Galois connections in Computational Intelligence: a short survey

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Abstract—**The construction of Galois connections between different structures provides a number of advantages, both from the theoretical and the applied standpoints. In this paper, we survey some works on Galois connections focused essentially on certain aspects of Computational Intelligence.**

Keywords: Galois connection, Computational Intelligence

I. INTRODUCTION

The notion of Galois connection originated in pure mathematics. Historically, the first occurrence of a Galois connection appeared in the works of Évariste Galois, hence its name, on the solvability by radicals of polynomial equations. The main idea was to link the algebraic solution to a polynomial equation to the structure of the group of permutations associated with the roots of the polynomial; in other works, he "moved" the problem of studying solutions to a polynomial to the realm of group theory. This is the essence of Galois connection, a passing between two (apparently disparate) worlds.

The term *Galois connection* was coined by Ore in 1944 in the context of complete lattices, and then Kan introduced the *adjunctions* in the context of category theory in 1958. Apart from some particularities, both notions are closely related and, in fact, are interchangeable.

Applications of Galois connections to Computer Science can be traced back to the eighties: in [1] Galois connections are used to proof the correctness of a compiler, to solve a a data type coercion problem; and to Scott's inverse limit construction for recursively defined domains. Much more recently, we still can find further applications, for instance, [2] introduces a binary relational combinator which mirrors the linguistic structure in expressions such as "the smallest such number", "the best approximation", "the longest such list" and exploits its potential for calculating programs by optimization, in particular, to specifications written in the form of Galois connections, in which one of the adjoints delivers the optimal solution being sought.

An interesting example of Galois connection arises in the field of Formal Concept Analysis (FCA), which can be seen both as a mathematical theory aiming at restructuring lattice theory (as stated by Rudolph Wille) and as a technology of data processing which complements collective intelligence and helps visualising the hidden information in apparently unstructured and distributed data [3]. Different computational intelligence tools which convert data into formal contexts and then analyse those contexts providing a concept lattice as a form of visualised knowledge have been developed.

In this work, we will stress on the usefulness of Galois connection as a tool for Computational Intelligence, and we will survey some recently published papers in which Galois connections play an important role.

The structure of this paper is the following: in Section II we recall the idea of Galois connection, and introduce the different versions that are in use, both in the crisp and in the fuzzy cases; then, Section III focuses on recent applications of Galois connections in the field of Evolutive Computation; later, in Section IV, the focus is put on applications related to Neural Computation; and, then, in Section V, we consider applications within the area of Fuzzy Computation; finally, in Section VII, we state some conclusions and future work.

II. DEFINITIONS

The standard notion of Galois connection is defined between two partially ordered sets. However, not all the authors consider the same definition of Galois connection and it is remarkable that not all of them are equivalent. In fact, there are four different notions of Galois connection, the most often used being the "right Galois connection" (also known as antitone Galois connection) and the "adjunction" (also known as isotone Galois connections).

Definition 1: Let $\mathbb{A} = (A, \leq)$ and $\mathbb{B} = (B, \leq)$ be posets, $f: A \to B$ and $g: B \to A$ be two mappings. The pair (f, g) is called a

- *Right Galois Connection between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by $(f, g): \mathbb{A} \leftarrowtail \mathbb{B}$ if, for all $a \in A$ and $b \in B$ it holds that

$$a \leq g(b) \text{ if only if } b \leq f(a)$$

- *Left Galois Connection between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by $(f, g): \mathbb{A} \rightarrowtail \mathbb{B}$ if, for all $a \in A$ and $b \in B$ it holds that

$$g(b) \leq a \text{ if only if } f(a) \leq b$$

- *Adjunction between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by $(f, g): \mathbb{A} \leftrightharpoons \mathbb{B}$ if, for all $a \in A$ and $b \in B$ it holds that

$$a \leq g(b) \text{ if only if } f(a) \leq b$$

- *Co-Adjunction between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by $(f, g): \mathbb{A} \rightleftharpoons \mathbb{B}$ if, for all $a \in A$ and $b \in B$ it holds that

$$g(b) \leq a \text{ if only if } b \leq f(a)$$

It is noteworthy that this definition is also compatible with the case of $\mathbb{A} = (A, \leq)$ and $\mathbb{B} = (B, \leq)$ being preordered sets.

