Quantified Equilibrium Logic and the First Order Logic of Here-and-There *

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Tech. Report MA-06-02 (Univ. of Málaga, Spain)

Abstract

This report continues the study of quantified equilibrium logic, **QEL**, introduced in [25, 26], and its monotonic base logic, here-and-there. We present a slightly modified version of **QEL** where the so-called *unique name assumption* or UNA is not assumed from the outset but may be added as a special requirement for specific applications. We also consider here an alternative axiom set for first-order here-and-there. The new system appears to be simpler as well as making it easier to derive some simple semantic validities.

In addition, based on the modified semantics for first-order here-and-there we present **QEL** and investigate some of its properties. We look in particular at two issues. First, we consider the relation of **QEL** to non-ground answer set programming. Specifically we show that in the quantified case equilibrium models corresponds precisely to the open answer sets of [12], while the earlier version of open answer sets discussed in [11] can be captured in **QEL** with UNA. Secondly, we propose a concept of strong equivalence for theories in **QEL** generalising the usual concept for propositional theories. The strong equivalence theorem of [16] is extended to the first-order case by showing that equivalence in the first-order logic of here-and-there is a necessary and sufficient condition for strong equivalence. We relate this to the concept of strong equivalence for non-ground logic programs studied in [3].

1 Introduction

Equilibrium logic was introduced in [21] as a general nonmonotonic formalism extending the semantics of answer sets for logic programs [6]; it was further studied and applied in [22, 16, 24] and elsewhere. For an overview of the main features and properties, see [23]. Equilibrium logic is based on a simple, minimal model construction in the nonclassical logic of *here-and-there*, **HT**. When

*This work is partially supported by CICyT project TIC-2003-9001.

a second negation operator is present in the language, as in the usual case of answer set semantics, the underlying logic is the least strong negation extension of **HT**, denoted by $\mathbf{N}_5$. As equilibrium logic is defined for arbitrary propositional formulas it yields an extension of the usual syntax of (ground) answer set programs. It also provides a useful logical foundation for answer set programming (ASP), the fast developing paradigm for declarative programming based on the answer set semantics [1].

A costly component of the computation of answer sets is the process of *grounding* a program containing variables by instantiating the variables with constants from the language to yield a ground propositional program for which candidate models are then generated and tested. For future generation systems it is desirable to develop mechanisms for program transformation and simplification that would allow for program optimisation and perhaps even partial evaluation prior to grounding. For this reason interest has recently grown in the logical and mathematical foundations of non-ground or first-order programs. For example [18, 3] have studied the property of strong equivalence and related notions for non-ground programs, while in [26] a first-order version of equilibrium logic was presented and its relation to non-ground programs under answer set semantics was studied. In a different direction, the concept of *open answer set* for non-ground programs was defined in [11, 10] and decidable classes of programs were identified via embeddings into fixpoint logic.

This report continues the work of [26] on first-order, or, as we shall say, *quantified* equilibrium logic (or **QEL** for short) and its relation to non-ground answer set programming. The report has three main contributions. First, we present a slightly different version of **QEL** where the so-called *unique name assumption* or UNA is not assumed from the outset but may be added as a special requirement for specific applications. The motivation for relaxing the UNA is to make equilibrium logic more flexible for certain kinds of applications. For instance in the area of reasoning for the Semantic Web there has been considerable interest recently in combining reasoning about ontologies with nonmonotonic rules under answer set semantics. In this domain the UNA may be undesirable or even in a sense incorrect, and in approaches to so-called hybrid knowledge bases that integrate classical logic with ASP the UNA has been dropped [27, 12]. A more specific plea for QEL without UNA is also made in [4]. In our earlier version [26] the UNA was present as a built-in feature of the Kripke model semantics of the underlying logic of quantified here-and-there. Here we present a modified version of the semantics without UNA. However we show that the logic $\mathbf{QHT}^s$ corresponding to the new models is equivalent in terms of satisfiability and validity to the logic described previously in [26].

Secondly, we consider here an alternative axiom set for $\mathbf{QHT}^s$. The new system appears to be simpler as well as making it easier to derive some simple semantic validities; we give an example in Section 3.2. Much of the report is given over to proving the completeness of the new axiom schemata.

Thirdly, based on the modified semantics for $\mathbf{QHT}^s$ we present quantified equilibrium logic **QEL** and investigate some of its properties. We look in particular at two issues. First, we consider the relation of **QEL** to non-ground answer set programming. Specifically we show that in the quantified case equilibrium models corresponds precisely to the open answer sets of [12], while the earlier version of open answer sets discussed in [11] can be captured in **QEL** with UNA. Secondly, we propose a concept of strong equivalence for theories in **QEL** generalising the usual concept for propositional theories. The strong equivalence theorem of [16] is extended to the first-order case by showing that equivalence in the logic $\mathbf{QHT}^s$ is a necessary and sufficient condition for strong equivalence.

2 Background: known logics and systems

We start by recalling some well-known logical systems. To simplify the presentation, for the time being we consider languages without strong negation. The addition of strong negation will be treated as a separate issue in Section 6. For the propositional language we consider the following connectives: $\wedge$ for conjunction, $\vee$ for disjunction, $\rightarrow$ for implication, $\neg$ for weak or intuitionistic negation. We also consider the connective $\leftrightarrow$ defined as $\alpha \leftrightarrow \beta =_{def} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. For the first-order language we add the usual universal and existential quantifiers $\forall$ and $\exists$, respectively.

We work with first order languages with function symbols, $\mathcal{L} = \langle C, F, P\rangle$, built over a set of *constants*, C, a set of *functions*, F, and a set of *predicates*, P; the three sets of symbols are disjoints and each predicate symbol and each function symbol has an assigned arity. *Atoms* and *formulas* are constructed as usual; *closed formulas*, or *sentences*, are those where no variable appears outside the scope of a quantifier. A *theory* is a set of sentences. Variable-free terms, atoms, formulas, or theories are also called *ground*.

If D is a non-empty set, we denote by $\mathrm{At}_D(C, P)$ the set of atomic sentences of $\langle C \cup D, F, P\rangle$ (if $D = \varnothing$, we obtain the set of atomic sentence of the language $\mathcal{L} = \langle C, F, P\rangle$);[1] and we denote by $\mathrm{T}_D(C, F)$ the set of ground terms of $\langle C \cup D, F, P\rangle$. If $\mathcal{L} = \langle C, F, P\rangle$ and $\mathcal{L}' = \langle C', F', P'\rangle$, we say that $\mathcal{L} \subseteq \mathcal{L}'$ if $C \subseteq C'$, $F \subseteq F'$ and $P \subseteq P'$.

2.1 Intuitionistic Logic

In this section we analyse the axiomatic systems for the logic used in this paper. All the systems are extensions of intuitionistic logic denoted by **Int**. We are going to work with the following axiomatic system for **Int**, where D is an infinite and countable set such that $C \cap D = \varnothing$.

Axioms: I1 $\varphi \rightarrow (\psi \rightarrow \varphi)$

I2 $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$

I3 $\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$

[1] We can think of the objects in D as additional constants. As we shall see below, for notational simplicity we do not distinguish between the objects in D and their names.

I4 $\varphi \wedge \psi \to \varphi$

I5 $\varphi \wedge \psi \to \psi$

I6 $\varphi \to \varphi \vee \psi$

I7 $\psi \to \varphi \vee \psi$

I8 $(\varphi \to \gamma) \to ((\psi \to \gamma) \to (\varphi \vee \psi \to \gamma))$

I9 $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$

I10 $\neg\varphi \to (\varphi \to \psi)$

I11 $\forall x\varphi(x) \to \varphi(t)$, for every $t \in \mathrm{T}_D(C, F)$

I12 $\varphi(t) \to \exists x\varphi(x)$, for every $t \in \mathrm{T}_D(C, F)$

Inference rules:

- *Modus Ponens:* From φ and $\varphi \to \psi$, conclude ψ.
- $\forall$-*Introduction:* From $\psi \to \varphi(c)$, where $c \in C \cup D$ does not occur in ψ, conclude $\psi \to \forall x\varphi(x)$.
- $\exists$-*Elimination:* From $\varphi(c) \to \psi$, where $c \in C \cup D$ does not occur in ψ, conclude $\exists x\varphi(x) \to \psi$.

