Multi-adjoint lattices from adjoint triples with involutive negation

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Abstract

We focus primarily on the use of involutive negations in adjoint triples and the satisfiability of the contraposition law. Instead of considering natural negations, such as $n(x) = x \to 0$, we consider an arbitrary involutive negation and an arbitrary adjoint triple. Then, we construct a multiadjoint lattice (an algebraic structure with several conjunctions and implications) with the help of two new adjoint triples defined from the original one and the involutive negation considered. Finally, we present several results that relate the different implications and conjunctions appearing in the mentioned multi-adjoint lattice in terms of the logical laws of contraposition, interchange and exportation.

Keywords: Fuzzy logic, Adjoint triple, Involutive Negation, Contraposition law, Interchange law, Exportation law, Modus Tollens.

1. Introduction

The notion of adjoint triple was introduced in [17] as a technical tool to increase flexibility of the language used both in fuzzy formal concept analysis and multi-adjoint logic programming by dismissing some of the requirements usually assumed on their underlying conjunctor and implicators. A number of algebraic structures are related to adjoint triples, for instance, implication triples, associatively tied implications, and biresiduated lattices; and a comparative study of expressivity and flexibility of adjoint triples was given in [5].

The consideration of different kinds of negations in fuzzy logic has been widely studied and investigated in the literature [9, 25]. In this respect, involutive negations (also called strong negations) are significant because of the satisfiability of the double negation law. However, considering involutive negations in fuzzy logic may conflict with some principles in logic, such as the contraposition law. For such a reason, the introduction of involutive negation operators in fuzzy logic still receives considerable attention in the recent years [3, 9, 24].

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This paper is related to the multi-adjoint framework for generalized logic programming, which proposes a relaxation of the syntax of fuzzy logic programming by allowing the simultaneous use of different implication symbols in a logic program. We consider the effect of introducing an involutive negation operator together with an adjoint triple and, in this sense, this work is closer in spirit to [4, 6] although, in our case, we are just interested in the logical and algebraic properties arising from its interaction.

The study of fuzzy frameworks on which certain logical laws are satisfied is an active research topic [1, 12, 23]. In this respect, we focus primarily on the contraposition law, but we will also study conditions under which other logical laws, such as the exchange law or the exportation law, are satisfied.

Depending on the particular generalization of connectives to fuzzy logic, many valid laws in classical logic are no longer valid [11, 21]. It is worth to take into account that most of the existing approaches to fuzzy logic are based on the notion of residuated lattice, therefore the relationships between connectives are subordinated to those imposed by the adjoint property (the core notion of residuated lattices).

As stated above, in this paper we focus on the double negation and the contraposition laws. The former is certainly modelled by means of involutive negations whereas, in the residuated lattice framework, the latter requires using the natural negation, which is not always involutive [9]. Some authors suggest working with Łukasiewicz connectives because both requirements above are satisfied, but real-world applications usually require connectives different from those ones. Our approach aims at developing a theoretical framework where it is possible to consider arbitrary residuated pairs, arbitrary involutive negations and still, being possible to apply "kind of" a contraposition law between the operators considered.

The underlying mathematical background of this paper is the multi-adjoint framework [20], an approach that has shown to be useful in several fields like Logic Programming [18], Formal Concept Analysis [13, 16] and Fuzzy Relation Equations [8]. The main difference between multi-adjoint lattices and residuated lattices is that, firstly, the former can contain several residuated pairs, i.e. several conjunctions and implications, and secondly, it allows to use non-commutative conjunctions, which implies the existence of two different residuated implications per each (non-commutative) conjunction.

Our approach contrasts with other studies about negations on multi-adjoint lattices, like [6], in that: given an adjoint triple, we construct other two adjoint triples provided an arbitrary involutive negation is available. The construction of the operators of these new adjoint triples is done in terms of a general version of the contraposition law, in which the negation need not be the natural (residuated) negation $n(x) = x \to 0$ and the implication used in the left-hand side of the equality need not be the same than that used in the right-hand side; note that the multi-adjoint framework naturally provides a number of the operators to be considered as candidates in the functional equation corresponding to the logical law, in this case, $x \to_1 y = n(y) \to_2 n(x)$.

The paper starts by showing some preliminaries in Section 2. Then, the main contribution is presented by defining firstly a multi-adjoint lattice to model the contraposition law in Section 3 and, subsequently, a sequence of results that relate the different implications appearing in the mentioned multi-adjoint lattice in Section 4. Finally, Section 5 shows the conclusions and future work.

2. Preliminaries

As stated in the introduction, our approach is based on the notion of adjoint triple, which generalizes the well-known notion of residuated pair widely used in fuzzy logic.

Definition 1 ([17]). *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be three posets. The mappings $\&\colon P_1 \times P_2 \to P_3$, $\searrow\colon P_2 \times P_3 \to P_1$, and $\nearrow\colon P_1 \times P_3 \to P_2$ form an* adjoint triple *among P_1, P_2 and P_3 whenever, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$, the following holds:*

$$x \leq_1 y \searrow z \quad \text{if and only if} \quad x \,\&\, y \leq_3 z \quad \text{if and only if} \quad y \leq_2 x \nearrow z \tag{1}$$

The main difference with respect to residuated pairs is that, in the case of adjoint triples, the operator used to model the conjunction ($\&$) is not necessarily commutative. This fact implies that we need two implications, denoted in Definition 1 by $\searrow$ and $\nearrow$, to model the adjoint property (1).

The following result recalls some properties of the operators in adjoint triples; particularly, $\&$ resembles conjunctions, and $\nwarrow$ and $\swarrow$ resemble implications.

Lemma 1. *If $(\&, \swarrow, \nwarrow)$ is an adjoint triple w.r.t. P_1, P_2, P_3, then*

1. *$\&$ is order-preserving on both arguments.*
2. *$\swarrow$, $\nwarrow$ are order-preserving on the first argument and order-reversing on the second argument.*

In this paper we consider a special case of adjoint triples, specifically those where the posets $(P_1, \leq_1), (P_2, \leq_2)$ and $(P_3, \leq_3)$ coincide and have the structure of bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$; where 0 and 1 denote the lowest and greatest element in L. In this case, operators in adjoint triples have some boundary properties.

Lemma 2. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined on a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$. Then, for all $x \in L$, the following holds:*

- $0 \,\&\, x = x \,\&\, 0 = 0$
- $x \searrow 1 = x \nearrow 1 = 1$
- $0 \searrow x = 0 \nearrow x = 1$

The notion of multi-adjoint lattice is given as follows.

Definition 2 ([17]). *Given a bounded lattice* $(L, \leq, \wedge, \vee, 0, 1)$*, a multi-adjoint lattice is a tuple*

$$(L, (\&_1, \swarrow_1, \nwarrow_1), (\&_2, \swarrow_2, \nwarrow_2), \dots, (\&_k, \swarrow_k, \nwarrow_k))$$

where $(\&_i, \swarrow_i, \nwarrow_i)$ *is an adjoint triple on* $(L, \leq)$ *for each* $i \in \{1, \dots, k\}$*.*

In this paper we consider multi-adjoint lattices enriched with an involutive negation. Let us recall that an involutive negation on a lattice $(L, \leq, \wedge, \vee, 0, 1)$ is an operator $n\colon L \to L$ such that $n(0) = 1$, $n(1) = 0$ and $n(n(x)) = x$ for all $x \in L$.

3. Defining n-multi-adjoint lattices

In this section, we focus on how to integrate a negation operator into an adjoint triple. The main difference with respect to the existing approaches [6] is that, instead of considering just the negation and the operator in one adjoint triple, we construct a multiadjoint lattice. The idea is to begin by building a new implication in terms of the contraposition law and, then, complete the construction of an additional adjoint triple; it is worth remarking that this idea can be implemented in two different (but related) ways, depending on the implication used in the contraposition law (either $\searrow$ or $\nearrow$).

