On Hoffman's celebrated cycling LP example ⋆

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Sincerely dedicated to Alan J. Hoffman on the 50th birthday of his example.

Abstract

In this short note we answer two questions that naturally arise while dealing with Hoffman's celebrated 50-years-old cycling example for the primal simplex method to solve linear programs, where an angle θ and a scaling factor ω are adjustable parameters in his example. In particular, we determine what conditions have to be imposed on ω for cycling to occur with $\theta = 2\pi/5$, and what on θ with $|\omega| = \tan(\theta)$.

Key words: linear programming, simplex method, cycling

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1 A motivating historical misunderstanding

Hoffman's celebrated 50-years-old cycling example is a linear program for which cycling occurs when Dantzig's primal simplex method (cf. [1]) with certain pivoting rule is used to solve it. There are two adjustable parameters involved, namely an angle θ and a scaling factor ω. From now on, we shall shorten the following trigonometric expressions as

$$c \doteq \cos(\theta), s \doteq \sin(\theta), t \doteq s/c, \quad \text{and} \quad C \doteq \cos(2\theta), S \doteq \sin(2\theta), T \doteq S/C.$$

Using this notation, Hoffman's example can be stated as follows:

$$\begin{array}{lll} (D) & \max & -[0,0,0,(c-1)/c,\omega,0,2\omega,4s^2,-2\omega C,4s^2,\omega(1-2c)]y, \\ & \text{s.t} & \begin{bmatrix} 1\,0\,0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0\,1\,0 & c & -\omega c & C & -2\omega c^2 & C & 2\omega c^2 & c & \omega c \\ 0\,0\,1 & \frac{st}{\omega} & c & \frac{2s^2}{\omega} & C & -\frac{2s^2}{\omega} & C & -\frac{st}{\omega} & c \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ & y \in \mathbb{R}^{11}, \; y \geq \mathrm{O}. & \end{array}$$

Note that we have used the following unsymmetric primal-dual pair of linear programs with a non-standard notation (and that we have deliberately exchanged the usual roles of b and c, x and y, n and m, and (P) and (D), as e.g. in [9, §2]):

$$\begin{array}{llll} (P) & \min c^T x \quad , \; x \in \mathbb{R}^n & (D) & \max b^T y \quad , \; y \in \mathbb{R}^m \\ & \text{s.t} \;\; A^T x \geq b & & \text{s.t} \;\; Ay = c \; , \; y \geq \mathrm{O}, \end{array}$$

where $A \in \mathbb{R}^{n \times m}$ with $m \geq n$ and $\operatorname{rank}(A) = n$. We denote with $\mathcal{F}$ and $\mathcal{G}$ the feasible region of (P) and (D), respectively. The notation used here is

$$r_j \doteq a_j^T x - b_j, \quad x \doteq A_{\mathcal{B}}^{-T} b_{\mathcal{B}}, \quad \text{and} \quad y_{\mathcal{B}} \doteq A_{\mathcal{B}}^{-1} c,$$

where a_j is the jth column of A. As usual, $\mathcal{B}$ is the *ordered* index set of basic variables and $\mathcal{N}$ that of the non-basic ones, and the current iteration is noted by a superindex $^{(k)}$ when it is not clear from context. We assume that the reader is capable of applying *to problem* (D) the primal simplex algorithm (starting from a vertex $y \in \mathcal{G}$ such that $|\mathcal{B}| = n$) and we do not repeat it here.

As Alan recently confirmed to us, his original report [6] had not been published until August 2003, date in which a collection of his selected papers done by Micchelli [8] has seen the light of day. Hence the usual source of information has been Dantzig's classical [2, pp. 228–229], although Alan told us that his example appeared in the first editions of Gass' book [4]. It has also been appeared far more recently, in Dantzig and Thapa's book [3, pp. 149–151] and Lee's paper [7, p. 99], but the key point is that $\theta = 2\pi/5$ and $\omega > (1-c)/(1-2c)$ is the unique requisite in all its published occurrences. We are interested in particularizing Hoffman's example to the case $|\omega| = t$, because it is the only value of $\omega \in \mathbb{R}$ such that $\|a_j\|_2$ is constant for all $j \in 1{:}\, m$, an important feature to design a cycling example for certain sparse linear programming algorithm in which we have been involved for several years [5].