When deductions are considered, in the inference rules for quantifiers, the constant c can not occur in the premises

If φ is deducible from Γ in this system we write: $\Gamma \vdash_{\mathbf{Int}} \varphi$.

2.1.1 Kripke semantics for intuitionistic logic

An *intuitionistic $\mathcal{L}$-structure* $\mathcal{I}$ comprises a tuple

$$\mathcal{I} = \langle (W, \leq), (\{D_w\}_{w \in W}, \sigma), \{I_w\}_{w \in W} \rangle$$

where: $(W, \leq)$ is a partial ordered set; every D_w is a non-empty set and $D_w \subseteq D_{w'}$ if $w \leq w'$; if $D = \bigcup_{w \in W} D_w$, then $\sigma \colon \mathrm{T}_D(C, F) \to D$ is recursively defined[2] and verifies that $\sigma(d) = d$ for all $d \in D$ and $\sigma(t) \in D_w$ if $t \in \mathrm{T}_{D_w}(C, F)$; and for every $w \in W$, $I_w \subseteq \mathrm{At}_{D_w}(\varnothing, \varnothing, P)$ and $I_w \subseteq I_{w'}$ if $w \leq w'$. The satisfiability relation is defined as follows:

- $\mathcal{M}, w \models p(t_1, \ldots, t_n)$ iff $p(\sigma(t_1), \ldots, \sigma(t_n)) \in I_w$ for every $t_1, \ldots, t_n \in \mathrm{T}_{D_w}(C, F)$.
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$.
- $\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$.
- $\mathcal{M}, w \models \varphi \to \psi$ iff $\mathcal{M}, w' \not\models \varphi$ or $\mathcal{M}, w' \models \psi$ for every $w' \geq w$.
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w' \not\models \varphi$ for every $w' \geq w$.

[2] That is, for every $a \in C$, $\sigma(a) \in D_w$ for all w, for every $f \in F$ with arity n, a mapping $f^{\mathcal{I}} \colon D^n \to D$ is defined verifying $f^{\mathcal{I}}(d_1, \ldots, d_n) \in D_w$ provided $d_1, \ldots, d_n \in D_w$.; so, the recursive definition is given by $\sigma(f(t_1, \ldots, t_n)) = f^{\mathcal{I}}(\sigma(t_1), \ldots, \sigma(t_n))$.

- $\mathcal{M}, w \models \forall x\varphi(x)$ iff $\mathcal{M}, w' \models \varphi(d)$ for all $w' \geq w$ and $d \in D_{w'}$.
- $\mathcal{M}, w \models \exists x\varphi(x)$ iff $\mathcal{M}, w \models \varphi(d)$ for some $d \in D_w$.

This is one of the standard ways to present Kripke semantics for **Int**, similar to that of [30]; for alternative but equivalent presentations, see eg. [29]. The semantics can be intuitively explained as follows. W is a set of states or 'possible worlds' partially ordered by $\leq$. Each world w is associated a *domain* D_w whose elements persist at 'later' worlds, ie. $D_w \subseteq D_{w'}$ if $w \leq w'$. For each world σ assigns elements and tuples of the domain to terms built from constant and function symbols in the original language extended with constants for each domain element; the semantic condition ensures that a function defined on terms in some domain is assigned a value in that domain. In addition, each world w is assigned an *interpretation* I_w which takes the form of a set of closed atoms in the language of D_w. These correspond to the atoms verified at world w. As usual in Kripke semantics, an atom verified at world w remains true at any 'later' world $w' \geq w$. For the truth relation one requires that an atomic sentence is true at a world w just in case its interpretation is an element of I_w. This relation is extended recursively to all sentences in the usual manner.

Truth of a sentence in a model is defined as follows: $\mathcal{M} \models \varphi$ iff $\mathcal{M}, w \models \varphi$ for all $w \in W$. The sentence φ is valid in **Int** if it is true in all models and it is denoted by $\models_{\mathbf{Int}} \varphi$. A formula φ is deduced from Γ if every model of Γ is a model of φ and it is denoted by $\Gamma \models_{\mathbf{Int}} \varphi$.

Int is strongly complete for the above semantics in the sense that $\Gamma \vdash_{\mathbf{Int}} \varphi$ if and only if $\Gamma \models_{\mathbf{Int}} \varphi$.

2.2 Propositional Here-and-There Logic

In the propositional case, the *logic of here-and-there* can be characterised by adding to intuitionistic propositional logic the following axiom schema ŁUK, introduced by Łukasiewicz [19]:

ŁUK $(\neg\alpha \rightarrow \beta) \rightarrow (((\beta \rightarrow \alpha) \rightarrow \beta) \rightarrow \beta)$

Added to the intuitionistic propositional calculus, the 3-valued logic of Heyting [13] and Gödel [7] is obtained, and we denote this logic by **HT**. Other axiomatic systems for here-and-there have been described in the literature [15]; for example, the axiom of Łukasiewicz can be replaced by the following axiom of Hosoi [14], in slightly simplified form:

HOS $\alpha \vee (\neg\beta \vee (\alpha \rightarrow \beta))$

This logic can be characterised by a Kripke semantics for rooted frames with just two elements, say 'h' and 't': $(\{h, t\}, \leq)$ with $h \leq t$. So models for **HT** can be regarded as pairs $\langle H, T\rangle$ where H and T are sets of atoms with $H \subseteq T$. The satisfiability relation is defined as in intuitionistic logic by considering $I_h = H$ and $I_t = T$; validity is defined in the same way.

3 Quantified Here-and-There Logics

We now turn to quantified versions of the here-and-there logic and consider both the cases of static and non-static domains.

3.1 Kripke semantics with expanding domains

We denote by **QHT** the Kripke semantics for here-and-there logic obtained by considering the interpretation of quantifiers as in intuitionistic logic. So a *(non-static) here-and-there $\mathcal{L}$-structure* (in short, QHT-$\mathcal{L}$-structure) is a tuple $\mathcal{I} = \langle (D_h, D_t, \sigma), H, T\rangle$ such that:

- $\varnothing \neq D_h \subseteq D_t$.
- $\sigma\colon \mathrm{T}_{D_t}(C, F) \to D_t$ is such that $\sigma(d) = d$ if $d \in D_t$ and $\sigma(t) \in D_h$ if $t \in \mathrm{T}_{D_h}(C, F)$.
- $H \subseteq \mathrm{At}_{D_h}(\varnothing, \varnothing, P)$, $T \subseteq \mathrm{At}_{D_t}(\varnothing, \varnothing, P)$ and $H \subseteq T$.

The satisfaction relation and validity are defined as in intuitionistic logic.

3.2 Kripke semantics with static domains

Another possibility is to consider the Kripke semantics for here-and-there logic taking the same domain for both worlds. We denote by $\mathbf{QHT}^s$ the Kripke semantics for here-and-there logic obtained by considering *static* domains.

A *static here-and-there $\mathcal{L}$-structure* (in short QHT^s-$\mathcal{L}$-structure) is a tuple $\mathcal{I} = \langle (D, \sigma), H, T\rangle$ such that: D is a non-empty set, $\sigma\colon \mathrm{T}_D(C, F) \to D$ is such that $\sigma(d) = d$ if $d \in D$ and $H \subseteq T \subseteq \mathrm{At}_D(\varnothing, \varnothing, P)$. The satisfiability relation and validity are defined as in Intuitionistic logic.

This logic is stronger than the non-static versions; for example, in the general case the formula $\forall x(p(x) \vee q(a)) \to (\forall x p(x) \vee q(a))$ is not valid in **QHT** ($\langle(\{a\}, \{a, b\}, id), \{p(a)\}, \{p(a), q(a)\}\rangle$ is not a model for it) but it is valid in $\mathbf{QHT}^s$.

Lemma 1 $\models_{\mathbf{QHT}^s} \forall x(p(x) \vee q(a)) \to (\forall x p(x) \vee q(a))$.

Proof: Let us consider the structure $\mathcal{M} = \langle (D, \sigma), H, T\rangle$.

