $\{\bot\} \preceq_H f(\top) \neq \varnothing$.

Point 1. Consider a subset $S$ of $\Psi(f)$ and $a = \bigvee S$. Let us show that $a \in \Psi(f)$. We know that for each $c \in S$, $\{c\} \preceq_H f(c)$ holds, i.e. there is $x_c \in f(c)$ such that $c \leq x_c$. But, $f$ is H-monotone and, thus, from $c \leq a$, $\{c\} \preceq_H f(c) \preceq_H f(a)$ follows. That is, there is $y_c \in f(a)$ such that $c \leq x_c \leq y_c$. Let $y = \bigvee_{c \in S} y_c$. As $f$ is $\vee$-closed, $y \in f(a)$ follows. Therefore, $a = \bigvee_{c \in S} c \leq \bigvee_{c \in S} y_c = y$, $\{a\} \preceq_H f(a)$ and, thus, $a \in \Psi(f)$. Therefore, $\Psi(f)$ is $\vee$-closed.

Point 2. From point 1, $\Psi(f)$ has greatest element $a$ and, thus, by Proposition 3.8, $f$ has $a$ as greatest fixed-point. □

Proposition A.8. *For a multi-valued function $f$,*

1. *if $f$ is deflationary then each $\top$-orbit is decreasing;*
2. *each decreasing $\top$-orbit converges to a fixed-point of $f$ (if no fixed-point exists then there is no orbit);*
3. *if $f$ is H-monotone and deflationary then for any maximal fixed-point of $f$ there is a $\top$-orbit converging to it.*

*Proof.* Let $(x_\alpha)_{\alpha\in I}$ be an orbit of $f$. Recall that for ordinal $\alpha$ then $x_{\alpha+1} \in f(x_\alpha) \neq \varnothing$. As $f$ is deflationary, $f(x_\alpha) \preceq_H \{x_\alpha\}$. But, by definition of $\preceq_H$, for $x_{\alpha+1} \in f(x_\alpha)$, $x_{\alpha+1} \leq x_\alpha$. For a limit ordinal $\lambda$, $x_\lambda = \bigvee_{\alpha<\lambda} x_\alpha$, $\varnothing \neq f(x_\lambda) \preceq_H \{x_\lambda\}$ and, thus, there is $x_{\lambda+1} \in f(x_\lambda)$ such that $x_{\lambda+1} \leq x_\lambda$.

For the second point, as $(x_\alpha)_{\alpha\in I}$ is a decreasing sequence and $|I| > |L|$, by Proposition 2.1 there is an ordinal $\alpha$ such that $x_\alpha = x_{\alpha+1} \in f(x_\alpha)$. That is, $x_\alpha$ is a fixed-point of $f$.

Finally, for the third point, assume $\bar{x} \in f(\bar{x})$ is a maximal fixed-point of $f$. Now, let us show on (transfinite) induction on $\alpha$ that there is a decreasing orbit $(x_\alpha)_{\alpha\in I}$ of $f$ s.t. $\bar{x} \leq x_\alpha$ for all $\alpha$.

$\alpha = 0$. $\bar{x} \leq \top = x_0$.

$\alpha$ **successor ordinal.** By induction, $\bar{x} \leq x_\alpha$. As $f$ is H-monotone and deflationary, $f(\bar{x}) \preceq_H f(x_\alpha) \preceq_H \{x_\alpha\}$. But, $\bar{x} \in f(\bar{x})$, so we can choose $x_{\alpha+1} \in f(x_\alpha)$ s.t. $\bar{x} \leq x_{\alpha+1} \leq x_\alpha$.

$\alpha$ **limit ordinal.** By induction, $\bar{x} \leq x_\beta$ holds for all $\beta < \alpha$, which implies that $\bar{x} \leq \bigvee_{\beta<\alpha} x_\beta = x_\alpha$.

The sequence $(x_\alpha)_{\alpha\in I}$ is decreasing and, thus, by Proposition 2.1 there is an ordinal $\alpha$ such that $x_\alpha = x_{\alpha+1} \in f(x_\alpha)$. So, $x_\alpha$ is a fixed-point of $f$ with $\bar{x} \leq x_\alpha$. As $\bar{x}$ is maximal, $x_\alpha = \bar{x}$. □

Proposition A.9. *For $f\colon L \to 2^L$, $h(x) = x \otimes f(x)$ is deflationary. Furthermore, if $f$ is H-monotone, then*

1. *$h$ is H-monotone;*
2. *$x \in f(x)$ implies $x \in h(x)$;*
3. *$x \in h(x)$ implies $\{x\} \preceq_H f(x)$;*
4. *if $x$ is a maximal fixed point of $h$ then $x$ is a maximal fixed point of $f$.*
5. *if $x$ is a maximal fixed point of $f$ and $f$ is also deflationary then $x$ is a maximal fixed point of $h$.*

*Proof.* Consider $f$ and $h$. If $f(x) = \varnothing$ then $\varnothing = h(x) \preceq_H \{x\}$. Otherwise, for $y \in h(x)$, $y \leq x$. Therefore, $h(x) \preceq_H \{x\}$ and, thus, $h$ is deflationary. Now, suppose $f$ H-monotone. Point 1. Easy: $h$ is a combination of H-monotone functions. Point 2. If $x \in f(x)$ then by definition of $h$, $x = x \wedge x \in h(x)$. Point 3. If $x \in h(x)$ then for some $y \in f(x)$, $x = x \wedge y$. Therefore, $x \leq y$ and, thus, $\{x\} \preceq_H f(x)$. Point 4. Assume $x$ is a maximal fixed-point of $h$, i.e. $x \in h(x) = x \otimes f(x)$. Therefore, there is $y \in f(x)$ such that $x \leq y$. As $f$ is H-monotone, $f(x) \preceq_H f(y)$. That is, there is $z \in f(y)$ such that $y \leq z$ and, thus, $y = y \wedge z$. Therefore, $y \in h(y)$. As $x$ is maximal and $x \leq y$, $y = x$ follows and, thus, $x \in f(x)$. To prove that $x$ is a maximal fixed-point of $f$, assume there is $x \leq y$ such that $y \in f(y)$. By Point 2., $y \in h(y)$ and, thus, as $x$ is a maximal fixed-point of $h$, $y = x$ follows; Point 5. Assume $x$ is a maximal fixed-point of $f$. By Point 2. $x \in h(x)$. To prove that $x$ is a maximal fixed-point of $h$, assume there is $x \leq y$ such that $y \in h(y)$. Then by Point 3. $\{y\} \preceq_H f(y)$ and, thus, $y \in \Psi(f)$. By Proposition 3.7, $y \in f(y)$, and, thus, as $x$ is a maximal fixed-point of $f$, $y = x$ follows. $\square$