Taking into account the dual order, by which $\mathbb{A}^{\partial} = (A, \geq)$, it is not difficult to check that the following conditions are equivalent:

1) $(f,g)\colon \mathbb{A} \rightharpoonup \mathbb{B}$.
2) $(f,g)\colon \mathbb{A}^{\partial} \rightharpoonup \mathbb{B}^{\partial}$.
3) $(f,g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}^{\partial}$.
4) $(f,g)\colon \mathbb{A}^{\partial} \rightleftharpoons \mathbb{B}$.

Focusing on one particular notion, say Right Galois Connection, there are many other equivalent definitions in terms of particular properties of the components of the connection f and g.

The different characterizations for each of the notions of (right- or left-) Galois connection and (co-)adjunction are summarized in Table I (taken from [4]), where we assume that the standard notions of (crisp) order theory are known by the reader. The only non-standard notation is that of p-max(C) to refer to the set of maximum elements of subset C of a preordered set (note that the absence of antisymmetry leads to the possible existence of several different elements fulfilling the properties of being a maximum for C).

Galois connections in the fuzzy case

A more interesting framework to work with Galois connections for Computational Intelligence is to consider potential extensions of the notion to the fuzzy case. As usual, we will consider a *complete residuated lattice* $\mathbb{L} = (L, \leq, \top, \bot, \otimes, \rightarrow)$ as underlying structure for considering the generalization to a fuzzy framework. As usual, supremum and infimum will be denoted by $\vee$ and $\wedge$, respectively

An $\mathbb{L}$-*fuzzy set* is a mapping from the universe set, say X, to the lattice L, i.e. $X\colon U \rightarrow L$, where $X(u)$ means the degree in which u belongs to X. We will denote L^A to refer to the set of all mappings from A to L.

Given X and Y two $\mathbb{L}$-fuzzy sets, X is said to *be included in Y*, denoted as $X \subseteq Y$, if $X(u) \leq Y(u)$ for all $u \in U$. The *subsethood degree* $S(X,Y)$, by which X is a subset of Y, is defined by $S(X,Y) = \bigwedge_{u \in U} \big(X(u) \rightarrow Y(u) \big)$.

The first notion of fuzzy Galois connection was given by Bělohlávek, and it can be rewritten as follows:

Definition 2 ([5]): An ($\mathbb{L}$-)fuzzy Galois connection between A and B is a pair of mappings $f\colon L^A \rightarrow L^B$ and $g\colon L^B \rightarrow L^A$ such that, for all $X \in L^A$ and $Y \in L^B$ it holds that $S(X, g(Y)) = S(Y, f(X))$.

This definition generalizes the standard notion of Galois connection between powersets to the fuzzy case, but not the Galois connection between (pre-)ordered structures. For this, one needs to consider suitable extensions of the notion of poset and preordered set to the fuzzy case.

An $\mathbb{L}$-*fuzzy binary relation on U* is an $\mathbb{L}$-fuzzy subset of $U \times U$, that is $\rho_U\colon U \times U \rightarrow L$, and it is said to be:

- *Reflexive* if $\rho_U(a,a) = \top$ for all $a \in U$.
- $\otimes$-*Transitive* if $\rho_U(a,b) \otimes \rho_U(b,c) \leq \rho_U(a,c)$ for all $a, b, c \in U$.

- *Symmetric* if $\rho_U(a,b) = \rho_U(b,a)$ for all $a, b \in U$.
- *Antisymmetric* if $\rho_U(a,b) = \rho_U(b,a) = \top$ implies $a = b$, for all $a, b \in U$.

We can now introduce the notions of *fuzzy poset* and *fuzzy preposet* as follows:

Definition 3: An $\mathbb{L}$-*fuzzy poset* is a pair $\mathbb{U} = (U, \rho_U)$ in which ρ_U is a reflexive, antisymmetric and transitive $\mathbb{L}$-fuzzy relation on U.

An $\mathbb{L}$-*fuzzy preposet* is a pair $\mathbb{U} = (U, \rho_U)$ in which ρ_U is a reflexive and transitive $\mathbb{L}$-fuzzy relation on U.

From now on, when no confusion arises, we will omit the prefix "$\mathbb{L}$-".