We consider a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$ as the underlying structure to be assumed hereafter.

Definition 3. *Let* $(\&, \searrow, \nearrow)$ *be an adjoint triple on* L*, and let* n *be an involutive negation on* L*.*

- *Its* associated $(.)_n$-adjoint triple $(\&, \searrow, \nearrow)_n$ *is the triple* $(\&_n, \searrow_n, \nearrow_n)$ *given by the operators defined, for all* $x, y \in L$*, by:*

$$\begin{aligned} x \nearrow_n y &= n(x \,\&\, n(y)) \\ x \,\&_n\, y &= n(x \nearrow n(y)) \\ x \searrow_n y &= n(y) \searrow n(x)\,. \end{aligned}$$

- *Its* associated $(.)^n$-adjoint triple $(\&, \searrow, \nearrow)^n$ *is the triple* $(\&^n, \searrow^n, \nearrow^n)$ *given by the operators defined, for all* $x, y \in L$*, by:*

$$\begin{aligned} x \nearrow^n y &= n(y) \nearrow n(x) \\ x \,\&^n\, y &= n(y \searrow n(x)) \\ x \searrow^n y &= n(n(y) \,\&\, x)\,. \end{aligned}$$

Let us note that the operators in the associated $(.)_n$- and $(.)^n$-adjoint triples are defined by using very well known laws and constructions in logic:

- The implications $\searrow_n$ and $\nearrow^n$ are, respectively, the reciprocal implication wrt n of the implications $\searrow$ and $\nearrow$.

- The implications $\nearrow_n$ and $\searrow^n$ correspond to the *material implications* modulo De Morgan laws; note that the construction assumes tacitly the non-commutativity of the conjunction. These implications are also known as *s-implications* [2, 22].

- Finally, the conjunctions $\&_n$ and $\&^n$ are defined in the standard way from a logical system formed by a negation and an implication.

The following result shows that the term 'adjoint triple' in the previous definition is properly used.

Lemma 3. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple on L, and let n be an involutive negation on L. Then both $(\&_n, \searrow_n, \nearrow_n)$ and $(\&^n, \searrow^n, \nearrow^n)$ are adjoint triples as well.*

Proof: Let us begin by showing that $(\&^n, \searrow^n, \nearrow^n)$ satisfies the chain of equivalences in Equation (1). Given $x, y, z \in L$, we have

$$\begin{aligned}
x \leq y \searrow^n z &\iff x \leq n(z) \searrow n(y) \\
&\iff x \,\&\, n(z) \leq n(y) \\
&\overset{(*)}{\iff} y \leq n(x \,\&\, n(z)) \\
&\iff y \leq x \nearrow^n z
\end{aligned}$$

where step $(*)$ holds because n is both antitone and involutive. Let us prove the other equivalence. For all $x, y, z \in L$ we have

$$\begin{aligned}
x \leq y \searrow^n z &\iff x \leq n(z) \searrow n(y) \\
&\iff x \,\&\, n(z) \leq n(y) \\
&\iff n(z) \leq x \nearrow n(y) \\
&\overset{(*)}{\iff} n(x \nearrow n(y)) \leq z \\
&\iff x \,\&^n\, y \leq z
\end{aligned}$$

where step $(*)$ again holds because n is both antitone and involutive.

In order to prove that $(\&_n, \searrow_n, \nearrow_n)$ is an adjoint triple we proceed similarly. $\square$

The following result is a natural consequence of the definition.

Lemma 4. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L and let $n\colon L \to L$ be an involutive negation, then the following equalities hold for all $x, y \in L$:*

1. $x \searrow_n y = n(y) \searrow n(x)$;
2. $x \nearrow^n y = n(y) \nearrow n(x)$;
3. $x \searrow y = n(y) \searrow_n n(x)$;
4. $x \nearrow y = n(y) \nearrow^n n(x)$;
5. $x \searrow^n y = n(y) \nearrow_n n(x)$;
6. $x \nearrow_n y = n(y) \searrow^n n(x)$.

Proof: Items 1 and 2 are by definition; items 3 and 4 by n being involutive. For item 5 we have the following

$$x \searrow^n y = n(n(y) \,\&\, x) = n\big(n(y) \,\&\, nn(x)\big) = n(y) \nearrow_n n(x)$$

Item 6 follows similarly. $\square$

The previous result provides six possible solutions to the equation

$$I_1(x, y) = I_2\big(n(y), n(x)\big)$$

and all of them can be encapsulated within the following multi-adjoint lattice

$$\big(L, \leq, (\&, \searrow, \nearrow), (\&_n, \searrow_n, \nearrow_n), (\&^n, \searrow^n, \nearrow^n)\big)\,.$$

The following proposition states that the previous construction contains essentially all the information associated with the adjoint triple, in that the successive iterations of the $(.)_n$ and $(.)^n$ constructions can be all expressed in terms of those conjunctions and implications.

Theorem 1. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L with an involutive negation n. Then the following equalities hold:*

1. $(\&, \searrow, \nearrow)^{nn} = (\&, \searrow, \nearrow)$
2. $(\&, \searrow, \nearrow)^n{}_n = (\overline{\&_n}, \nearrow_n, \searrow_n)$
3. $(\&, \searrow, \nearrow)^n{}_n{}^n = (\overline{\&}, \nearrow, \searrow)$
4. $(\&, \searrow, \nearrow)^n{}_n{}^n{}_n = (\overline{\&^n}, \nearrow^n, \searrow^n)$
5. $(\&, \searrow, \nearrow)^n{}_n{}^n{}_n{}^n = (\&_n, \searrow_n, \nearrow_n)$
6. $(\&, \searrow, \nearrow)^n{}_n{}^n{}_n{}^n{}_n = (\&, \searrow, \nearrow)$
7. $(\&, \searrow, \nearrow)_{nn} = (\&, \searrow, \nearrow)$
8. $(\&, \searrow, \nearrow)_n{}^n = (\overline{\&^n}, \nearrow^n, \searrow^n)$
9. $(\&, \searrow, \nearrow)_n{}^n{}_n = (\overline{\&}, \nearrow, \searrow)$
10. $(\&, \searrow, \nearrow)_n{}^n{}_n{}^n = (\overline{\&_n}, \nearrow_n, \searrow_n)$
11. $(\&, \searrow, \nearrow)_n{}^n{}_n{}^n{}_n = (\&^n, \searrow^n, \nearrow^n)$
12. $(\&, \searrow, \nearrow)_n{}^n{}_n{}^n{}_n{}^n = (\&, \searrow, \nearrow)$

where the overlined operators denote the dual operation, namely $\overline{*}$ is defined as $x \,\overline{*}\, y = y * x$.