1. If $\mathcal{M}, t \models \forall x(p(x) \vee q(a))$ then $\mathcal{M}, t \models p(d) \vee q(a)$ for all $d \in D$; we distinguish two cases: if $\mathcal{M}, t \models q(a)$, then $\mathcal{M}, t \models \forall x p(x) \vee q(a)$; if $\mathcal{M}, t \not\models q(a)$, then necessarily $\mathcal{M}, t \models p(d)$ for all $d \in D$ and consequently $\mathcal{M}, t \models \forall x p(x)$ and $\mathcal{M}, t \models \forall x p(x) \vee q(a)$. Therefore, $\mathcal{M}, t \models \forall x p(x) \vee q(a)$.
2. If $\mathcal{M}, h \models \forall x(p(x) \vee q(a))$, then $\mathcal{M}, h \models p(d) \vee q(a)$ and $\mathcal{M}, t \models p(d) \vee q(a)$ for all $d \in D$; we distinguish three cases: if $\mathcal{M}, h \models q(a)$, then $\mathcal{M}, h \models \forall x p(x) \vee q(a)$; if $\mathcal{M}, h \not\models q(a)$, then necessarily $\mathcal{M}, h \models p(d)$ for all $d \in D$ and therefore $\mathcal{M}, t \models p(d)$ for all $d \in D$ and it follows that $\mathcal{M}, h \models \forall x p(x)$ (at this point we make use of the fact that $D = D(h) = D(t)$) and $\mathcal{M}, h \models \forall x p(x) \vee q(a)$. Therefore $\mathcal{M}, h \models \forall x p(x) \vee q(a)$.

Consequently, $\mathcal{M} \models \forall x(p(x) \vee q(a)) \to (\forall x p(x) \vee q(a))$ and therefore $\models_{\mathbf{QHT}^s}$ $\forall x(p(x) \vee q(a)) \to (\forall x p(x) \vee q(a))$. □

3.3 Independence from the language

If $\mathcal{M} = \langle (D_h, D_t, \sigma), H, T \rangle$ is a $\mathbf{QHT}$-$\mathcal{L}'$-structure and $\mathcal{L} \subset \mathcal{L}'$, we denote by $\mathcal{M}\restriction_{\mathcal{L}}$ to the restriction of $\mathcal{M}$ to the sublanguage $\mathcal{L}$:

$$\mathcal{M}\restriction_{\mathcal{L}} = \langle (D_h, D_t, \sigma\restriction_{\mathcal{L}}), H\restriction_{\mathcal{L}}, T\restriction_{\mathcal{L}} \rangle$$

Theorem 2 *Let Π be a theory in $\mathcal{L}$ and $\mathcal{M}$ a $\mathbf{QHT}$-$\mathcal{L}'$-model of Π. Then $\mathcal{M}\restriction_{\mathcal{L}}$ is a $\mathbf{QHT}$-$\mathcal{L}$-model.*

Theorem 3 *Let $\varphi \in \mathcal{L}$ and $\mathcal{L}' \supset \mathcal{L}$. Then φ is valid (respt. satisfiable) in $\mathbf{QHT}_{\mathcal{L}}$ if and only if is valid (respt. satisfiable) in $\mathbf{QHT}_{\mathcal{L}'}$.*

Corollary 4 *Π_1 and Π_2 are equivalent in $\mathbf{QHT}_{\mathcal{L}}$ if and only if they are equivalent in $\mathbf{QHT}_{\mathcal{L}'}$.*

We have the same results for $\mathbf{QHT}^s$.

3.4 Relation to the work of [26]

In [26] quantified here-and-there logic was intruduced using a smaller class of structures: first, we take a set C' such that $C \subset C'$ and $C' \setminus C$ is infinite and denumerable; the domain is $D = \mathrm{T}(C', F)$. It is clear that $\mathrm{T}_D(C, F) = D$. So the structures used in [26] are just $\langle (D, id), H, T \rangle$.[3] In this section, we are going to prove that both semantics are equivalent.

Definition 5 *For a HT^s-$\mathcal{L}$-structure $\mathcal{M}$ the set $\mathrm{Th}(\mathcal{M})$ is defined as follows:*

$$\mathrm{Th}(\mathcal{M}) = \{\varphi \in \mathcal{L} \textit{ and sentence} \mid \mathcal{M} \models \varphi\}$$

Two HT^s-$\mathcal{L}$-structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are said (elementary) equivalent *if $\mathrm{Th}(\mathcal{M}_1) = \mathrm{Th}(\mathcal{M}_2)$, and it is denoted by $\mathcal{M}_1 \equiv \mathcal{M}_2$.*

Theorem 6 *For every HT^s-$\mathcal{L}$-structure $\mathcal{M} = \langle (D, \sigma), H, T \rangle$ there exists another HT^s-$\mathcal{L}$-structure, $\mathcal{M}' = \langle (D', id), H', T' \rangle$, such that $\mathcal{M} \equiv \mathcal{M}'$.*

Proof: We can assume that $C \cap D = \varnothing$ and D is infinite and denumerable. The new domain is defined as follows: $C' = C \cup D$, $D' = \mathrm{T}(C', F)$. The set of atoms H' is defined as follows: $H' = \bigcup_{n \in \mathbb{N}} H_n$, where $H_0 = H$ and

$$H_{i+i} = \bigcup_{\substack{p(\ldots,\overset{\frown}{\underset{i+1}{d}},\ldots) \in H_i \\ \mathrm{Ar}(p) \geq i+1}} \{p(\ldots, t, \ldots) \colon \sigma(t) = d\}$$

[3]The notation used in [26] is slightly different, it has been adapted to the nomenclature of the present work.

T' is defined in the same way. Now we must prove that $\mathcal{M} \models \varphi$ if and only if $\mathcal{M}' \models \varphi$; this is a consequence of the property: $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}', w \models \varphi$ for every w and this is easy to prove by induction over φ.$\Box$

Corollary 7 *The semantics for QHT^s introduced here is equivalent to the semantics in [26] in the following sense: validity and satisfiability agree and additionally if Γ is elementary representable in one logic it is also elementary representable in the other.*

4 Axioms for QHT

In this section we prove that the following axioms, when added to the intuitionistic calculus, characterise the logic **QHT**.

FOHT $\forall x \neg\neg\alpha(x) \to \exists x(\alpha(x) \to \forall x \alpha(x))$

We will denote by $\vdash_{\mathbf{QHT}}$ to the inference relation of $\mathbf{Int} \cup \{HOS, FOHT\}$.

The axiom FOHT is weaker than the Double Negation Shift axiom

DNS $\forall x \neg\neg\alpha(x) \to \neg\neg\forall x \alpha(x)$

which characterises the class of intuitionisic models such that, for each world w, there exists an $w' \geq w$ which is maximal [5]. This of course holds in the case of **QHT**:

$$\begin{array}{l}
\vdash_{\mathbf{Int}} \forall x \neg\neg\alpha(x) \to \exists x(\alpha(x) \to \forall x \alpha(x)) \\
\vdash_{\mathbf{Int}} \forall x \neg\neg\alpha(x) \to \exists x(\neg\neg\alpha(x) \to \neg\neg\forall x \alpha(x)) \\
\vdash_{\mathbf{Int}} \forall x \neg\neg\alpha(x) \to (\forall x \neg\neg\alpha(x) \to \neg\neg\forall x \alpha(x)) \\
\vdash_{\mathbf{Int}} \forall x \neg\neg\alpha(x) \to \neg\neg\forall x \alpha(x)
\end{array}$$

Recently, Skvortsov in [28] has studied a parametrized family of intemediate logics, including **QHT** and has provided axiomatisation for them. Additionally, we are going to prove that if the following axiom is added to the previous system, then we obtain a complete axiomatic for $\mathbf{QHT}^s$:

CD $\neg\neg\exists x \alpha(x) \to \exists x \neg\neg\alpha(x)$

Görnemann proved in [8] that the axiom

CD' $\forall x(\alpha(x) \vee \beta) \to (\forall x \alpha(x) \vee \beta)$

characterises the Kripke models whose domain is a constant map, and thus it can replace to CD in our system. We will denote by $\vdash_{\mathbf{QHT}^s}$ the inference relation of $\mathbf{Int} \cup \{HOS, FOHT, CD\}$.

Lemma 8 *The formula FOHT is valid in* $\mathbf{QHT}$ *and* $\mathbf{QHT}^s$*, and CD is valid in* $\mathbf{QHT}^s$*.*

Theorem 9 (Soundness) *If* $\Gamma \vdash_{\mathbf{QHT}} \alpha$ *then* $\Gamma \models_{\mathbf{QHT}} \alpha$. *If* $\Gamma \vdash_{\mathbf{QHT}^s} \alpha$ *then* $\Gamma \models_{\mathbf{QHT}^s} \alpha$.