Let us tell how our misunderstanding arose, because it constitutes the main motivation of this work. In Hoffman's example we have to apply the following rules when operating with the primal simplex method:

(1) The entering variable p is the non-basic whose reduced cost is minimum.

(2) The leaving variable q is determined with the classical min-ratio test, with ties resolved in favour of the eligible index appearing leftmost in $\mathcal{B}$.
(3) Variables p and q are interchanged in the corresponding ordered sets.

For $\theta = 2\pi/5$, t is greater than $(1-c)/(1-2c)$ but surprisingly there was no cycle, for the negative reduced costs r_j in iteration 1 are those corresponding to y_5, y_6 and y_8 but the most negative one is not r_5. This led us to think that there must be some typo or even to doubt whether expressions as $\tan\theta/\omega$ and the like (as appear in [2, p. 229] and [3, pp. 150–151]) should be interpreted as $\tan(\theta)/\omega$ or as $\tan(\theta/\omega)$, but Lee's article [7] (that we were not aware of and to which Alan kindly called our attention) confirmed that the former was the right interpretation. Moreover, Alan also wrote us

> *I simply underestimated how big ω had to be. This was called to the attention of Saul Gass when he published the example in his book on linear programming, Saul told me, but I didn't do anything about it.*

so our interpretation "for *all* $\omega > (1-c)/(1-2c)$" was wrong, and the right interpretation must be "for *some* $\omega > (1-c)/(1-2c)$". This fact, that was also noticed by Saul Gass, lead us to the conclusion that additional conditions have to be imposed for cycling to occur.

The aim of this short note is to answer the following questions (that naturally arise and that have not been addressed before, to the best of our knowledge):

(1) What conditions have to be imposed on ω for cycling to occur in Hoffman's cycling example with $\theta = 2\pi/5$?
(2) What conditions have to be imposed on θ for cycling to occur in Hoffman's cycling example with $|\omega| = \tan(\theta)$?

We shall address the fomer in §2, and the latter in §3.

2 Answering the first question: θ set up to $2\pi/5$

In Hoffman's cycling example, primal simplex method is started from $\mathcal{B}^{(0)} = [1, 2, 3]$, thus $x^{(0)} = [0; 0; 0]$, $r^{(0)} = -b$ and $y_{\mathcal{B}}^{(0)} = c$. Using the criterion given in §1, the basis in the kth iteration ($k \in \mathbb{N}$) has to be

$$\mathcal{B}^{(k)} = \left[1, 2\left(\left\lceil \frac{k}{2} \right\rceil \mod 5 + 1\right), 2\left(\left\lfloor \frac{k}{2} \right\rfloor \mod 5 + 1\right) + 1\right],$$

hence $\mathcal{B}^{(10)} = \mathcal{B}^{(0)}$ and Hoffman's cycle takes place. The tableaux with $\theta = 2\pi/5$ for iterations 0, 1 and 2 can be found in [2, p. 229] and [3, pp. 150–151], so we do not repeat them here.

When $\theta = 2\pi/5$ we have $t = \sqrt{5 + 2\sqrt{5}} \approx 3.08$. In iteration 0, the condition for $r_4 = (c-1)/c$ to be the most negative reduced cost is $\omega > 0$. In fact, if such condition holds then r_4 is the only negative reduced cost. In iteration 1, there are three negative reduced costs with $\omega > 0$, namely $r_5 = \omega(2c-1)/c$, $r_6 = -2st$ and $r_8 = (c-1)/c$; for $r_5 < r_8$ the usual condition $\omega > (1-c)/(1-2c) = (5+\sqrt{5})/4 \approx 1.81$ must hold, but for $r_5 < r_6$ it turns out that

$$\omega > 2s^2/(1-2c) = 5/2 + \sqrt{5} \approx 4.74.$$

Therefore, $\omega > 2s^2/(1-2c)$ —instead of $\omega > (1-c)/(1-2c)$— is the condition for cycling to occur with $\theta = 2\pi/5$. It is an astonishing thing that such a condition for y_5 to be the entering variable in iteration 1 has been in the shadow for fifty years! Anyway, $|\omega| = t$ does not satisfy this inequality, hence our interest in the second question.

3 Answering the second question: $|\omega|$ set up to $\tan(\theta)$

Firstly, note that we have to recompute the tableaux from [2] or [3] because they are only valid for $\theta = 2\pi/5$. Now let us show what conditions arise on θ.