Proof:

1.

$$x \,\&^{nn}\, y = n\big(y \searrow^n n(x)\big) = nn\big(nn(x) \,\&\, y)\big) = x \,\&\, y$$
$$x \searrow^{nn} y = n\big(n(y) \,\&^n\, x\big) = nn(x \searrow nn(y)) = x \searrow y$$
$$x \nearrow^{nn} y = n(y) \nearrow^n n(x) = nn(x) \nearrow nn(y) = x \nearrow y$$

2.

$$x \,\&^n{}_n y = n\big(x \nearrow^n n(y)\big) = n\big(nn(y) \nearrow n(x))\big) = n(y \nearrow n(x)) = y \,\&_n x$$
$$x \searrow^n{}_n y = n(y) \searrow^n n(x) = n\big(nn(x) \,\&\, n(y)\big) = n\big(x \,\&\, n(y)\big) = x \nearrow_n y$$
$$x \nearrow^n{}_n y = n\big(x \,\&^n n(y)\big) = nn\big(n(y \searrow n(x)\big) = n(y) \searrow n(x) = x \searrow_n y$$

3.

$$x \,\&^n{}_n{}^n y = n\big(y \nearrow_n n(x)\big) = nn\big(y \,\&\, nn(x)\big) = y \,\&\, x$$
$$x \searrow^n{}_n{}^n y = n\big(n(y) \overline{\&}_n x\big) = n\big(x \,\&_n n(y)\big) = nn\big(x \nearrow nn(y)\big) = x \nearrow y$$
$$x \nearrow^n{}_n{}^n y = n(y) \searrow_n n(x) = nn(x) \searrow nn(y) = x \searrow y$$

The rest of the items are proved similarly. □

Remark: Note that the notation used for the resulting conjunctions and implications is always *relative to the initial adjoint triple*, and make no sense when applied to a single operator.

Definition 4. *Let* $(\&, \searrow, \nearrow)$ *be an adjoint triple defined in a lattice* L *and let* n *be an involutive negation in* L*. Then, the* n-multi-adjoint lattice of $(\&, \searrow, \nearrow)$ *is defined by*

$$(L, \leq, (\&, \searrow, \nearrow), (\&_n, \searrow_n, \nearrow_n), (\&^n, \searrow^n, \nearrow^n)) \ .$$

The following example shows the construction of the n-multiadjoint lattices in the particular cases of the well-known product and Gödel t-norms together with their residuated implication.

Example 1.

- *Consider the product t-norm in [0,1] together with its residuated implication:*

$$x \,\&\, y = x \cdot y \qquad\qquad x \to y = \begin{cases} 1 & \text{if } y \geq x \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

  Despite the fact that there is just one residuated implication, given the standard negation $n(x) = 1 - x$*, it is worth noting that a new adjoint triple (with two different implications) is generated.*

  The corresponding $(.)_n$*-adjoint triple is given by the operators:*

$$x \nearrow_n y = 1 - x + xy;$$
$$x \,\&_n\, y = \begin{cases} 0 & \text{if } 1 - y \geq x \\ \frac{x+y-1}{x} & \text{otherwise} \end{cases} \quad ;$$
$$x \searrow_n y = \begin{cases} 1 & \text{if } y \geq x \\ \frac{1-x}{1-y} & \text{otherwise} \end{cases}$$

  However, the commutativity of $\&$ *implies that the* $(.)^n$*-adjoint triple essentially coincides with the previous one; specifically,*

$$x \,\&^n\, y = y \,\&_n\, x \qquad x \searrow^n y = x \nearrow_n y \qquad x \nearrow^n y = x \searrow_n y \,.$$

- *Consider the Gödel t-norm in* $[0,1]$ *given and its residuated implication:*

$$x \mathbin{\&} y = \min\{x,y\} \qquad\qquad x \to y = \begin{cases} 1 & \text{if } y \geq x \\ y & \text{otherwise.} \end{cases}$$

Given the standard negation $n(x) = 1 - x$*, the corresponding* $(.)_n$*-adjoint triple is given by the operators:*

$$\begin{aligned} x \nearrow_n y &= \max\{y, 1-x\}; \\ x \mathbin{\&}_n y &= \begin{cases} 0 & \text{if } 1-y \geq x \\ y & \text{otherwise} \end{cases}; \\ x \searrow_n y &= \begin{cases} 1 & \text{if } y \geq x \\ 1-x & \text{otherwise} \end{cases} \end{aligned}$$

Once again, the commutativity of $\&$ *implies that*

$$x \mathbin{\&}^n y = y \mathbin{\&}_n x \qquad x \searrow^n y = x \nearrow_n y \qquad x \nearrow^n y = x \searrow_n y\,.$$

The case of the Łukasiewicz adjoint pair is treated separately, since together with the standard negation $n(x) = 1 - x$ does not generate new adjoint triples (this is just a straightforward computation). We illustrate this case by considering a negation different from the natural one.

Example 2. *Consider the Łukasiewicz t-norm in [0,1] and its residuated implication:*

$$x \mathbin{\&} y = \max\{0, x+y-1\} \qquad\qquad x \to y = \min\{1, 1+y-x\}$$

Given the involutive negation $n(x) = \sqrt{1-x^2}$*, the corresponding* $(.)_n$*-adjoint triple is given by the operators:*

$$\begin{aligned} x \nearrow_n y &= \min\left\{1, \sqrt{1 - \left(\sqrt{1-y^2} + x - 1\right)^2}\right\}; \\ x \mathbin{\&}_n y &= \max\left\{0, \sqrt{1 - \left(\sqrt{1-y^2} - x + 1\right)^2}\right\}; \\ x \searrow_n y &= \min\left\{1, \sqrt{1-x^2} - \sqrt{1-y^2} + 1\right\}. \end{aligned}$$

Once again, the commutativity of $\&$ *implies that*

$$x \mathbin{\&}^n y = y \mathbin{\&}_n x \qquad x \searrow^n y = x \nearrow_n y \qquad x \nearrow^n y = x \searrow_n y\,.$$

4. General properties of n-multi-adjoint lattices

In this section we show that n-multi-adjoint lattices satisfy further properties than the contraposition law. Most algebraic structures oriented to multi-valued logic (e.g., residuated lattice) are oriented to the satisfiability of certain equalities arisen from equivalences of classical logic; in residuated lattices and MV-algebras there are several books devoted to this issue (see [11, 22]). The main differences of such studies and the one based on multi-adjoint lattices is that, whereas in residuated lattices and MV-algebras there is a one-to-one correspondence between classical logic tautologies and equations (which may or may not hold), in multi-adjoint lattices each classical logic tautology may be interpreted as different equations. For example, the tautology $p \to (q \to p)$, that implies to study the equality $x \to (y \to x) = 1$ in residuated lattices, implies the study of 36 equalities in the n-multi-adjoint lattice, one per each combination (with repetition) of two elements in the set of implications in the framework (in this case 6).

This study begins with the description of the contraposition rule in this algebraic environment. Then, we continue with the interpretation of operators in the n-multi adjoint lattice as conjunctions and implications. We show that, under certain circumstances, the involutive negation n can be considered as a natural negation defined from an implication and falsity (i.e., 0) by following the standard definition $x \to 0$. Subsequently, we analyze the associativity of the operators $\&, \&^n$ and $\&_n$ and finally, the exportation and exchange principles.

4.1. L-fuzzy implications, L-fuzzy conjunctions and n-multi-adjoint lattices

The most general approach to the notion of implications and conjunctions in the L-fuzzy setting is, perhaps, that proposed in [10], in which the definition requires only to fulfil some boundary conditions and a natural monotonicity.

Definition 5. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a complete lattice.*

- *An operator $C\colon L \times L \to L$ is called L-fuzzy conjunction if the following properties hold:*
  - *C is monotonic in both arguments. for all $x, y, z, t \in L$.*
  - *$C(0,1) = C(1,0) = 0$.*
  - *$C(1,1) = 1$.*
- *An operator $I\colon L \times L \to L$ is called L-fuzzy implication if the following properties hold:*
  - *I is antitonic in the first argument and monotonic in the second one.*
  - *$I(1,1) = I(0,0) = 1$.*
  - *$I(1,0) = 0$.*

Let us note firstly that not every conjunction and implication in an adjoint triple is an L-fuzzy conjunction or, respectively, L-fuzzy implication, since the boundary conditions might not hold (see Lemmas 1 and 2). Conditions to ensure the satisfiability of the boundary conditions for conjunctions and implications in adjoint triples can be found in [5].