To prove the completeness of the systems, we will follow the Henkin method for intuitionistic logic. For this reason some details are omitted but they can be found, for instance, in [30].

Lemma 10 *If* Γ *is a set of formulas and* φ *is another formula such that* $\Gamma \not\vdash_{\mathbf{QHT}} \varphi$*, then there exists a theory* Γ_h *in an extension of the language,* $\langle D_h, F, P\rangle$*,* $C \subset D_h$*, such that* $\Gamma \subset \Gamma_h$*,* $\Gamma_h \not\vdash_{\mathbf{QHT}} \varphi$*,* Γ_h *is closed for* $\vdash_{\mathbf{QHT}}$*, and it is* prime*, that is, it verifies the following properties: if* $\alpha \vee \beta \in \Gamma_h$*, then either* $\alpha \in \Gamma_h$ *or* $\beta \in \Gamma_h$*; and if* $\exists x\alpha(x) \in \Gamma_h$*, then* $\alpha(d) \in \Gamma_h$ *for some* $d \in D_h$*.*

For $\vdash_{\mathbf{QHT}^s}$ *we have the same property: if* $\Gamma \not\vdash_{\mathbf{QHT}^s} \varphi$*, then there exists a theory* Γ_h^s *in an extension of the language,* $\langle D_h, F, P\rangle$*,* $C \subset D_h$*, such that* $\Gamma \subset \Gamma_h$*,* $\Gamma_h \not\vdash_{\mathbf{QHT}^s} \varphi$*,* Γ_h *is closed for* $\vdash_{\mathbf{QHT}}$*, and it is* prime.

The theories Γ_h y Γ_h^s allow us to construct the here part of the (counter)-model. To define the there-part, we need a second extension and it depends of the axiomatic system. For the static version, the language is not extended.

Lemma 11 *Let us consider a set of formulas* Γ*, other formula* φ *and the extension* Γ_h^s *defined in the previous lemma. We consider also the theory* Γ_t^s *that is maximally consistent and it is closed for* $\vdash_{\mathbf{QHT}^s}$ *(it exists by the Lindenbaum lemma). Then,* Γ_t^s *is a Henkin theory, that is, for every formula* $\exists x\alpha(x)$ *in the language, there exists* $c \in D_h$ *such that* $\exists x\alpha(x) \rightarrow \alpha(c) \in \Gamma_t^s$*.*

Proof. We firstly prove the following property: if $\Gamma_t^s \vdash \alpha$, then $\Gamma_h^s \vdash \neg\neg\alpha$. If $\Gamma_t^s \vdash \alpha$, then $\Gamma_t^s \not\vdash \neg\alpha$ and $\Gamma_h^s \not\vdash \neg\alpha$; thus $\Gamma_h^s \vdash \neg\neg\alpha$, because $\neg\alpha \vee \neg\neg\alpha$ is valid en **HT** and Γ_h^s is prime.

To prove the Henkin property we considerer three cases:

1. If $\Gamma_h^s \vdash \exists x\alpha(x)$, then $\Gamma_h^s \vdash \alpha(c)$ and $\Gamma_h^s \vdash \exists x\alpha(x) \rightarrow \alpha(c)$ and therefore $\Gamma_t^s \vdash \exists x\alpha(x) \rightarrow \alpha(c)$.

2. If $\Gamma_t^s \not\vdash \exists x\alpha(x)$, then $\Gamma_t^s \vdash \neg\exists x\alpha(x)$ and:

$$
\begin{array}{ll}
\Gamma_t^s \vdash \neg\exists x\alpha(x) & \\
\quad\ \vdash \neg\exists x\alpha(x) \rightarrow \forall x\neg\alpha(x) & \text{(Int)} \\
\Gamma_t^s \vdash \forall x\neg\alpha(x) & \\
\Gamma_h^s \vdash \neg\neg\forall x\neg\alpha(x) & \\
\Gamma_h^s \vdash \neg\alpha(c) \rightarrow \forall x\neg\alpha(x) & \\
\Gamma_t^s \vdash \neg\alpha(c) \rightarrow \forall x\neg\alpha(x) & \\
\Gamma_t^s \vdash \neg\forall x\neg\alpha(x) \rightarrow \neg\neg\alpha(c) & \\
\Gamma_t^s \vdash \exists x\alpha(x) \rightarrow \alpha(c) &
\end{array}
$$

The last line is consequence of the maximality consistency of Γ_t^s.

3. If $\Gamma_h^s \nvdash \exists x\alpha(x)$ and $\Gamma_t^s \vdash \exists x\alpha(x)$, then $\Gamma_h^s \vdash \neg\neg\exists x\alpha(x)$ and:

$$
\begin{array}{ll}
\Gamma_h^s \vdash \neg\neg\exists x\alpha(x) & \\
\quad \vdash \neg\neg\exists x\alpha(x) \to \exists x\neg\neg\alpha(x) & \text{CD} \\
\Gamma_h^s \vdash \exists x\neg\neg\alpha(x) & \\
\Gamma_h^s \vdash \neg\neg\alpha(c) & \\
\Gamma_t^s \vdash \alpha(c) & \\
\Gamma_t^s \vdash \exists x\alpha(x) \to \alpha(c) & \square
\end{array}
$$

However, to extend Γ_h to a Henkin theory with respect $\vdash_{\mathbf{QHT}}$, we need to extend the language with new constants, as in classical logic.

Lemma 12 *Let us consider a set of formulas Γ, other formula φ and the extension Γ_h defined before. Then there exists a theory Γ_t^s in an extension of the language, $\langle D_t, F, P\rangle$, $C \subset D_h \subset D_t$ such that: it is closed for $\vdash_{\mathbf{QHT}}$, it is maximally consistent and it is a Henkin theory.*

We already can construct the models that we are going to use in the following lemma. For the static case $\mathcal{M} = \langle (D_h, id), H, T\rangle$, where H and T are the sets of ground atomic formulas in Γ_h^s and Γ_t^s respectively. For the non-static case, $\mathcal{M} = \langle (D_h, D_t, id), H, T\rangle$, where H and T are the sets of ground atomic formulas in Γ_h and Γ_t respectively. The following lemma and its proof is valid for both axiomatic systems, because it only use HOS and DNS and the properties of Henkin theories.

Lemma 13 *For every formula ψ, in the initial language, and $w \in \{h, t\}$, $\mathcal{M}, w \models \psi$ iff $\Gamma_w \vdash \psi$.*

Proof. We prove the lemma by induction over ψ. By the definition of H and T the result is obvious for atoms. Assume that the result is valid for any formula with degree less than n and let ψ be a formula with degree n. For $\psi = \alpha \wedge \beta$, $\psi = \alpha \vee \beta$, $\psi = \neg\alpha$ and $\psi = \exists x\alpha(x)$ the proof is as for intuitionistic logic.
For $\psi = \alpha \to \beta$:

- By the induction hypothesis, $\mathcal{M}, t \models \alpha \to \beta$ iff: if $\Gamma_t \vdash \alpha$, then $\Gamma_t \vdash \beta$. Thus: if $\Gamma_t \vdash \alpha$, then $\Gamma_t \vdash \beta$ and using the axiom $\beta \to (\alpha \to \beta)$ we conclude that $\Gamma_t \vdash \alpha \to \beta$; if $\Gamma_t \nvdash \alpha$, then $\Gamma_t \vdash \neg\alpha$ (because Γ_t is maximally consistent) and using the axiom $\neg\alpha \to (\alpha \to \beta)$ we conclude that $\Gamma_t \vdash \alpha \to \beta$. The converse is trivial.
- By the induction hypothesis, $\mathcal{M}, h \models \alpha \to \beta$ iff: ir $\Gamma_h \vdash \alpha$, then $\Gamma_h \vdash \beta$ and if $\Gamma_t \vdash \alpha$, then $\Gamma_t \vdash \beta$. We consider three cases: if $\Gamma_h \vdash \alpha$, then $\Gamma_h \vdash \beta$ and using the axiom $\beta \to (\alpha \to \beta)$ we conclude that $\Gamma_h \vdash \alpha \to \beta$; if $\Gamma_h \vdash \neg\alpha$ then with the axiom $\neg\alpha \to (\alpha \to \beta)$ we conclude that $\Gamma_h \vdash \alpha \to \beta$; finally, if $\Gamma_h \nvdash \alpha$ and $\Gamma_h \nvdash \neg\alpha$, then $\Gamma_h \vdash \neg\neg\alpha$, $\Gamma_t \vdash \alpha$, $\Gamma_t \vdash \beta$, $\Gamma_t \nvdash \neg\beta$ and $\Gamma_h \nvdash \neg\beta$ and so, using the Hosoi axiom, $\alpha \vee (\neg\beta \vee (\alpha \to \beta))$, we conclude that $\Gamma_h \vdash \alpha \to \beta$. The converse is trivial.