We can now recall the extension to the fuzzy case provided by Yao and Lu, also used in [6],which can be stated as follows:

Definition 4 ([7]): Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy preposets. A pair of mappings $f\colon A \rightarrow B$ and $g\colon B \rightarrow A$ forms a *Galois connection* between $\mathbb{A}$ and $\mathbb{B}$, denoted $(f,g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ if, for all $a \in A$ and $b \in B$, the equality $\rho_A(a, g(b)) = \rho_B(f(a), b)$ holds.

A further step towards generalization to the fuzzy realm is possible when considering fuzzy equivalence relations in each of the involved sets instead of the mere equality relation. This leads to a notion of fuzzy Galois connection in which the mappings f and g can be seen, in some sense, as fuzzy mappings instead of being crisp ones.

The additional consideration of an underlying fuzzy equivalence relation suggests considering the following notions:

Definition 5:

(i) A *fuzzy structure* $\mathcal{A} = \langle A, \approx_A \rangle$ is a set A endowed with a fuzzy equivalence relation $\approx_A$.
(ii) A *morphism* between two fuzzy structures $\mathcal{A}$ and $\mathcal{B}$ is a mapping $f\colon A \rightarrow B$ such that for all $a_1, a_2 \in A$ the following inequality holds: $(a_1 \approx_A a_2) \leq (f(a_1) \approx_B f(a_2))$. In this case, we write $f\colon \mathcal{A} \rightarrow \mathcal{B}$, and we say that f is *compatible* with $\approx_A$ and $\approx_B$.

We can now introduce the notion of *fuzzy preordered structure* as follows:

Definition 6: Given a fuzzy structure $\mathcal{A} = \langle A, \approx_A \rangle$, the pair $\mathbb{A} = \langle \mathcal{A}, \rho_A \rangle$ will be called a $\otimes$-$\approx_A$-*fuzzy preordered structure* or simply *fuzzy preordered structure* (when there is no risk of confusion), if ρ_A is a fuzzy relation that is $\approx_A$-reflexive, $\otimes$-$\approx_A$-antisymmetric and $\otimes$-transitive, where

(i) $\approx_A$-*reflexive* means $(a_1 \approx_A a_2) \leq \rho_A(a_1, a_2)$ for all $a_1, a_2 \in A$.
(ii) $\otimes$-$\approx_A$-*antisymmetric* means $\rho_A(a_1, a_2) \otimes \rho_A(a_2, a_1) \leq (a_1 \approx_A a_2)$ for all $a_1, a_2 \in A$.

If the underlying fuzzy structure is not clear from the context, we will sometimes write a fuzzy preordered structure as a triplet $\mathbb{A} = \langle A, \approx_A, \rho_A \rangle$.

A reasonable approach to introduce the notion of Galois connection between fuzzy preordered structures $\mathbb{A}$ and $\mathbb{B}$ would be the following

Definition 7 ([8]): Let $\mathbb{A}$ and $\mathbb{B}$ be two fuzzy preordered structures. Given two morphisms $f\colon \mathcal{A} \rightarrow \mathcal{B}$ and $g\colon \mathcal{B} \rightarrow \mathcal{A}$, the pair (f,g) is said to be a *Galois connection* between $\mathbb{A}$ and