The following proposition shows that, if the conjunction and both implications in $(\&, \searrow, \nearrow)$ are an L-fuzzy conjunction and two L-fuzzy implications, respectively, then the other two conjunctions and four implications in the respective n-multi-adjoint lattice are indeed L-fuzzy conjunctions and L-fuzzy implications as well.

Proposition 1. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L with an involutive negation n, such that $\searrow$ and $\nearrow$ are L-fuzzy implications and $\&$ is an L-fuzzy conjunction. Then,*

- *The implications $\searrow^n, \nearrow^n, \searrow_n$ and $\nearrow_n$ in the associated n-multi-adjoint lattice are L-fuzzy implications as well.*
- *The conjunctions $\&_n$ and $\&^n$ in the associated n-multi-adjoint lattice are L-fuzzy conjunctions as well.*

Proof: Monotonicity and antitonicity conditions come from Lemma 1. Let us check the boundary conditions. For $\searrow_n$ we have:

$$\begin{aligned} 1 \searrow_n 1 &= n(1) \searrow n(1) = 0 \searrow 0 = 1 \\ 0 \searrow_n 0 &= n(0) \searrow n(0) = 1 \searrow 1 = 1 \\ 1 \searrow_n 0 &= n(0) \searrow n(1) = 1 \searrow 0 = 0 \end{aligned}$$

For $\searrow^n$ we have:

$$\begin{aligned} 1 \searrow^n 1 &= n(n(1) \,\&\, 1)) = n(0 \,\&\, 1)) = n(0) = 1 \\ 0 \searrow^n 0 &= n(n(0) \,\&\, 0)) = n(1 \,\&\, 0)) = n(0) = 1 \\ 1 \searrow^n 0 &= n(n(0) \,\&\, 1)) = n(1 \,\&\, 1)) = n(1) = 0 \end{aligned}$$

For $\&_n$ we have:

$$\begin{aligned} 1 \,\&_n\, 1 &= n(1 \nearrow n(1))) = n(1 \nearrow 0)) = n(0) = 1 \\ 0 \,\&_n\, 1 &= n(0 \nearrow n(1))) = n(0 \nearrow 0)) = n(1) = 0 \\ 1 \,\&_n\, 0 &= n(1 \nearrow n(0))) = n(1 \nearrow 1)) = n(1) = 0 \end{aligned}$$

The boundary conditions for $\nearrow^n$, for $\nearrow_n$, and for $\&^n$ are proved similarly. $\square$

4.2. Neutral elements in a n-multi-adjoint triple

One additional property usually required to fuzzy conjunctions is to have the element 1 as a neutral element. Contrariwise to the case of residuated lattice, in an adjoint triple the top element of L need not be the neutral element of $\&$. This section elaborates on the consequences obtained from the fact that 1 is (left or right) neutral element for $\&$.

Lemma 5. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple in a lattice L such that 1 is a left (resp. right) neutral element of $\&$ then, $1 \nearrow x = x$ (resp. $1 \searrow x = x$).*

Proof: The inequality $x \leq 1 \nearrow x$ comes from the following equivalence obtained by the adjoint property and that 1 is a left neutral element of $\&$:

$$1 \,\&\, x \leq x \iff x \leq 1 \nearrow x$$

The other inequality $1 \nearrow x \leq x$ is obtained similarly by:

$$1 \nearrow x \leq 1 \nearrow x \iff 1 \,\&(1 \nearrow x) \leq x \iff 1 \nearrow x \leq x$$

□

Either left of right neutrality of 1 implies that the implications $\searrow$ and, respectively, $\nearrow$ can be directly related to the lattice ordering.

Proposition 2 ([5]). *Let $(\&, \searrow, \nearrow)$ be an adjoint triple a lattice L such that 1 is a left (resp. right) neutral element of $\&$ then, for all $x, y \in L$ we have that*

$$x \searrow y = 1 \quad \big(\textit{resp.} \quad x \nearrow y = 1\big) \qquad \textit{iff} \qquad x \leq y$$

The following result shows that if 1 is a neutral element of $\&$ then, it is a left neutral element of $\&_n$ and a right neutral element of $\&^n$. As a consequence, both implications $\searrow_n$ and $\nearrow^n$ can be also directly related to the lattice ordering.

Proposition 3. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple in a lattice L with an involutive negation n, and such that 1 is a left (resp. right) neutral element of $\&$ then, 1 is a left (resp. right) neutral element of $\&_n$ (resp. $\&^n$). As a consequence, for all $x, y \in L$ we have that*

$$x \searrow_n y = 1 \quad \big(\textit{resp.} \quad x \nearrow^n y = 1\big) \qquad \textit{iff} \qquad x \leq y$$

Proof: Firstly, since 1 is a left neutral element of $\&$ then $1 \nearrow x = x$. Secondly, for all $x \in L$ we have:

$$1 \,\&_n\, x = n(1 \nearrow n(x)) = n(n(x)) = x$$

That is, 1 is a left neutral element of $\&_n$. Finally, the rest of the statement is a direct consequence of Proposition 3. □

In general, we cannot ensure that if 1 is the neutral element of $\&$ (both sides), then 1 is the neutral element of $\&_n$ and $\&^n$ as well, as the following example shows.

Example 3. *Consider the n-multiadjoint lattice for the product t-norm given in Example 1. Note that 1 is the neutral element of $\&$, however if $0 \neq x \neq 1$ we have $x \,\&_n\, 1 = 1 \neq x$.*

Despite the neutral element of $\&$ is not preserved by $\&^n$ and $\&_n$, we can prove that 1 is the only possible neutral element of $\&_n$ and $\&^n$.

Proposition 4. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in lattice L with an involutive negation n, and such that 1 is the neutral element of $\&$. If e is the right (resp. left) neutral element of $\&_n$ (resp. $\&^n$) then $e = 1$.*

Proof: Let us assume that e is a right neutral element of $\&_n$ then, for all $x \in L$, we have

$$x \&_n e = x \iff n(x \nearrow n(e)) = x \iff x \nearrow n(e) = n(x)$$

In particular, for $x = 1$ we have

$$1 \nearrow n(e) = n(e) = n(1)$$

where the first equality follows by Lemma 5. As a result, by n involutive, $e = 1$. $\square$

The following result shows that the operators $\&_n$ and $\&^n$ in the n-multi-adjoint lattice have zero divisors.

Proposition 5. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L with an involutive negation n, and such that 1 is the neutral element of $\&$. Given $x, y \in L$ then,*

$$x \&_n y = x \&^n y = 0 \qquad \iff \qquad y \leq n(x)\,.$$

Proof: Let us assume that $x \&_n y = 0$ for two element $x, y \in L$. Then, we have the following equivalences:

$$x \&_n y = 0 \iff n(x \nearrow n(y)) = 0 \iff x \nearrow n(y) = 1$$

Now, by Proposition 3, this is equivalent to $x \leq n(y)$, or to $y \leq n(x)$. $\square$

Note that as a consequence of the previous proposition, we have

$$x \&_n n(x) = x \&^n n(x) = 0$$

for all $x \in L$, which establishes an interesting relationship between $\&_n$, $\&^n$ and the excluded middle law.

4.3. The negation n in the n-multi-adjoint triple

Negations in adjoint triples were defined in [6] in the form $n(x) = x \nearrow a$ and $n(x) = x \searrow a$ for certain $a \in L$. The following result shows that, if 1 is a neutral element of $\&$, then negations of the type $x \searrow a$ (or $x \nearrow a$) are not involutive unless $a = 0$.

