Set functors and $L$-fuzzy set categories: towards a fuzzy programming paradigm

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Abstract. In this work we generalise previous constructions of fuzzy set categories, introduced in [5], by considering \(L\)-fuzzy sets in which the values of the characteristic functions run on a completely distributive lattice, rather than in the unit real interval. Later, these \(L\)-fuzzy sets are used to define the \(L\)-fuzzy categories, which are proven to be rational. In the final part of the paper, it is shown that the \(L\)-fuzzy functors given by the extension principles can be extended to a monad.

1 Introduction

Fuzziness is more the rule than the exception in practical problems, for usually there is no well-defined best solution for a given problem. Fuzzy set theory is based on the idea that many non-mathematical properties cannot be adequately described in terms of crisp sets comprising those elements that fulfill a given property. Therefore the notion of membership is considered as a gradual property for fuzzy sets.

Several heuristic approaches have been suggested to extend logic programming to the fuzzy case. However, the lack of a foundational base is an obstacle for a wider acceptance of these models, and further, formal approaches typically build upon conventional terms. For instance, restricting to finitely many truth values, a many-valued predicate calculus was proposed in [6].

This paper is motivated by the use of categorical methods in many-valued logic programming, our long term goal being the generalisation of the categorical unification algorithm given by Rydeheard and Burstall in [9]. Specifically, the generalisation of terms can be achieved by composing monads, for in the classical case most general unifiers are coequalisers in the Kleisli category associated with the term monad. Following this idea, our approach to the fuzzification of a “set of terms” will be considering a “fuzzy set” of terms”, therefore the different generalisations of the powerset functor become important, specially if they can be extended to a monad.

In [3] it was shown how set functors can be composed to providing monads, and some motivation to investigate techniques for constructing new monads from given ones was presented. In this work we introduce a number of set functors, which extend the crisp powerset functor, together with their extension principles. Then, \(L\)-fuzzy set categories are defined for each of these extended powerset functors and the rationality of the extension principle is proved in the categorical sense, ie the associated \(L\)-fuzzy set categories are equivalent to the category of sets and mappings. Finally, in the last section, it is shown that each of these new set functors can be extended to a monad.

2 Preliminary definitions

Recall the standard definition of the (crisp) powerset of a set \(X\) by means of characteristic functions

\[
\mathcal{P}(X) = \{A \mid A \subseteq X\} = \{A \mid A: X \rightarrow \{0,1\}\}
\]

With this definition, each mapping \(A: X \rightarrow \{0,1\}\) defines a subset of \(X\) as the inverse image \(A^{-1}(1)\).

This definition can be relaxed by allowing each element to have a degree of membership, using the real unit interval as the codomain of
the extended characteristic functions, that is, the fuzzy power set is defined as:
\[ \mathcal{F}(X) = \{ A \mid A: X \to [0, 1] \} \]

Goguen [4] further generalizes this construction by allowing the range of these extended characteristic functions to be a completely distributive lattice, and defining the L-fuzzy power set as follows:
\[ \mathcal{L}(X) = \{ A \mid A: X \to L \} \]

### Extension principles

Given \( X, Y \in \text{Set} \) and a mapping \( f: X \to Y \), it is possible to define a mapping between the corresponding power sets \( f: \mathcal{P}(X) \to \mathcal{P}(Y) \) by means of the direct image of \( f \), that is, given \( A \in \mathcal{P}(X) \) then \( f(A) = f(A) \in \mathcal{P}(Y) \).

The extension of \( f \) given above admits different generalizations when we are working in the fuzzy case according to the optimistic or pessimistic interpretation of the fuzziness degree:

1. Maximal extension principle:
   \[ \tilde{f}_{M}: \mathcal{F}(X) \to \mathcal{F}(Y) \] is defined such that given \( A \in \mathcal{F}(X) \) then \( \tilde{f}_{M}(A)(y) = \sup \{ A(x) \mid x \in f^{-1}(y) \text{ and } A(x) > 0 \} \) if the set is non-empty and \( \tilde{f}_{M}(A)(y) = 0 \) otherwise.

2. Minimal extension principle:
   \[ \tilde{f}_{m}: \mathcal{F}(X) \to \mathcal{F}(Y) \] is defined such that given \( A \in \mathcal{F}(X) \) then \( \tilde{f}_{m}(A)(y) = \inf \{ A(x) \mid x \in f^{-1}(y) \text{ and } A(x) > 0 \} \) if the set is non-empty and \( \tilde{f}_{m}(A)(y) = 0 \) otherwise.

It is straightforward to show that both extensions \( f_{M} \) and \( f_{m} \) coincide with the direct image extension in the case of crisp subsets, that is, given \( A \in \mathcal{P}(X) \subset \mathcal{F}(X) \), then \( \tilde{f}_{M}(A) = \tilde{f}_{m}(A) = f(A) \in \mathcal{P}(Y) \subset \mathcal{F}(Y) \).

The maximal and minimal extension principles just introduced can be further generalised to the L-fuzzy power sets, just changing the calculations of suprema and infima by the lattice join and meet operators. In the following, we will use the set \( I = \{ x \in X \mid x \in f^{-1}(y) \text{ and } A(x) > 0 \} \):

1. **Maximal L-fuzzy extension principle:**
   \[ \tilde{f}_{M}: \mathcal{L}(X) \to \mathcal{L}(Y) \] is defined in such a way that given \( A \in \mathcal{L}(X) \) then \( \tilde{f}_{M}(A)(y) = \bigvee_{I} A(x) \) if the set \( I \) is nonempty and \( \tilde{f}_{M}(A)(y) = 0 \) otherwise.