Galois Connections	
Right Galois Connections between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \hookleftarrow \mathbb{B}$	Left Galois Connections between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \rightharpoondown \mathbb{B}$
$b \leq f(a) \Leftrightarrow a \leq g(b)$ for all $a \in A$ and $b \in B$	$f(a) \leq b \Leftrightarrow g(b) \leq a$ for all $a \in A$ and $b \in B$
f and g are antitone and $g \circ f$ and $f \circ g$ are inflationary	f and g are antitone and $g \circ f$ and $f \circ g$ are deflationary
$f(a)^{\downarrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$ $g(b)^{\downarrow} = f^{-1}(b^{\uparrow})$ for all $b \in B$	$f(a)^{\uparrow} = g^{-1}(a^{\downarrow})$ for all $a \in A$ $g(b)^{\uparrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$
f is antitone and $g(b) \in \text{p-max}\, f^{-1}(b^{\uparrow})$ for all $b \in B$	f is antitone and $g(b) \in \text{p-min}\, f^{-1}(b^{\downarrow})$ for all $b \in B$
g is antitone and $f(a) \in \text{p-max}\, g^{-1}(a^{\uparrow})$ for all $a \in A$	g is antitone and $f(a) \in \text{p-min}\, g^{-1}(a^{\downarrow})$ for all $a \in A$
Adjunctions	
Adjunction between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \leftharpoondown \mathbb{B}$	co-Adjunction between $\mathbb{A}$ and $\mathbb{B}$ $(f,g)\colon \mathbb{A} \rightleftharpoons \mathbb{B}$
$f(a) \leq b \Leftrightarrow a \leq g(b)$ for all $a \in A$ and $b \in B$	$b \leq f(a) \Leftrightarrow g(b) \leq a$ for all $a \in A$ and $b \in B$
f and g are isotone, $g \circ f$ is inflationary and $f \circ g$ is deflationary	f and g are isotone, $g \circ f$ is deflationary and $f \circ g$ is inflationary
$f(a)\uparrow = g^{-1}(a^{\uparrow})$ for all $a \in A$ $g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ for all $b \in B$	$f(a)^{\downarrow} = g^{-1}(a^{\downarrow})$ for all $a \in A$ $g(b)^{\uparrow} = f^{-1}(b^{\uparrow})$ for all $b \in B$
f is isotone and $g(b) \in \text{p-max}\, f^{-1}(b^{\downarrow})$ for all $b \in B$	f is isotone and $g(b) \in \text{p-min}\, f^{-1}(b^{\uparrow})$ for all $b \in B$
g isotone and $f(a) \in \text{p-min}\, g^{-1}(a^{\uparrow})$ for all $a \in A$	g is isotone and $f(a) \in \text{p-max}\, g^{-1}(a^{\downarrow})$ for all $a \in A$

$\mathbb{B}$ (briefly, $(f,g)\colon \mathbb{A} \leftharpoondown \mathbb{B}$) if the following conditions hold for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$:

(G1) $(a_1 \approx_A a_2) \otimes \rho_A(a_2, g(b)) \leq \rho_B(f(a_1), b)$
(G2) $(b_1 \approx_B b_2) \otimes \rho_B(f(a), b_1) \leq \rho_A(a, g(b_2))$

The following proposition proves that the previous definition behaves as expected, namely, it satisfies the standard equality for Galois connections.

Proposition 1 ([8]): Consider two fuzzy preordered structures $\mathbb{A} = \langle \mathcal{A}, \rho_A \rangle$ and $\mathbb{B} = \langle \mathcal{B}, \rho_B \rangle$, and two mappings $f\colon A \to B$ and $g\colon B \to A$. It holds that the pair (f,g) is a Galois connection between $\mathbb{A}$ and $\mathbb{B}$ if and only if both mappings are *morphisms* and $\rho_A(a, g(b)) = \rho_B(f(a), b)$ for all $a \in A$ and $b \in B$.

III. GALOIS CONNECTIONS IN EVOLUTIONARY COMPUTATION

A relevant field in Computational Intelligence is the Evolutionary Computation, which embraces a wide range of algorithms for global optimization. Their main feature is that they are inspired by biological evolution. Genetics Algorithms (GA) form the most popular kind of Evolutionary Algorithm, which seek the solution of a optimization problem in the form of strings of numbers (usually binary chains) by applying operators such as mutation, crossover and selection.

GAs lack a rigorous explanation of exactly why and on what functions they perform well. There is, however, a chief approach to studying the power of GAs, which is by considering the schemata they are processing. In [9], the authors define two basic operations for schematas, the expansion and compression, and, using these operators that constitute a Galois connection, they define the notions of the schematic completion and the schematic lattice. They use the schematic completion to observe the building blocks (schema with below average order, below average defining length and above average fitness) during the course of a GA. Finally, they introduce methods to explicitly calculate the schemata present in a population and the identification underlying lattice structures involved in schema processing.

Galois connections and, in particular, FCA techniques have been also used in order to improve the efficiency of this kind of algorithms in particular applications. One example is the approach proposed in [10] to Feature Location (process of finding the set of software artifacts that realize a particular feature) that target models as the feature realization artifacts. It is based on a GA that generates alternative model fragments that can be the realizations of the feature being located. Then, it uses FCA techniques to cluster the model fragments by their common attributes and to generate feature candidates. The

feature candidates are assessed comparing them to a search query that describes the target feature. The closer ones are selected to engendrate the next generations.