Lemma 6. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple on a lattice L such that 1 is the neutral element of $\&$ and consider $a \in L$. If the operator $n(x) = x \nearrow a$ (resp. $n(x) = x \searrow a$) is involutive, then $a = 0$.*

Proof: Since n is involutive, it follows that it is bijective; moreover, by definition, we have that n is decreasing. This leads to the fact that $n(1) = 0$. Now, by Lemma 5 we have that $n(1) = 1 \nearrow a = a$ and, $a = 0$. $\square$

The following result shows that, under certain circumstances, the negation n used to define the n-multiadjoint lattice turns out to be a natural residuated negation with respect to its associated implications $\nearrow^n$, $\searrow^n$, $\nearrow_n$, $\searrow_n$.

Proposition 6. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L such that 1 is the right (resp. left) neutral element of $\&$, and let n be an involutive negation. Then, for all $x \in L$,*

$$x \searrow_n 0 = x \nearrow^n 0 = n(x) \qquad (\textit{resp. } x \nearrow^n 0 = x \searrow^n 0 = n(x)) .$$

Proof: The proof just requires a checking. For all $x \in L$ we have:

$$x \searrow_n 0 = n(0) \searrow n(x) = 1 \searrow n(x) = n(x).$$

On the other hand, for all $x \in L$ we have:

$$x \nearrow_n 0 = n\big(x \,\&\, n(0)\big) = n\big(x \,\&\, 1\big) = n(x).$$

The equalities for $x \nearrow^n 0$ and $x \searrow^n 0$ can be proved similarly. $\square$

The following result states how to express the different implications in terms of negation and conjunction, therefore, leading to the notion of negation normal form.

Proposition 7. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L and let n be an involutive negation in L, then the following equalities hold for all $x, y \in L$:*

1. $n(x \nearrow y) = x \,\&_n\, n(y)$
2. $n(x \nearrow^n y) = n(y) \,\&_n\, x$
3. $n(x \nearrow_n y) = x \,\&\, n(y)$
4. $n(x \searrow y) = n(y) \,\&^n\, x$
5. $n(x \searrow_n y) = x \,\&^n\, n(y)$
6. $n(x \searrow^n y) = n(y) \,\&\, x$

Proof:

1. $x \,\&_n\, n(y)) = n\big(x \nearrow nn(y)\big) = n(x \nearrow y)$
2. $n(y) \,\&_n\, x = n\big(n(y) \nearrow n(x)\big) = n(x \nearrow^n y)$
3. $x \,\&\, n(y) = nn\big(x \,\&\, n(y)\big) = n(x \nearrow_n y)$

The rest of items are proved similarly. $\square$

4.4. Associativity, exchange and exportation principles

The validity of the principles of exchange $p \to (q \to r) = q \to (p \to r)$ and exportation $p \to (q \to r) = (p \,\&\, q) \to r$ have already been studied in the more general framework of fuzzy logic. Note firstly that in adjoint triples, due to the existence of two implications, both principles can be extended in more than one single way. The following result relates the associativity of $\&$ in an adjoint triple $(\&, \searrow, \nearrow)$ to the satisfiability of some exchange and exportation principles constructed with $\searrow$ and $\nearrow$, in line with other existing studies [7].

Theorem 2. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple on a lattice L. Then, the following items are equivalent:*

i) *$\&$ is associative.*
ii) *$x \nearrow (y \searrow z) = y \searrow (x \nearrow z)$ for all $x, y, z \in L$.*
iii) *$x \nearrow (y \nearrow z) = (y \,\&\, x) \nearrow z$ for all $x, y, z \in L$.*
iv) *$x \searrow (y \searrow z) = (x \,\&\, y) \searrow z$ for all $x, y, z \in L$.*

Proof: i) $\Rightarrow$ ii) Let us assume that $\&$ is associative then, for all $x, y, z \in L$ we have the following chain of equivalent inequalities:

$$\begin{aligned}
x \nearrow (y \searrow z) &\leq x \nearrow (y \searrow z)\\
x \,\&\, \big(x \nearrow (y \searrow z)\big) &\leq (y \searrow z)\\
\big(x \,\&\, \big(x \nearrow (y \searrow z)\big)\big) \,\&\, y &\leq z\\
x \,\&\, \big(\big(x \nearrow (y \searrow z)\big) \,\&\, y\big) &\leq z\\
\big(x \nearrow (y \searrow z)\big) \,\&\, y &\leq (x \nearrow z)\\
x \nearrow (y \searrow z) &\leq y \searrow (x \nearrow z).
\end{aligned}$$

Since the first inequality is satisfied for all $x, y, z \in L$, we have the inequality $x \nearrow (y \searrow z) \leq y \searrow (x \nearrow z)$. To prove $y \searrow (x \nearrow z) \leq x \nearrow (y \searrow z)$ we can proceed similarly.

ii) $\Rightarrow$ i) Let us assume that $x \nearrow (y \searrow z) = y \searrow (x \nearrow z)$ for all $x, y, z \in L$. Then we have the following chain of equivalent inequalities for $x, y, z \in L$:

$$\begin{aligned}
x \,\&(y \,\&\, z) &\leq x \,\&(y \,\&\, z)\\
y \,\&\, z &\leq x \nearrow \big(x \,\&(y \,\&\, z)\big)\\
y &\leq z \searrow \big(x \nearrow \big(x \,\&(y \,\&\, z)\big)\big)\\
y &\leq x \nearrow \big(z \searrow \big(x \,\&(y \,\&\, z)\big)\big)\\
x \,\&\, y &\leq z \searrow \big(x \,\&(y \,\&\, z)\big)\\
(x \,\&\, y) \,\&\, z &\leq x \,\&(y \,\&\, z).
\end{aligned}$$

Since the first inequality is satisfied for all $x, y, z \in L$, we have the inequality $(x \,\&\, y) \,\&\, z \leq x \,\&(y \,\&\, z)$. To prove $x \,\&(y \,\&\, z) \leq (x \,\&\, y) \,\&\, z$ we can proceed similarly.

$i) \Rightarrow iii)$ Let us assume that & is associative then, for all $x, y, z \in L$ we have the following chain of equivalent inequalities:

$$\begin{aligned}
x \nearrow (y \nearrow z) &\le x \nearrow (y \nearrow z)\\
x \,\&\, \big(x \nearrow (y \nearrow z)\big) &\le y \nearrow z\\
y \,\&\, \big(x \,\&\, \big(x \nearrow (y \nearrow z)\big)\big) &\le z\\
(y \,\&\, x) \,\&\, \big(x \nearrow (y \nearrow z)\big) &\le z\\
x \nearrow (y \nearrow z) &\le (y \,\&\, x) \nearrow z.
\end{aligned}$$

Since the first inequality is satisfied for all $x, y, z \in L$, we have the inequality $x \nearrow (y \nearrow z) \le (y \,\&\, x) \nearrow z$. To prove $(y \,\&\, x) \nearrow z \le x \nearrow (y \nearrow z)$ we can proceed similarly.

$iii) \Rightarrow i)$ Let us assume that $x \nearrow (y \nearrow z) = (y \,\&\, x) \nearrow z$ for all $x, y, z \in L$. Then we have the following chain of equivalent inequalities for $x, y, z \in L$:

$$\begin{aligned}
x \,\&(y \,\&\, z) &\le x \,\&(y \,\&\, z)\\
y \,\&\, z &\le x \nearrow \big(x \,\&(y \,\&\, z)\big)\\
z &\le y \nearrow \big(x \nearrow \big(x \,\&(y \,\&\, z)\big)\big)\\
z &\le (x \,\&\, y) \nearrow \big(x \,\&(y \,\&\, z)\big)\\
(x \,\&\, y) \,\&\, z &\le x \,\&(y \,\&\, z).
\end{aligned}$$

Since the first inequality is satisfied for all $x, y, z \in L$, we have the inequality $(x \,\&\, y) \,\&\, z \le x \,\&(y \,\&\, z)$. To prove $x \,\&(y \,\&\, z) \le (x \,\&\, y) \,\&\, z$ we can proceed similarly.