For $\psi = \forall x\alpha(x)$:

- With the axiom $\forall x\alpha(x) \rightarrow \alpha(s)$ and the hypothesis of induction it is immediate that: if $\Gamma_w \vdash \forall x\alpha(x)$ then $\mathcal{M}, w \models \alpha(s)$ for all $s \in \mathrm{T}(D_w, F)$.
- By the induction hypothesis, if $\mathcal{M}, t \models \forall x\alpha(x)$ then $\Gamma_t \vdash \alpha(s)$ for all $s \in \mathrm{T}(D_t, F)$; for Γ_t is a Henkin theory, $\exists x\neg\alpha(x) \rightarrow \neg\alpha(c) \in \Gamma_t$ for some $c \in D_t$ thus, using classical equivalence, $\alpha(c) \rightarrow \forall x\alpha(x) \in \Gamma_t$ and $\forall x\alpha(x) \in \Gamma_t$.
- By the induction hypothesis, if $\mathcal{M}, h \models \forall x\alpha(x)$ then $\Gamma_h \vdash \alpha(s)$ for all $s \in \mathrm{T}(D_h, F)$ and $\Gamma_t \vdash \alpha(s)$ for all $s \in \mathrm{T}(D_t, F)$. From the previous item, $\Gamma_t \vdash \forall x\alpha(x)$ and thus $\Gamma_h \not\vdash \neg\forall x\alpha(x)$ (otherwise $\Gamma_t \vdash \neg\forall x\alpha(x)$) and $\Gamma_h \vdash \neg\neg\forall x\alpha(x)$ (because $\neg A \vee \neg\neg A$ is valid in HT and Γ_h is prime);

$$\begin{array}{ll}
\Gamma_h \vdash \neg\neg\forall x\alpha(x) & \\
\quad\vdash \neg\neg\forall x\alpha(x) \rightarrow \exists x(\alpha(x) \rightarrow \forall x\alpha(x)) & \text{FOHT} \\
\Gamma_h \vdash \exists x(\alpha(x) \rightarrow \forall x\alpha(x)) & \\
\Gamma_h \vdash \alpha(c) \rightarrow \forall x\alpha(x) & \text{(It is prime)} \\
\Gamma_h \vdash \alpha(c) & \\
\Gamma_h \vdash \forall x\alpha(x) & \qquad\Box
\end{array}$$

Theorem 14 (Completeness) *If* $\Gamma \models_{\mathbf{QHT}} \varphi$ *then* $\Gamma \vdash_{\mathbf{QHT}} \varphi$. *If* $\Gamma \models_{\mathbf{QHT}^s} \varphi$ *then* $\Gamma \vdash_{\mathbf{QHT}^s} \varphi$.

Proof. The completeness follows inmediately from the previous lemma: $\mathcal{M}, h \models \psi$ for every $\psi \in \Gamma$, for $\Gamma \subset \Gamma_h$, but $\mathcal{M}, h \not\models \varphi$, for $\varphi \notin \Gamma_h$. $\Box$

In fact, we have proved an stronger version of the completeness theorem: the axiomatic systems are complete for models where the domains are extension of the constants and the assignment σ is the identity. Therefore, the general semantics introduced in this paper and that presented in[26] are equivalent with respect to validity

The axioms given here for $\mathbf{QHT}^s$ are somewhat simpler to work with than those presented earlier in [26], We give an example to show how an obviously valid statement can be derived from the axioms and inference rules.

Example: $\neg\forall x\alpha(x) \rightarrow \exists x\neg\alpha(x)$ is valid in $\mathbf{QHT}^s$. To see that it is also a theorem, we will use the following well-known valid rules and theorems in intuitionistic logic:

IR1 $A \rightarrow B \vdash \neg B \rightarrow \neg A$

IR2 $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

IT1 $A \rightarrow \neg\neg A$

IT2 $\exists x\neg\neg\neg A(x) \rightarrow \exists x\neg A(x)$

Let c be a constant that does not occur in $\alpha(x)$

1.	$\neg\alpha(c) \to \exists x \neg\alpha(x)$	I12
2.	$\neg\exists x \neg\alpha(x) \to \neg\neg\alpha(c)$	IR1: 1
3.	$\neg\exists x \neg\alpha(x) \to \forall x \neg\neg\alpha(x)$	∀I:2
4.	$\forall x \neg\neg\alpha(x) \to \neg\neg\forall x \alpha(x)$	DNS
5.	$\neg\exists x \neg\alpha(x) \to \neg\neg\forall x \alpha(x)$	IR2: 3,4
6.	$\neg\neg\neg\forall x \alpha(x) \to \neg\neg\exists x \neg\alpha(x)$	IR1: 5
7.	$\neg\forall x \alpha(x) \to \neg\neg\neg\forall x \alpha(x)$	IT1
8.	$\neg\forall x \alpha(x) \to \neg\neg\exists x \neg\alpha(x)$	IR2: 7,6
9.	$\neg\neg\exists x \neg\alpha(x) \to \exists x \neg\neg\neg\alpha(x)$	CD
10.	$\neg\forall x \alpha(x) \to \exists x \neg\neg\neg\alpha(x)$	IR2: 8,9
11.	$\exists x \neg\neg\neg\alpha(x) \to \exists x \neg\alpha(x)$	IT2
12.	$\neg\forall x \alpha(x) \to \exists x \neg\alpha(x)$	IR2: 10,11

Ono proved in [20] that the system obtained by extending the intuitionistic calculus with the axioms ŁUK and CD' is complete for $\mathbf{QHT}^s$. Another complete calculus can be obtained with the axioms ŁUK and

SQHT $\exists x(\alpha(x) \to \forall x \alpha(x))$.[4]

4.1 Adding decidable equality

The completeness result is general, so we may assume that Γ contains the identity axioms. However, we cannot assume that the interpretation of the equality symbol in the language is identity in the worlds. If the equality predicate is interpreted with the condition for every $w \in \{h,t\}$

- $\mathcal{M}, w \models t_1 = t_2$ iff $\sigma(t_1) = \sigma(t_2)$ for every $t_1, t_2 \in \mathrm{T}_D(C,F)$

then we need to consider the axiom of "decidible equality" to obtain a complete calculus[4]

DE $\forall x \forall y(x = y \vee x \neq y)$.

Below we shall consider the result of adding to $\mathbf{QHT}^s$ the usual axioms for equality plus the condition DE and we denote the resulting logic by $\mathbf{QHT}^s_=$.

5 Quantified Equilibrium Logic

We turn now to the definition of quantified equilibrium logic. We will use as a basis quantified here-and-there logic with static domains and decidable equality,

[4] This will be discussed in more detail in a forthcoming article with Vladimir Lifschitz.

ie. $\mathbf{QHT}^s_=$. For theories without equality we use simply $\mathbf{QHT}^s$. We do not exclude that for some kinds of reasoning problems the non-static variant of here-and-there might also provide a suitable base logic. However, for the purpose of obtaining a logical foundation for the answer set semantics for non-ground programs it seems that $\mathbf{QHT}^s$ or $\mathbf{QHT}^s_=$ are fully adequate.

Let us mention some additional conditions that might be applied to $\mathbf{QHT}^s$ models for specific applications. By a universal theory we mean a theory that is equivalent to a set of prenex formulas all of whose quantifiers are universal. In [26] it was noted that any theory Π can be extended to an equivalent universal theory by the addition of new (Skolem) functions and constants.

Let $\mathcal{M} = \langle (D, \sigma), H, T \rangle$ be a model of a universal theory Π. We say that:

- $\mathcal{M}$ is a PNA-model (parameter names assumption model) if the restriction of σ to terms is surjective.
- $\mathcal{M}$ is a UNA-model (unique names assumption model) if the restriction of σ to terms is injective.
- $\mathcal{M}$ is a SNA-model (standard names assumption model) if the restriction of σ to terms a bijection.

For general theories, we can apply the same notions considering the interpretation in the extended language obtained by skolemization. So, when applying equilibrium logic we can make use these restrictions, when appropriate, depending on the intended application.