In [11], FCA techniques and GA are combined for the automatic definition of fuzzy classification systems. The task of classification has been widely researched by computational intelligence community. Among the methods proposed for the task of classification, special attention has been given to the Genetic Fuzzy Systems (GFS) with a large number of proposals found in the literature [12]. The approach proposed in [11], by using FCA techniques, extract rules directly from data avoiding the random extraction of rules with low classification power. This approach to extract rules presents polynomial complexity. The obtained rules can be used with any genetic approach that performs rule selection improving the accuracy rates for most of the datasets.

GA have been also proposed in [13] as alternative to FCA techniques to obtain frequent item sets and large bite item sets.

In [14], an FCA-based algorithm for choosing the most appropriate consensus function for each case is provided. There have been various attempts, solutions, and approaches towards constructing an appropriate consensus tree based on a given set of phylogenetic trees, but it is not always clear, for a given dataset, which of these would create the most relevant consensus tree. The authors propose the use of FCA to address this problem.

IV. GALOIS CONNECTIONS IN NEURAL NETWORKS AND MACHINE LEARNING

Neural Computing is another very popular subfield of biologically inspired computing. Artificial neural networks (ANNs) are computing systems, inspired by the biological neural networks, that learn from examples, generally without task-specific programming.

Concepts are the base of human thinking and FCA. Perceptions and cognitions are one among the natural ability of human brain that can be modeled to an extent with neural networks. Therefore, one can easily find papers that relates the techniques of FCA and ANNs. We highlight below the most relevant and current.

In [15], the authors propose an approach to generating neural network architecture based on the covering relation (graph of the diagram) of a lattice coming from antitone Galois connections (standard concept lattice) or isotone Galois connections. Selecting an appropriate network architecture is a crucial problem when looking for a solution based on a neural network. The method proposed by the authors provides an architecture fitted to the problem and directly derived from the dataset.

ANNs also suffer from poor interpretability of learning results, which could be critical in applications such as medical decision making. Another advantadge of the approach proposed in [15] is that allows to explain why objects are assigned to particular classes.

One way of human brain to learn and memorize the new concepts is via its association with previously learned con-cepts. ANNs implements this associations through either feedforward or recurrent networks. These networks store the set of patterns as memories and identify the corresponding associated pattern in the memory for the given input pattern. The most popular and widely used associative memory model is Bidirectional Associative Memory (BAM), which is a recurrent network that is capable of modeling the hetero-association. A BAM is a complete bipartite graph whose vertexes are neurons. It perfectly fits with the notion of concept in FCA. In [16], the authors model the associative memory activity using FCA techniques: patterns are associated with the help of object-attribute relations and the memory is represented using the formal concepts generated using FCA.

In the other direction, there is a wide set of papers that propose the use of ANNs to solve FCA problems. Thus, for instance, [17] proposes an enhancement of FCA by Lattice Computing techniques. The authors introduce a novel Galois connection toward defining tunable metric distances as well as tunable inclusion measure functions between formal concepts induced from hybrid data. The formal concepts are interpreted as descriptive decision making knowledge (rules) induced from the training data. In addition, a simple KNN algorithm for inducing formal concepts is introduced there as an extension of the Karnaugh map technique from digital electronics.

V. GALOIS CONNECTIONS IN FUZZY COMPUTATION

The groundbreaking work for extending the concept of Galois connection to a fuzzy setting is due to Bělohlávek [5]. In this seminal paper, a fuzzy Galois connection was introduced as a pair of crisp mappings between the sets of fuzzy sets of two universes which satisfies certain adjoint-like property. Since then many other papers have been published including further approaches to both antitone and isotone Galois connections [18]–[24].

An outstanding issue in all the generalized notions of Galois connection is its actual construction, namely, the problem of constructing the residual (also known as right adjoint) mapping of a given $f\colon A \to B$. An easy-going possibility is to apply a suitable version of the Freyd's adjoint theorem (coming from category theory) which characterizes when such a residual exists if both A and B have the same structure.