$i) \Rightarrow iv)$ similar to $i) \Rightarrow iii)$

$iv) \Rightarrow i)$ similar to $iii) \Rightarrow i)$ □

As in our n-multi-adjoint lattices we have three adjoint triples, the previous result leads to the following:

- If $\&^n$ is associative, then

- $x \nearrow^n (y \nearrow^n z) = (y \,\&^n\, x) \nearrow^n z$
- $x \searrow^n (y \searrow^n z) = (x \,\&^n\, y) \searrow^n z$
- $x \nearrow^n (y \searrow^n z) = y \searrow^n (x \nearrow^n z)$

- If $\&_n$ is associative, then

- $x \nearrow_n (y \nearrow_n z) = (y \,\&_n\, x) \nearrow_n z$
- $x \searrow_n (y \searrow_n z) = (x \,\&_n\, y) \searrow_n z$
- $x \nearrow_n (y \searrow_n z) = y \searrow_n (x \nearrow_n z)$

Sadly enough, requiring that & is associative is not sufficient to guarantee that $\&_n$ and $\&^n$ are associative as well, as shown in the following example.

Example 4. *Let us consider the* $(.)_n$*-adjoint triple associated to the product t-norm given in Example 1. Then, & is clearly associative but* $\&_n$ *is not, since:*

$$0.5 \,\&_n (1 \,\&_n\, 0.5) = 0.5 \,\&_n\, 0.5 = 0$$

but

$$(0.5 \,\&_n\, 1) \,\&_n\, 0.5 = 1 \,\&_n\, 0.5 = 0.5$$

Despite that associativity of $\&_n$ and $\&_n$ does not hold in general, there are some relationships between $\&$, $\&_n$ and $\&_n$ that, somehow, resemble associativity.

Proposition 8. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L and let n be an involutive negation in L. If $\&$ is associative, then the following equalities hold for all $x, y, z \in L$:*

1. $x \&_n (y \&_n z) = (y \& x) \&_n z$,
2. $(x \&^n y) \&^n z = x \&^n (z \& y)$,
3. $x \&_n (y \&^n z) = (x \&_n y) \&^n z$.

Proof: The first and second equalities are proved similarly. Let us prove the first equality, let $x, y, z \in L$, then

$$\begin{aligned} x \&_n (y \&_n z) &= x \&_n \big(n(y \nearrow n(z))\big) = n\big(x \nearrow nn\big(y \nearrow n(z)\big)\big) \\ &= n\big(x \nearrow (y \nearrow n(z))\big) \overset{(*)}{=} n\big((y \& x) \nearrow n(z)\big) = (y \& x) \&_n z \end{aligned}$$

Equality (*) uses Theorem 2 in the application of the exportation principle.

For the third equality let us consider $x, y, z \in L$, then

$$\begin{aligned} x \&_n (y \&^n z) &= x \&_n \big(n(z \searrow n(y))\big) = n\big(x \nearrow nn\big(z \searrow n(y)\big)\big) \\ &= n\big(x \nearrow (z \searrow n(y))\big) = n\big(z \searrow (x \nearrow n(y))\big) = n\big(z \searrow n(x \&_n y)\big) \\ &= (x \&_n y) \&^n z \end{aligned}$$

□

The exchange property can be also defined for conjunctions [7] which, as in the case of the exchange property for implications, resembles a certain type of commutativity.

Definition 6. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple on a lattice L. We say that $\&$ satisfies the left (resp. right) exchange principle if the following equality holds for all $x, y, z \in L$:*

$$x \& (y \& z) = y \& (x \& z) \qquad \Big(\textit{resp. } (x \& y) \& z = (x \& z) \& y\Big)$$

The exchange property for conjunctions can be characterized in terms of the exportation and exchange property for implications, as in Theorem 2.

Theorem 3. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple on a lattice L. The following items are equivalent:*

i) *$\&$ satisfies the left exchange principle*
ii) *$x \nearrow (y \nearrow z) = y \nearrow (x \nearrow z)$ for all $x, y, z \in L$.*
iii) *$x \searrow (y \nearrow z) = (y \& x) \searrow z$ for all $x, y, z \in L$.*

Proof: $i) \Rightarrow ii)$ Let us assume that & satisfies the left exchange principle. Then, we have the following chain of equivalent inequalities for all $x, y, z \in L$.

$$\begin{aligned}
x \nearrow (y \nearrow z) &\leq x \nearrow (y \nearrow z) \\
x \,\&\, \big(x \nearrow (y \nearrow z)\big) &\leq (y \nearrow z) \\
y \,\&\, \big(x \,\&\, \big(x \nearrow (y \nearrow z)\big)\big) &\leq z \\
x \,\&\, \big(y \,\&\, \big(x \nearrow (y \nearrow z)\big)\big) &\leq z \\
y \,\&\, \big(x \nearrow (y \nearrow z)\big) &\leq (x \nearrow z) \\
x \nearrow (y \nearrow z) &\leq y \nearrow (x \nearrow z).
\end{aligned}$$

Since the first inequality is satisfied for all $x, y, z \in L$, we have the inequality $x \nearrow (y \nearrow z) \leq y \nearrow (x \nearrow z)$.

$ii) \Rightarrow i)$ Let us assume now that $x \nearrow (y \nearrow z) = y \nearrow (x \nearrow z)$ for all $x, y, z \in L$ and let us show that & satisfies the left exchange principle. Consider the following chain of equivalent inequalities:

$$\begin{aligned}
x \,\&(y \,\&\, z) &\leq x \,\&(y \,\&\, z) \\
y \,\&\, z &\leq x \nearrow \big(x \,\&(y \,\&\, z)\big) \\
z &\leq y \nearrow \big(x \nearrow \big(x \,\&(y \,\&\, z)\big)\big) \\
z &\leq x \nearrow \big(y \nearrow \big(x \,\&(y \,\&\, z)\big)\big) \\
x \,\&\, z &\leq y \nearrow \big(x \,\&(y \,\&\, z)\big) \\
y \,\&(x \,\&\, z) &\leq x \,\&(y \,\&\, z).
\end{aligned}$$

Since the first inequality is satisfied for all $x, y, z \in L$, we have the inequality $y \,\&(x \,\&\, z) \leq x \,\&(y \,\&\, z)$.

$i) \Leftrightarrow iii)$ is similar to the previous ones. □

Theorem 4. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple on a lattice L. The following items are equivalent:*

i) *& satisfies the right exchange principle*
ii) $x \searrow (y \searrow z) = y \searrow (x \searrow z)$ *for all* $x, y, z \in L$.
iii) $x \nearrow (y \searrow z) = (x \,\&\, y) \nearrow z$ *for all* $x, y, z \in L$.

Proof: The proof is similar to the proof of Theorem 3. □

It is worth noting that the satisfiability of both exchange principles of & is not enough to guarantee both exchange principles of $\&_n$ and $\&^n$.

Example 5. *Let us consider again the $(.)_n$-adjoint triple associated to the product t-norm (see Example 1). The operator & satisfies obviously both exchange principles, however $\&_n$ does not satisfy the right exchange principle since:*

$$(1 \,\&_n\, 0.8) \,\&_n\, 0.4 = 0.8 \,\&_n\, 0.4 = 0.25$$

$$(1 \,\&_n\, 0.4) \,\&_n\, 0.8 = 0.4 \,\&_n\, 0.8 = 0.75$$

Despite of the previous example, there is a strong relationship between the exchange principles of $\&, \&_n$ and $\&^n$.