Definition 15 *In the collection of* $\mathbf{QHT}^s_=$*-*$\mathcal{L}$*-structures we define the order* $\trianglelefteq$ *as follows:* $\langle (D, \sigma), H, T \rangle \trianglelefteq \langle (D', \sigma'), H', T' \rangle$ *if* $D = D'$, $\sigma = \sigma'$, $T = T'$ *and* $H \subseteq H'$.

Definition 16 *Let* Π *be a set of sentences and* $\mathcal{M}$ *a model of* Π.

1. $\mathcal{M} = \langle (D, \sigma), H, T \rangle$ *is said to be* total *if* $H = T$.
2. $\mathcal{M}$ *is said to be an* equilibrium *model of* Π *if it is minimal under* $\trianglelefteq$ *among models of* Π*, and it is total.*

Theorem 17 *Let* Π *be a theory in* $\mathcal{L}$ *and* $\mathcal{M}$ *an equilibrium model of* Π *in* $\mathcal{L}'$ *for some* $\mathcal{L}' \supset \mathcal{L}$. *Then* $\mathcal{M}\restriction_{\mathcal{L}}$ *is an equilibrium model of* Π *in* $\mathcal{L}$.

Quantified equilibrium logic (**QEL**) is the (non-monotonic) logic determined by truth in all equilibrium models. We now consider some important properties of **QEL** starting with its relation to non-ground answer set programs.

5.1 Relation to answer set programming

We consider here function-free first-order languages $\mathcal{L} = \langle C, P \rangle$ without equality and we treat non-ground disjunctive logic programs with (default) negation allowed in rule heads and bodies under the answer set semantics [17]. Syntactically, a program Π consist of sets of rules r of the form

$$a_1 \vee \ldots \vee a_k \vee \neg a_{k+1} \vee \ldots \vee \neg a_l \leftarrow b_1, \ldots, b_m, \neg b_{m+1}, \ldots, \neg b_n \tag{1}$$

where the a_i ($i \in \{1, \ldots, l\}$) and b_j ($j \in \{1, \ldots, n\}$) are atoms, called head (body, respectively) atoms of r, in $\mathcal{L} = \langle C, P \rangle$. By $C_\Pi \subseteq C$ we denote the constants appearing in Π. Note that negation in rule heads is an extension of the original answer set semantics. A rule with $k = l$ and $m = n$ is called *positive.* Rules where each variable appears in at least one positive body atom are called *safe*; a program is *safe* if all its rules are safe. We shall treat rules of form (1) as ordinary logical formulas in $\mathcal{L}$ of the form:

$$b_1 \wedge \ldots \wedge b_m \wedge \neg b_{m+1} \ldots \wedge \neg b_n \rightarrow a_1 \vee \ldots \vee a_k \vee \neg a_{k+1} \vee \ldots \vee \neg a_l \tag{2}$$

Answer set semantics in its usual form is defined via classical interpretations and the concept of program reduct. Given a language $\mathcal{L} = \langle C, P \rangle$, a (classical) $\mathcal{L}$-structure consists of a pair $\mathcal{I} = \langle U, I \rangle$, where $U = (D, \sigma)$ is the *universe* consisting of a non-empty domain D and a function σ which assigns a domain value to each element of $C \cup D$, ie $\sigma \colon C \cup D \rightarrow D$ such that $\sigma(d) = d$ for all $d \in D$. Given a universe (D, σ) we apply the terms PNA, UNA and SNA as in the previous subsection. An $\mathcal{L}$-interpretation I over D is defined as a subset of $At_D(C, P)$. The satisfaction of a formula φ in an interpretation $\mathcal{I}$ is defined in the usual way and denoted by $\mathcal{I} \models \varphi$. We can define a subset relation for $\mathcal{L}$-structures $\mathcal{I}_1 = \langle (D, \sigma_1), I_1 \rangle$ and $\mathcal{I}_2 = \langle (D, \sigma_2), I_2 \rangle$ over the same domain by setting $\mathcal{I}_1 \subseteq \mathcal{I}_2$ if $I_1 \subseteq I_2$.[5] When defining answer sets we refer to subset minimality of models understood as minimality wrt $\subseteq$ among all models over the same domain.

We now define the concept of the *grounding* of a program Π wrt to a universe:

Definition 18 *The* grounding $gr_U(\Pi)$ *of* Π *wrt to a universe* U *– sometimes called pre-interpretation – denotes the set of all rules obtained as follows: For* $r \in \Pi$*, replace (i) each constant* c *appearing in* r *with* $\sigma(c)$ *and (ii) each variable with some element in* D*.*

Observe that thus $gr_U(\Pi)$ is a ground program over the atoms in $At_D(\varnothing, P)$.

For a ground program Π and first-order structure $\mathcal{I}$, the *reduct* $\Pi^{\mathcal{I}}$ consists of all rules

$$a_1 \vee a_2 \vee \ldots \vee a_k \leftarrow b_1, \ldots, b_m$$

obtained from rules of the form (1) in Π such that $\mathcal{I} \models a_i$ for all $k < i \leq l$ and $\mathcal{I} \not\models b_j$ for all $m < j \leq n$.

Answer set semantics is usually defined in terms of *Herbrand structures* over $\mathcal{L} = \langle C, P \rangle$. Herbrand structures have a fixed universe, the so-called *Herbrand universe* $\mathcal{H} = (C, id)$, where id is the identity function. For a Herbrand structure

[5] Note that this is not the substructure or submodel relation in classical model theory, which holds between a structure and its restriction to a subdomain.

$\mathcal{I} = \langle \mathcal{H}, I \rangle$, I can be viewed as a subset of the *Herbrand base*, $\mathcal{H}_B$, which consists of the ground atoms of $\mathcal{L} = \langle C, P \rangle$. Note that by definition of $\mathcal{H}$, Herbrand structures are SNA-structures.

Definition 19 *Let* Π *be a logic program over* $\mathcal{L}$.[6] *An Herbrand structure* $\mathcal{I}$ *is called an* answer set *of* Π *if* $\mathcal{I}$ *is subset minimal among the structures satisfying all rules in* $gr_{\mathcal{H}}(\Pi)^{\mathcal{I}}$ *.*

A variation of this semantics, the open answer set semantics, considers open domains [11], thereby relaxing the PNA:[7] An *extended Herbrand structure* is a first-order structure based on a universe $U = (D, id)$, where $D \supseteq C$. Note that since the assignment is the identity function, the UNA applies.

Definition 20 *An extended Herbrand structure* $\mathcal{I} = \langle U, I \rangle$ *is called an* open answer set *of* Π *if* $\mathcal{I}$ *is subset minimal among the structures satisfying all rules in* $gr_U(\Pi)^{\mathcal{I}}$*.*

The next variant of answer set semantics we discuss here, introduced by Heymans et al. [12], relaxes the UNA, ie. arbitrary first-order $\mathcal{L}$-structures are allowed:

Definition 21 *A first-order* $\mathcal{L}$*-structure* $\mathcal{I} = \langle U, I \rangle$ *is called a* generalised open answer set *of* Π *if* $\mathcal{I}$ *is subset minimal among the structures satisfying all rules in* $gr_U(\Pi)^{\mathcal{I}}$*.*

We now relate the different notions of answer set to equilibrium models. First we recall the basic property of equilibrium models for propositional theories. In the propositional case we represent here-and-there models as pairs $\langle H, T \rangle$ of sets of atoms with $H \subseteq T$. The ordering $\trianglelefteq$ among models is defined as above where now only the sets H and T are considered. Equilibrium models of a propositional theory Π are then total models (ie. where $H = T$) that are $\trianglelefteq$-minimal among **HT**-models of Π [21].

Proposition 22 *[21] Let* Π *be a ground logic program. A total model* $\langle T, T \rangle$ *is an equilibrium model of* Π *iff* T *is an answer set of* Π*.*

We return now to the first-order case. When we interpret non-ground logic programs in $\mathbf{QHT}^s$ or $\mathbf{QHT}^s_=$ we assume that all program rules of form (2) are universally quantified wrt to the variables appearing in them. In other words we treat the universal closure of the program. The following is adapted from [26].