But when the ordered-like structure is just on the domain A, this theorem cannot be applied since, firstly, the missing structure on B needs to be built. This has been one of the recent research lines of our team, in which a number of results have been obtained by considering different underlying settings. Namely, in [25] we worked with crisp functions between a crisp poset (resp. preordered set) and an unstructured set as a first step to undertake the topic in the fuzzy framework.

As a generalization of Bělohlávek's approach, in [7], fuzzy Galois connections between the so-called fuzzy posets (i.e. crisp universes endowed with fuzzy order relations [26]) were introduced. Our first approach started from this definition but since the property of antisymmetry of fuzzy order relations seems to be rather restrictive, fuzzy preorder relations were preferred. Thus, in [6], we considered a mapping $f\colon \mathbb{A} \to$

B from a fuzzy preposet $\mathbb{A} = \langle A, \rho_A \rangle$ into an unstructured set B, and then characterize those situations in which B can be endowed with a fuzzy preorder relation and a mapping $g\colon B \to A$ can be defined such that the pair (f, g) becomes a Galois connection.

The second approach that we adopted consists of replacing the standard equality between elements of the universe by a similarity relation, in order terms, a fuzzy equivalence relation to reflect the degree in which two elements in the universe are similar. The mappings that make sense between these *fuzzy* structures are those compatible with the similarity. In order to consider the definition of a Galois connection in this new framework, it is necessary to add an extra fuzzy binary relation in the domain and the codomain. In [8] our problem is then to find a right adjoint to a mapping $f\colon \langle A, \approx_A, \rho_A \rangle \to \langle B, \approx_B \rangle$ in which the fuzzy equivalence $\approx_B$ is already given and has to be preserved.

Concerning the study of IF-THEN rules to describe dependences between graded attributes values in data collections, (also called fuzzy attribute implications or similarity-based functional dependences) isotone Galois connections on fuzzy sets have proven to be very useful [27]. Novel parameterizations based on systems of Galois connections allows to emphasize antecedents in fuzzy attribute implications or to admit a graded notion of satisfaction in the rules. This approach is more general than the parameterizations based on linguistic hedges which have been used previously in [28], [29]. The use of Galois connections brings more versatility into the applications of the IF-THEN rules in data analysis in that experts are able to specify parameterizations of rules from a rich family of parameterizations. Moreover, even in the borderline case of classical functional dependencies (which can be seen as a particular case of the graded ones when the structure of degrees is the two-element Boolean algebra) parameterization by systems of isotone Galois connections brings new types of semantics, in contrast with the earlier approaches by hedges, which yield no nontrivial parameterization in the crisp setting.

A promising task in this direction could be to find methods for constructing families of parameterizations by user requirements and to investigate properties of such families of isotone Galois connections. It may be related with the construction of residual mappings that has been described in our papers. Likewise, further investigation may focus on connections of the presented theory to the general approaches to multiadjoint concept lattices and related structures.

Other authors have emphasized that fuzzy Galois connections are the core of generalizations of FCA [30] which are oriented to knowledge discovery and information management under uncertainty. In the near future research regarding fuzzy implications the existence of such Galois connections should stand foremost among the applications of such implications. Some examples of applications that can be explored under this perspective are the management of contingency tables for error assessment [31], analysis of Gene Expression Data [32] or discovery of semantic web services [33].

VI. Miscellaneous applications

A. *Mathematical morphology*

In a few words, Mathematical Morphology (MM) is a formal theory for the study of shape. It is based on two operators (*erosion* and *dilation*) whose composition generates the operators of *opening* and *closing*. Obviously, MM is mainly used in the analysis of digital images, specifically to investigate the interaction, in terms of erosion and dilation, between an image and a given structuring element. Our interest in MM lies on the fact that the operators of erosion and dilation form an isotone Galois connection.

Although the original definition of MM was oriented to B/W images, several extensions have been developed to grayscale and color images. In fact, the underlying theory of MM has been extended to complete lattices to the so-called L-fuzzy Mathematical Morphology [34].

The fact that the main constructors in both FCA and MM are based on the notion of Galois connection suggest a possible relation between them. Actually, the relation exists not only between them, but also with other lattice-based formalisms such as fuzzy sets, rough sets or the F-transforms [35]. In particular, this paper shows how fuzzy MM techniques can be used to navigate a fuzzy concept lattice.