Proposition 9. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple on a lattice L, and let n be an involutive negation on L. Then $\&$ satisfies the left (resp. right) exchange principle if and only if $\&_n$ satisfies the left exchange principle ($\&^n$ satisfies the right exchange principle).*

Proof: Consider $x, y, z \in L$.

Assume the left exchange principle holds for $\&$, then:

$$\begin{aligned} x \,\&_n (y \,\&_n z) &= n\big(x \nearrow n(y \,\&_n z)\big) = n\big(x \nearrow (y \nearrow n(z))\big) \\ &\overset{(*)}{=} n\big(y \nearrow (x \nearrow n(z))\big) = y \,\&_n (x \,\&_n z). \end{aligned}$$

where the equality $(*)$ is a consequence of Theorem 3.

Assume now the left exchange principle for $\&_n$, then:

$$\begin{aligned} x \,\&(y \,\& z) &= nn\big(x \,\& nn(y \,\& nn(z))\big) = n\big(x \nearrow_n n(y \,\& nn(z))\big) = \\ &= n\big(x \nearrow_n (y \nearrow_n n(z))\big) \overset{(*)}{=} n\big(y \nearrow_n (x \nearrow_n n(z))\big) = y \,\&(x \,\& z). \end{aligned}$$

where the equality $(*)$ is again a consequence of Theorem 3. □

The exchange principle for $\&_n$ and $\&^n$ are also strongly linked together, in this case in left/right instances.

Proposition 10. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple on a lattice L, and let n be an involutive negation on n, then $\&^n$ satisfies the left exchange principle if and only if $\&_n$ satisfies the right exchange principle.*

Proof: Consider $x, y, z \in L$, and assume that $\&^n$ satisfies the left exchange principle. Then:

$$\begin{aligned} (x \,\&_n y) \,\&_n z &= n(x \nearrow n(y)) \,\&_n z = n\big(n(x \nearrow n(y)) \nearrow n(z)\big) \\ &= n\big(z \nearrow^n (x \nearrow n(y))\big) = n\big(z \nearrow^n (y \nearrow^n n(x))\big) \\ &\overset{(*)}{=} n\big(y \nearrow^n (z \nearrow^n n(x))\big) = (x \,\&_n z) \,\&_n y \end{aligned}$$

where the equality $(*)$ is a consequence of Theorem 3. The converse is similar. □

Under the assumption of a conjunction $\&$ satisfying the left (or right) exchange principle, we have also the following property.

Proposition 11. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple on a lattice L and let n be an involutive negation on L. Consider $x, y, z \in L$, then*

1. *If $\&$ satisfies the right exchange principle, then $x \,\&_n (y \,\&^n z) = (x \,\& z) \,\&_n y$.*
2. *If $\&$ satisfies the left exchange principle, then $(x \,\&_n y) \,\&^n z = y \,\&^n (x \,\& z)$.*

Proof: Consider $x, y, z \in L$, then

1. Assuming that the right exchange principle holds for $\&$ we have that:

$$x \,\&_n (y \,\&^n z) = n(x \nearrow n(y \,\&^n z)) = n\big(x \nearrow (z \searrow n(y))\big) = \stackrel{(*)}{=} n\big((x \,\&\, z) \nearrow n(y))\big) = (x \,\&\, z) \,\&_n y.$$

where the equality $(*)$ is a consequence of Theorem 4.

2. Assuming that the left exchange principle holds for $\&$ we have that:

$$(x \,\&_n y) \,\&^n z = n(z \searrow n(x \,\&_n y)) = n\big(z \searrow (x \nearrow n(y))\big) = \stackrel{(*)}{=} n\big((x \,\&\, z) \searrow n(y))\big) = y \,\&^n (x \,\&\, z).$$

where the equality $(*)$ is a consequence of Theorem 3. □

Note that the exportation principle can be instantiated in many different forms by combining conjunctions and implications from different adjoint triples. The last results determine a series of such instances in an n-multi-adjoint lattice that hold for the case the initial conjunction $\&$ is either associative or satisfies some exchange principles.

Proposition 12. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L and let n be an involutive negation in L. Then, The following items are equivalent:*

i) $\&$ *is associative,*
ii) $x \nearrow_n (y \nearrow_n z) = (x \,\&\, y) \nearrow_n z$ *for all* $x, y, z \in L$,
iii) $x \searrow^n (y \searrow^n z) = (x \,\&\, y) \searrow^n z$ *for all* $x, y, z \in L$,
iv) $x \nearrow (y \searrow_n z) = (x \,\&_n y) \searrow_n z$ *for all* $x, y, z \in L$,
v) $x \searrow (y \nearrow^n z) = (y \,\&^n x) \nearrow^n z$ *for all* $x, y, z \in L$.

Proof: Consider $x, y, z \in L$, then:

i) ⇒ ii)
$$\begin{aligned} x \nearrow_n (y \nearrow_n z) &= n\big(x \,\&\, n(y \nearrow_n z)\big) = n\big(x \,\&(y \,\&\, n(z))\big) \\ &= n\big((x \,\&\, y) \,\&\, n(z)\big) = (x \,\&\, y) \nearrow_n z. \end{aligned}$$

ii) ⇒ i)
$$\begin{aligned} x \,\&(y \,\&\, z) &= x \,\&\, n(y \nearrow_n n(z)) = n\big(x \nearrow_n (y \nearrow_n n(z))\big) \\ &= n\big((x \,\&\, y) \nearrow_n n(z)\big) = (x \,\&\, y) \,\&\, z. \end{aligned}$$

i) ⇒ iii)
$$\begin{aligned} x \nearrow (y \searrow_n z) &= x \nearrow (n(z) \searrow n(y)) \stackrel{(*)}{=} n(z) \searrow (x \nearrow n(y)) \\ &= n(z) \searrow n(x \,\&_n y) = (x \,\&_n y) \searrow_n z \end{aligned}$$

where the equality $(*)$ is a consequence of Theorem 2.

The other equalities are obtained similarly. □

Proposition 13. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L, and let n be an involutive negation on L. The following items are equivalent:*

i) $\&$ *satisfies the left exchange principle.*

ii) $x \nearrow (y \nearrow^n z) = (x \&_n y) \nearrow^n z$, *for all* $x, y, z \in L$.

iii) $x \searrow_n (y \nearrow_n z) = (y \&_n x) \searrow_n z$, *for all* $x, y, z \in L$.

iv) $x \searrow_n (y \searrow^n z) = y \searrow (x \nearrow^n z)$, *for all* $x, y, z \in L$.