Lemma 23 *[26] Let* Π *be a logic program over* $\mathcal{L} = \langle C, P \rangle$*. Then a* $\mathbf{QHT}^s$*-*$\mathcal{L}$*-structure* $\langle U, H, T \rangle$ *is a model of the univesral closure of* Π *iff* $\langle H, T \rangle$ *is a propositional* **QHT** *model of* $gr_U(\Pi)$*.*

[6] We assume here that all constants of $\mathcal{L}$ appear in Π, ie $C_\Pi = C$.

[7] Here and in the next two definitions we follow the terminology of [2].

This gives us immediately the following proposition, also a slight adaption of a theorem of [26].

Proposition 24 *Let* Π *be a logic program over* $\mathcal{L} = (C, P)$. *Let* $\langle U, T, T\rangle$ *be a total* $\mathbf{QHT}^s$*-model of the universal closure of* Π. *Then* $\langle U, T, T\rangle$ *is an equilibrium model of* Π *iff* $\langle T, T\rangle$ *is a propositional equilibrium model of* $gr_U(\Pi)$.

The relations to answer sets are now straightforward.

Proposition 25 *Let* Π *be a logic program over* $\mathcal{L} = \langle C, P\rangle$. *A total Herbrand model* $\langle U, T, T\rangle$ *of the universal closure of* Π *is an equilibrium model of* Π *iff* T *is an answer set of* Π.

Proof. Immediate from Definition 19 and Proposition 22. □

And from Proposition 24 we obtain:

Corollary 26 *Let* Π *be a logic program. A total* $\mathbf{QHT}^s_=$*-model* $\langle U, T, T\rangle$ *of* Π *is an equilibrium model of* Π *iff* $\langle U, T\rangle$ *is a generalised open answer set of* Π.

Proof. We note first that in the definition of (generalised) open answer set [11, 12] equality is interpreted as in the manner of Section 4.1 above, so we can apply static here-and-there with decidable equality. Now consider Definition 21. Evidently if $\langle U, T\rangle$ is a generalised open answer set of Π, then T is a propositional answer set of $gr_U(\Pi)$. By Propositions 22 and 24 it follows that $\langle U, T, T\rangle$ is an equilibrium model of Π. The converse direction is similar. □

Corollary 27 *Suppose the unique name assumption holds, so that all* $\mathbf{QHT}^s_=$*-structures and equilibrium models are UNA-structures. Let* Π *be a logic program. A total* $\mathbf{QHT}^s_=$*-model* $\langle U, T, T\rangle$ *of* Π *is an equilibrium model of* Π *iff* $\langle U, T\rangle$ *is an open answer set of* Π.

Proof. Similar to Corollary 26 except that we apply the UNA. □

The assumptions of Corollary 27 correspond to the version of equilibrium logic presented in [26].

5.2 Strong equivalence

The concept of strong equivalence for logic programs under answer set semantics and propositional theories in equilibrium logic was investigated in [16]. Subsequently, the study of strong equivalence and its variants has become a popular theme in ASP. Since strongly equivalent programs are inter-substitutable in all contexts, the concept is relevant for program optimisation and modularity.

Let Π_1 and Π_2 be two theories in $\mathcal{L}$. They are said to be *equivalent* if they have they have the same equilibrium models and *strongly equivalent* if for every theory Π in $\mathcal{L}'$, with $\mathcal{L}' \supseteq \mathcal{L}$, $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ have the same equilibrium models.

Theorem 28 *Let* Π_1 *and* Π_2 *be two theories (resp. theories with equality) in* $\mathcal{L}$. *They are strongly equivalent if and only if they are equivalent in* $\mathbf{QHT}^s$ *(resp.* $\mathbf{QHT}^s_=$*).*

Proof: With respect to $\mathbf{QHT}^s$ (resp. $\mathbf{QHT}^s_=$), if Π_1 and Π_2 are equivalent in $\mathcal{L}$ then they are equivalent in $\mathcal{L}'$ and thus $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ are equivalent.

Let us assume that $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ have the same equilibrium models for every theory Π in $\mathcal{L}'$ with $\mathcal{L}' \supseteq \mathcal{L}$. Suppose that $\mathcal{M} = \langle (D,\sigma), H, T\rangle$ is a model of Π_1 which is not a model of Π_2 and let us consider $\mathcal{M}' = \langle (D,\sigma), T, T\rangle$.

1. If $\mathcal{M}'$ is not a model of Π_2 it cannot be model of $\Pi_2 \cup T$ but this is impossible because $\mathcal{M}'$ is clearly an equilibrium model of $\Pi_1 \cup T$.

2. If $\mathcal{M}'$ is a model of Π_2 we take:

$$\Pi = H \cup \{\alpha \rightarrow \beta \colon \alpha, \beta \in T \smallsetminus H, \alpha \neq \beta\}$$

   Trivially, $\mathcal{M}'$ is a model of $\Pi_2 \cup \Pi$; moreover, it is an equilibrium model: if $\langle (D,\sigma), J, T\rangle \models \Pi_2 \cup \Pi$ with $J \subset T$, then $H \subset J$; if we take $\alpha \in J \smallsetminus H$ and $\beta \in T \smallsetminus J$, then $\alpha \rightarrow \beta \in \Pi$; but $\langle (D,\sigma), J, T\rangle \not\models \alpha \rightarrow \beta$, which is contradictory with $\langle (D,\sigma), J, T\rangle \models \Pi_2 \cup \Pi$. But in this case we obtain a contradiction, because $\mathcal{M}'$ is not an equilibrium model of $\Pi_1 \cup \Pi$, for $\mathcal{M} \lhd \mathcal{M}'$ and $\mathcal{M} \models \Pi_1 \cup \Pi$.

5.2.1 Strong equivalence for logic programs

We note first that if Π_1 and Π_2 are logic programs that are not strongly equivalent, the proof of Theorem 28 shows that there exists a set of program rules Π (of a simple form) such that $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ have different equilibrium models. Since by Proposition 21 equilibrium models coincide with generalised open answer sets, our definition and characterisation of strong equivalence, when applied to logic programs, captures the following notion of strong equivalence: two programs Π_1 and Π_2 are strongly equivalent iff $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ have the same generalised open answer sets, for any set of program rules Π.

Now it turns out that for logic programs without equality, if we replace generalised open answer sets by open answer sets, we capture precisely the same concept of strong equivalence.

Proposition 29 *Two logic programs Π_1 and Π_2 without equality over language $\mathcal{L}$ are strongly equivalent wrt open answer sets iff they are strongly equivalent wrt generalised open answer sets. Logical equivalence in $\mathbf{QHT}^s$ captures both notions.*[8]

Proof. Evidently if Π_1 and Π_2 have the same QHT^s-models they are strongly equivalent wrt either concept of answer set. Suppose that $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ have the same open answer sets for every theory Π in $\mathcal{L}'$ with $\mathcal{L}' \supseteq \mathcal{L}$. By Proposition 27 they have the same equilibrium models under the unique name assumption, UNA. As in the proof of Theorem 28 suppose that $\mathcal{M} = \langle (D,\sigma), H, T\rangle$ is a model of Π_1 which is not a model of Π_2. By Theorem 6 there is an equivalent

[8]The reason for the restriction to programs without equality lies in our use of Theorem 6, which does not hold for the decidable version of equality.

model $\mathcal{N} = \langle (D', \sigma'), H', T' \rangle$ where $\sigma' = id$. Clearly the restriction of σ' to $\mathcal{L}$-terms is an injection, so $\mathcal{N}$ is a UNA model. Now $\mathcal{N}$ is not a model of Π_2 and we continue as in the proof of Theorem 28 by considering $\mathcal{N}' = \langle (D', \sigma'), T', T' \rangle$. In the remainder of the proof we need only refer to models with the same universe (D', σ'), so we work entirely under the UNA assumption, showing that if Π_1 and Π_2 are not QHT^s equivalent they are not strongly equivalent wrt open answer sets. □

Strong equivalence for non-ground programs has also been defined and studied in [18, 3]. In the case of [3] the concept is similar to the one presented here except that the equivalence is with respect to ordinary answer sets rather than open answer sets and equality is not explicitly treated. In general the two concepts are different since not every open answer set need be an answer set. However for the safe programs studied exclusively in [3], ordinary and open answer sets do coincide, as shown for example in Proposition 3 of [2]. So we obtain:

Corollary 30 *Consider only safe rules and programs. Then* Π_1 *and* Π_2 *are strongly equivalent in the sense of [3] if and only if they are equivalent in the logic* $\mathbf{QHT}^s$.