To finish with, since the previous reference also used the notion of F-transform (i.e. fuzzy transform) [36], which provides an alternative technique for approximation models, it is worth to note that Galois connections also appear on the realm of F-transforms, since the pair functions given by the direct and inverse transform form an isotone Galois connection. Recently, this theory has been extended to the lattice-based case [37].

B. *Programming languages semantics*

Another interesting research field in which Galois connections play their role is in structuring abstraction in semantics. Static analysis based on abstract interpretation relies on abstract domains of program properties, and Galois connections are which provide the most widespread and useful formal tool for mathematically specifying abstract domains.

In [38] the notion of *constructive Galois connection* was introduced in order to define abstract domains in a style more oriented to mechanization, which enables the automatic generation of certified algorithms using systems such as Coq.

C. *Knowledge extraction*

The fixpoints of Galois connections form patterns in binary object-attribute relations, that are important in disparate data analysis areas (including, Boolean factor analysis, or frequent itemset mining). In FCA, fixpoints of the Galois connection which is defined by concept-forming operators, are particular clusters in cross-tables, defined by means of attribute sharing. This is a key notion in the theory and the problem of building the concept lattice is not trivial, as the number of concepts can be exponential in the size of the input context. From the very beginning of FCA, different algorithms have been proposed to generate the set of all formal concepts and Hasse diagrams of concept lattices as Ganter's Next Concept [39]

or Kuznetsov's Close-by-One (CbO) [40] which has been the basis for several variants and suggested improvements, such as Krajca et al's realisation of CbO [41], Andrews's In-Close [42] or Outrata and Vychodil's FCbO [43]. Recently, various algorithms have been discussed and compared, including some performances of previous algorithms such as In-Close2 and In-Close3 [44], [45]. The exploration of new algorithms for computing Galois connection fixpoints and its comparison with previous algorithms represent an issue of current and permanent interest in the FCA community and other areas.

D. Granular Computing

Since Galois connections are the mathematical foundation of several disciplines, it is interesting to explore the links which can be established at the theoretical level between different frameworks related to the so-called *granular computing*. In their position paper [46], Dubois and Prade consider four settings: possibility theory [47], formal concept analysis, extensional fuzzy sets [48] and rough sets [49]. All of them have been developed independently and, although they look very different, they are concerned with ideas of grouping items and handling similar notions around the idea of granular computing. The analysis carried out in [46] includes the parallel between possibility theory and formal concept analysis in the crisp case. The gap with rough sets is shortened by restricting to relations between objects. It is also analyzed the interactions in the non-Boolean case, specifically, between the theory of extensional fuzzy sets and representation of fuzzy extensions of equivalence relations with the gradual version of formal concept analysis. Likewise, by replacing an equivalence relation by a fuzzy similarity relation, rough sets have been extended to a fuzzy setting [50], which are also connected to extensional fuzzy sets in [51]. Therefore, since similar structures were found to be at work in such settings it may lead to mutual enrichments between such theories.

E. Mathematics

In this section, we would like to point out the use of Galois connections to provide alternative and shorter proofs of well-known theorems. For instance, in [52] two particular Galois connections are defined and, then, are used to prove in a short way the Knaster-Tarski's fixpoint theorem. It is worth to note that this theorem is important for the area of Logic Programming, and its fuzzy extensions, in that it is the essential tool in order to define the fixpoint semantics of a (fuzzy) logic program.

VII. CONCLUSIONS AND FUTURE WORK

In this work, firstly, we have surveyed different notions of Galois connections published up to now, focusing especially on the extensions to the fuzzy case; secondly, we have also surveyed on recent applications of the theory of Galois connections within the realm of Computational Intelligence. As a result, one can see, once again, that the study of theoretical notions is not at odds with the applications.

Our next goal is to study fuzzy Galois connections constituted of truly fuzzy mappings. In [53] we started the search for a more adequate notion involving fuzzy functions as components. The notion of relational fuzzy Galois connection is introduced and proved that the construction embeds Yao's notion of fuzzy Galois connection as a particular case. As future work, we are planning to continue the line initiated in [6], [8] and attempt the construction of the residual, in the sense of relational fuzzy Galois connections, to a given mapping between differently structured domain and codomain.

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