Proof: Consider $x, y, z \in L$. The result comes from the characterization given by Theorem 3 and the following chains of equalities:

i) $\Leftrightarrow$ ii)

$$\begin{aligned} x \nearrow (y \nearrow^n z) &= x \nearrow \big(n(z) \nearrow n(y)\big) = n(z) \nearrow \big(x \nearrow n(y)\big) \\ &= n\big(x \nearrow n(y)\big) \nearrow^n z = (x \&_n y) \nearrow^n z. \end{aligned}$$

and

$$\begin{aligned} x \nearrow (y \nearrow z) &= x \nearrow \big(n(z) \nearrow^n n(y)\big) = \big(x \&_n n(z)\big) \nearrow^n n(y) \\ &= n\big(x \nearrow z\big) \nearrow^n n(y) = y \nearrow (x \nearrow z) \end{aligned}$$

i) $\Leftrightarrow$ iii)

$$\begin{aligned} x \searrow_n (y \nearrow_n z) &= n\big(y \nearrow_n z\big) \searrow n(x) = (y \& n(z)) \searrow n(x) \\ &= n(z) \searrow (y \nearrow n(x)) = n(z) \searrow n(y \&_n x) \\ &= (y \&_n x) \searrow_n z. \end{aligned}$$

and

$$\begin{aligned} (y \& x) \searrow z &= n\big(y \nearrow_n n(x)\big) \searrow z = n(z) \searrow_n (y \nearrow_n n(x)) \\ &= (y \&_n n(z)) \searrow_n n(x) = x \searrow n(y \&_n n(z)) \\ &= x \searrow (y \nearrow z) \end{aligned}$$

i) $\Leftrightarrow$ iv)

$$\begin{aligned} x \searrow_n (y \searrow^n z) &= n\big(y \searrow^n z\big) \searrow n(x) = (n(z) \& y) \searrow n(x) \\ &= y \searrow (n(z) \nearrow n(x)) = y \searrow (x \nearrow^n z). \end{aligned}$$

and

$$\begin{aligned} (y \& x) \searrow z &= n\big(x \searrow^n n(y)\big) \searrow z = n(z) \searrow_n (x \searrow^n n(y)) \\ &= x \searrow (n(z) \nearrow^n n(y)) = x \searrow (y \nearrow z). \end{aligned}$$

$\square$

Proposition 14. *Let* $(\&, \searrow, \nearrow)$ *be an adjoint triple defined in a lattice* L, *and let* n *be an involutive negation on* L. *Consider* $x, y, z \in L$, *the following items are equivalent:*

i) $\&$ *satisfies the right exchange principle then,*

ii) $x \searrow (y \searrow_n z) = (y \&^n x) \searrow_n z$,

iii) $x \nearrow^n (y \searrow^n z) = (x \,\&^n\, y) \nearrow^n z$,
iv) $x \nearrow^n (y \nearrow_n z) = y \nearrow (x \searrow_n z)$.

Proof: Similar to the proof of proposition 14 but by considering the characterization given in Theorem 4.

□

Theorem 1, together with the three propositions above, allow to obtain straightforward characterizations for the cases in which $\&_n$ or $\&^n$ are either associative or satisfy the left/right exchange principle.

Corollary 1. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L, and let n be an involutive negation on L. Consider $x, y, z \in L$, then*

i) *The following items are equivalent:*
   - $\&_n$ *is associative,*
   - $x \nearrow (y \nearrow z) = (x \,\&_n\, y) \nearrow z$
   - $x \nearrow^n (y \nearrow^n z) = (x \,\&_n\, y) \nearrow^n z$
   - $x \nearrow_n (y \searrow z) = (x \,\&\, y) \searrow z$
   - $x \searrow_n (y \searrow^n z) = (x \,\&^n\, y) \searrow^n z$

ii) *The following items are equivalent:*
   - $\&^n$ *is associative,*
   - $x \searrow_n (y \searrow_n z) = (x \,\&^n\, y) \searrow_n z$
   - $x \searrow (y \searrow z) = (x \,\&^n\, y) \searrow z$
   - $x \nearrow^n (y \nearrow_n z) = (y \,\&_n\, x) \nearrow_n z$
   - $x \searrow^n (y \nearrow z) = (y \,\&\, x) \nearrow z$

Proof: By Proposition 13 and Theorem 1. □

Corollary 2. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L, and let n be an involutive negation on L. Consider $x, y, z \in L$, then*

i) *The following items are equivalent:*
   - $\&_n$ *satisfies the left exchange principle*
   - $x \nearrow_n (y \searrow^n z) = (x \,\&\, y) \searrow^n z$,
   - $x \searrow (y \nearrow z) = (y \,\&\, x) \searrow z$,
   - $x \searrow (y \nearrow^n z) = y \searrow_n (x \searrow^n z)$.

ii) *The following items are equivalent:*
   - $\&^n$ *satisfies the left exchange principle,*
   - $x \nearrow^n (y \nearrow z) = (y \,\&_n\, x) \nearrow z$,
   - $x \nearrow_n (y \searrow_n z) = (x \,\&_n\, y) \nearrow_n z$,

- $x \nearrow_n (y \searrow z) = y \searrow^n (x \nearrow z)$.

Proof: By Proposition 14 and Theorem 1. $\square$

Corollary 3. *Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L, and let n be an involutive negation on L. Consider $x, y, z \in L$, then*

i) *The following items are equivalent:*
   - $\&_n$ *satisfies the right exchange principle,*
   - $x \searrow_n (y \searrow z) = (x \&^n y) \searrow z$,
   - $x \searrow^n (y \nearrow^n z) = (y \&^n x) \searrow^n z$,
   - $x \searrow^n (y \nearrow z) = y \nearrow_n (x \searrow z)$.

ii) *The following items are equivalent:*
   - $\&^n$ *satisfies the right exchange principle,*
   - $x \searrow^n (y \nearrow_n z) = (y \& x) \nearrow_n z$,
   - $x \nearrow (y \searrow z) = (x \& y) \nearrow z$,
   - $x \nearrow (y \searrow_n z) = y \nearrow^n (x \nearrow_n z)$.

Proof: By Proposition 15 and Theorem 1. $\square$

5. Conclusions and future work

Given an adjoint triple $(\&, \searrow, \nearrow)$ on a bounded lattice L together with an involutive negation n on L, we have shown how to build a multiadjoint lattice, the n-multiadjoint lattice, which consists of three adjoint triples. The underlying idea is considering an algebraic structure where both contraposition laws (note we have two implications, $\nearrow$ and $\searrow$, in adjoint triples) and the double negation law are satisfied. When working with a multiadjoint lattice, the contraposition law can be interpreted in a broader sense, since there are several implications to be used. Specifically, the contraposition law for a negation n and an implication $\nearrow$ can be identified with the existence of an implication, say $\nearrow^n$, possibly different from $\nearrow$ and $\searrow$, such that $x \nearrow y = n(y) \nearrow^n n(x)$ for all $x, y \in L$; note that $\nearrow^n$ always exists for any involutive negation n and implication $\nearrow$ so both, the contraposition law and the double negation law are satisfied at once. Following the previous idea, given an adjoint triple $(\&, \searrow, \nearrow)$ on a lattice L and an involutive negation n on L, we have defined two new adjoint triples $(\&^n, \searrow^n, \nearrow^n)$ and $(\&_n, \searrow_n, \nearrow_n)$ on L, leading to the n-multiadjoint lattice

$$\big(L, \leq, (\&, \searrow, \nearrow), (\&^n, \searrow^n, \nearrow^n), (\&_n, \searrow_n, \nearrow_n)\big).$$

We have proven that this construction of adjoint triples is cyclic in the sense that no new operators are obtained when any of the constructions is further applied to either $(\&^n, \searrow^n, \nearrow^n)$ or $(\&_n, \searrow_n, \nearrow_n)$ and, then, we have studied algebraic properties of these operators.

As future work, we can follow different research lines. On the one hand, a deeper study of algebraic properties of n-multiadjoint lattices should be investigated; a mid-term goal in this line is the development of a formal logic theory based on the structure of n-multiadjoint lattices, as done for the case of residuated lattices and MV-algebras [11, 21]. On the other hand, it is already known that multiadjoint lattices can be considered as the underlying algebraic structure in fuzzy logic programming [19]; an application to generalized fuzzy logic programming [4] is envisaged in terms of the use of the contraposition law as a modus tollens inference rule in order to, for instance, create new equivalent programs under the answer set semantics [14, 15] from which we may obtain interesting inferences from negative information.

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