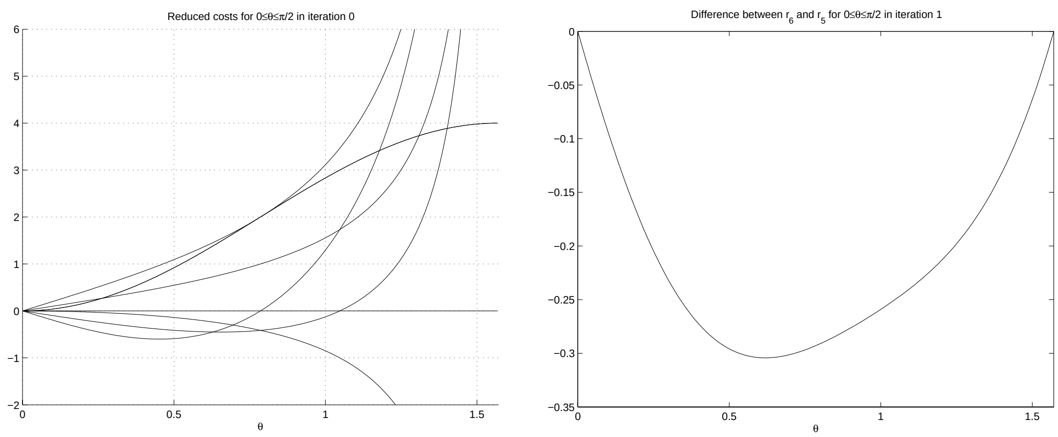

Fig. 1. Graphs to help in the study of reduced costs

With $\omega = t$, we can restrict ourselves to $\theta \in [0, \pi/2]$, because $r_4 \not< 0$ if $\theta \in [\pi/2, 3\pi/2]$ and $r_5 < r_4$ if $\theta \in [3\pi/2, 2\pi]$ in iteration 0. As shown in the left-hand side of Figure 1, the negative reduced costs in iteration 0 are $r_4 = (c-1)/c$, $r_9 = -2Ct$ and $r_{11} = t(1-2c)$. There is a value of θ from which r_4 is the most negative reduced cost, and such a value is $\pi/4 \approx 0.79$. But in iteration 1, it turns out that $x^{(1)} = [0; (1-c)/c^2; 0]$ and (see the right-hand side of Figure 1)

$$\frac{C(1-c)}{c^2} = r_6 < r_5 = \frac{s(2c-1)}{c^2}, \qquad \forall \theta \in [0, \pi/2],$$

so y_5 is not the entering variable in iteration 1. A totally analogous reasoning can be done with $\omega = -t$ for $\theta \in [3\pi/2, 2\pi]$ in this case, and then we can conclude that $|\omega| = t$ entails no cycling for any value of θ.

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References

[1] G. B. Dantzig. Maximization of a linear function of variables subject to linear inequalities. In T. C. Koopmans, ed., *Activity Analysis of Production and Allocation*, Monograph no. 13 of *Cowles Commision for Research in Economics*, pages 339–347. John Wiley and Sons, London, UK, 1951.

[2] G. B. Dantzig. *Linear Programming and Extensions*. Princeton University Press, Princeton, NJ, USA, 1963.

[3] G. B. Dantzig and M. Thapa. *Linear Programming II: Theory*. Springer-Verlag, Berlin, Germany, 2003.

[4] S. I. Gass. *Linear Programming: Methods and Applications*. McGraw-Hill, New York, NY, USA, third edition, 1969.

[5] P. Guerrero-García. *Range-Space Methods for Sparse Linear Programs* (Spanish). Ph.D. thesis, Dept. Matemática Aplicada, Univ. Málaga, Spain, July 2002.

[6] A. J. Hoffman. Cycling in the simplex algorithm. Tech. report 2974, National Bureau of Standards, Gaithersburg, USA, December 16th, 1953. Also in [8].

[7] J. Lee. Hoffman's circle untangled. *SIAM Rev.*, 39(1):98–105, March 1997.

[8] C. A. Micchelli, editor. *Selected Papers of Alan J. Hoffman*. World Scientific Publishing, New Jersey, USA, August 2003.

[9] M. R. Osborne. *Finite Algorithms in Optimization and Data Analysis.* John Wiley and Sons, Chichester, 1985.