Proof. Suppose that Π_1 and Π_2 are safe programs. Clearly, if they are $\mathbf{QHT}^s$-equivalent they are strongly equivalent. If they are not $\mathbf{QHT}^s$-equivalent we construct a set of rules Π as in the proof of Proposition 29. Since these rules are ground (in an expanded language) they are trivially safe. Therefore $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ are safe programs whose open answer sets coincide with their answer sets. So by Proposition 29 Π_1 and Π_2 are not strongly equivalent in the sense of [3]. □

6 Adding Strong Negation

The logic obtained adding a new connective, $\sim$, called *strong negation* and the following axioms to intuitionistic logic is known as Nelson's logic and it is denoted by **N**:

N1 $\sim(\alpha \rightarrow \beta) \leftrightarrow \alpha \wedge \sim\beta$

N2 $\sim(\alpha \wedge \beta) \leftrightarrow \sim\alpha \vee \sim\beta$

N3 $\sim(\alpha \vee \beta) \leftrightarrow \sim\alpha \wedge \sim\beta$

N4 $\sim\neg\alpha \leftrightarrow \alpha$

N5 $\sim\sim\alpha \leftrightarrow \alpha$

N6 $\sim\exists x\alpha \leftrightarrow \forall x\sim\alpha$

N7 $\sim\forall x\alpha \leftrightarrow \exists x\sim\alpha$

N8 $\sim\alpha \rightarrow \neg\alpha$, for atomic α

This system is taken from the calculus of Vorob'ev [31, 32] and actually, the axiom N8 is valid for every formula α and not only for atoms. If α is deduced from a set of formulas Γ in this system, we write $\Gamma \vdash_{\mathbf{N}} \alpha$. There are other equivalent systems; for instance, the axiom N8 can be replaced by the following one:

$$\alpha \to (\sim\alpha \to \beta)$$

To extend the Kripke semantics for this logic, the idea is to take the usual Kripke models for intuitionistic logic but to allow for sentences to be not only constructively verified at possible worlds or stages of the model, but also constructively falsified (equivalently their strong negations are verified). First we need to introduce the notion of literal: a *literal* in the propositional language is either a propositional variable or the strong negation of a propositional variable and a ground *literal* in the first order language is either a ground atomic formula or the strong negation of a ground atomic formula. We denote by $\mathrm{Lit}_D(C, F, P)$ the set of ground *literals* in the language $\langle C \cup D, F, P\rangle$. If L is an atom, we say that $\sim L$ is the *contrary* of L and *vice versa.*

In the Nelson logic, a structure $\mathcal{M}$ is a tuple

$$\mathcal{I} = \langle (W, \leq), (\{D_w\}_{w\in W}, \sigma), \{I_w\}_{w\in W}\rangle$$

with the same conditions as in intuitionistic logic but, in this case, for every $w \in W$, $I_w \subseteq \mathrm{Lit}_{D_w}(\varnothing, \varnothing, P)$ and $I_w \subseteq I_{w'}$ if $w \leq w'$. The satisfiability relation is extended to the formulas with strong negation as follows:

- $\mathcal{M}, \omega \models \sim(\varphi \wedge \psi)$ iff $\mathcal{M}, \omega \models \sim\varphi$ or $\mathcal{M}, \omega \models \sim\psi$.
- $\mathcal{M}, \omega \models \sim(\varphi \vee \psi)$ iff $\mathcal{M}, \omega \models \sim\varphi$ and $\mathcal{M}, \omega \models \sim\psi$.
- $\mathcal{M}, \omega \models \sim(\varphi \to \psi)$ iff $\mathcal{M}, \omega \models \varphi$ and $\mathcal{M}, \omega \models \sim\psi$.
- $\mathcal{M}, \omega \models \sim\neg\psi$ iff $\mathcal{M}, \omega \models \varphi$.
- $\mathcal{M}, \omega \models \sim\sim\psi$ iff $\mathcal{M}, \omega \models \varphi$.
- $\mathcal{M}, \omega \models \sim\forall x\varphi(x)$ iff $\mathcal{M}, \omega \models \sim\alpha(t)$ for some $t \in \mathrm{T}(D(\omega), \mathcal{F})$.
- $\mathcal{M}, \omega \models \sim\exists x\varphi(x)$ iff $\mathcal{M}, \omega' \models \sim\alpha(t)$ for every $\omega' \geq \omega$ and every $t \in \mathrm{T}(D(\omega'), \mathcal{F})$,

If the axioms N1-N8 are added to the the system **QHT** we obtain quantified here-and-there logic with strong negation and non-static domains, denoted by $\mathbf{QN}_5$. The structures for this logics are tuples $\mathcal{I} = \langle (D_h, D_t, \sigma), I_h, I_t\rangle$ where $I_h \subseteq \mathrm{Lit}_{D_h}(C, F, P)$, $I_t \subseteq \mathrm{Lit}_{D_t}(C, F, P)$ and $I_h \subseteq I_t$.

Finally, if the axioms N1-N8 are added to the system $\mathbf{QHT}^s$ we obtain quantified here-and-there logic with strong negation and static domains, denoted by $\mathbf{QN}_5^s$. The structures for this logics are tuples $\mathcal{I} = \langle (D, \sigma), I_h, I_t\rangle$ where $I_h \subseteq I_t \subseteq \mathrm{Lit}_D(C, F, P)$.

The soundness of the axiomatic systems for $\mathbf{QN}_5$ and $\mathbf{QN}_5^s$ is straighforward and completeness is proved using the method of Vorov'eb and Gurevich in [9] which we recall in the following lemma.

Lemma 31 *Assume that φ is in negation-normal form (ie where strong negation is driven-in to stand directly before atoms) and its predicate symbols are among $P_1, \ldots, P_k$; let $Q_1, \ldots, Q_k$ be other different predicate symbols such that P_i and Q_i have the same arity for every i. We define the formulas Φ and $\tau(\varphi)$ as follows: $\Phi = \bigwedge_{i=1}^{n} \forall \vec{x}(Q_i(\vec{x}) \rightarrow \neg P_i(\vec{x}))$; $\tau(\varphi)$ is the result of uniformly replacing each ocurrence of $\sim P_i$ in φ by Q_i. Then*

1. *If $\vdash_{\mathbf{QHT}} \Phi \rightarrow \tau(\varphi)$, then $\vdash_{\mathbf{QN}_5} \varphi$.*
2. *Conversely, if $\vdash_{\mathbf{QN}_5} \varphi$ and $P_1, \ldots, P_k$ are the predicate symbols in the proof, then $\vdash_{\mathbf{QHT}} \Phi \rightarrow \tau(\varphi)$.*

The proof follows step by step the proof of the corresponding result in [9]. From the lemma and the completeness of **QHT** the completeness of $\mathbf{QN}_5$ follows straighforwardly.

Theorem 32 (Soundness and Completeness)
$\Gamma \models_{\mathbf{QN}_5} \alpha$ *iff* $\Gamma \vdash_{\mathbf{QN}_5} \alpha$.
$\Gamma \models_{\mathbf{QN}_5^s} \alpha$ *iff* $\Gamma \vdash_{\mathbf{QN}_5^s} \alpha$.

We obtain a strong negation extension of quantified equilibrium logic by applying the usual equilibrium construction to $\mathbf{QN}_5^s$-models. This version, with the appropriate restrictions, captures the various concepts of answer set for non-ground programs when the second, strong negation is present in the language. The results on strong equivalence extend accordingly.

7 Conclusions

We have presented a new axiomatisation of the first order logic of here-and-there, for non-static and static domains. Based on the latter we defined quantified equilibrium logic **QEL** without the unique names assumption, UNA. We showed that in this version equilibrium logic captures the semantics of generalised open answer sets for non-ground logic programs due to [12], while the earlier version of **QEL** with UNA presented in [26] captures the open answer sets of [11].

We gave a definition of strong equivalence for theories in **QEL** and showed that this coincides with logical equivalence in $\mathbf{QHT}^s$ (resp. $\mathbf{QHT}^s_=$). Moreover we could infer that where equality is not present this characterisation continues to hold for the strong equivalence of logic programs wrt open answer sets and, in the case of safe programs and rules, wrt ordinary answer sets, precisely the concept considered in [3].

8 Acknowledgment

We are highly indebted to Vladimir Lifschitz for valuable discussions on the topic of this paper and very helpful remarks on an earlier draft of this report.

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