2. **Minimal L-fuzzy extension principle:**
   \[ \tilde{f}_{m}: \mathcal{L}(X) \to \mathcal{L}(Y) \] is defined in such a way that given \( A \in \mathcal{L}(X) \) then \( \tilde{f}_{m}(A)(y) = \bigwedge_{I} A(x) \) if the set \( I \) is nonempty and \( \tilde{f}_{m}(A)(y) = 0 \) otherwise.

### 3 Categories of L-fuzzy sets

The extension principles just stated suggest the possibility of extending the definition of \( \mathcal{L} \) to be a functor between classical sets and L-fuzzy sets but, obviously, the first step should be to define the appropriate concept of \( \mathcal{L} \)-fuzzy set category. The natural way to build a categorical structure on the classes of \( \mathcal{L} \)-fuzzy sets is to consider the arrows between \( \mathcal{L}(X) \) and \( \mathcal{L}(Y) \) as those given by any of the extension principles introduced above.

**Definition 1** (Category of \( \mathcal{L} \)-fuzzy sets).
Let \( L \) be a completely distributive lattice. The category of \( \mathcal{L} \)-fuzzy sets has as objects the class \( \{ \mathcal{L}(X) \mid X \in \text{Set} \} \). The set of morphisms between two objects \( \mathcal{L}(X) \) and \( \mathcal{L}(Y) \) is defined as
\[ \{ \tilde{f}_{M}: \mathcal{L}(X) \to \mathcal{L}(Y) \mid f: X \to Y \text{ is a mapping in Set} \} \]

It is straightforward to check that the previous construction is indeed a category, denoted \( \mathcal{L}\text{-Set} \), which has been previously used in [3, 10].

Now that we have a category of fuzzy sets, we can attempt the definition of a functor between \( \text{Set} \) and \( \mathcal{L}\text{-Set} \).

**Definition 2.** Let \( L \) be a completely distributive lattice. The covariant \( \mathcal{L} \)-fuzzy power-set functor \( \mathcal{L} \) is obtained by defining \( \mathcal{L}X = L^{X} \), i.e. the \( \mathcal{L} \)-fuzzy sets \( A: X \to L \), and following [4], for a morphism \( f: X \to Y \) in \( \text{Set} \), by defining
\[ \mathcal{L}f(A)(y) = \bigvee_{f(x)=y} A(x) \]

The functoriality of the previous construction can be seen in [4]. Also, note that the above definition of \( \mathcal{L}f \) is the same that the given by the extension principle, \( \tilde{f}_{M} \).

When working with the minimal extension principle, that is, changing \( f_{M} \) into \( f_{m} \), some problems arise when checking the axioms of category. This is mainly due to the fact that the minimal extension principle is not exactly the dual of the maximal one: in the case that the set \( I \) in the definition is empty, then both \( f_{M}(A)(y) \)
and \( \tilde{f}_m(A)(y) \) yield the minimum element in the
lattice; this is natural in the maximal extension principle, for \( \sqrt{\varnothing} = 0 \), but \( \land \varnothing \neq 0 \). This is why it
is interesting to consider, in analogy with the
results in [5], the \( \alpha \)-upper \( L \)-fuzzy set categories
and the \( \alpha \)-lower \( L \)-fuzzy set categories.

**Definition 3.** For all \( \alpha \in L \), the classes of the \( \alpha \)-upper \( L \)-fuzzy sets and the \( \alpha \)-lower \( L \)-fuzzy
sets, denoted \( \mathcal{L}_\alpha(X) \) and \( \mathcal{L}^\alpha(X) \) respectively,
are defined as follows:

\[
\mathcal{L}_\alpha(X) = \{ A \mid A \in \mathcal{L}(X), A(x) \geq \alpha \\
or A(x) = 0, \text{ for all } x \in X \}
\]

\[
\mathcal{L}^\alpha(X) = \{ A \mid A \in \mathcal{L}(X), A(x) \leq \alpha \\
or A(x) = 1, \text{ for all } x \in X \}
\]

**Definition 4 (Category of \( \mathcal{L}_\alpha \)-fuzzy sets).**
Let \( L \) be a completely distributive lattice and \( \alpha \in L \) with \( \alpha > 0 \). The category of \( \mathcal{L}_\alpha \)-fuzzy sets,
denoted \( \mathcal{L}_\alpha \)-Set, has as objects the class\( \{ \mathcal{L}_\alpha(X) \mid X \in \text{Set} \} \). The set of
morphisms between two objects \( \mathcal{L}_\alpha(X) \) and \( \mathcal{L}_\alpha(Y) \) is defined as

\[
\{ \tilde{f}_m : \mathcal{L}_\alpha(X) \to \mathcal{L}_\alpha(Y) \mid f : X \to Y \text{ is a mapping in Set} \}
\]

where \( \tilde{f}_m \) is the restriction of the mapping given
by the minimal extension principle to the \( L \)-fuzzy set \( \mathcal{L}_\alpha(X) \).

Checking that this construction verifies the
axioms of a category is not straightforward, as shown below. However, as we have adopted a
rather formal categorical machinery, we are re-
warded in that our proof is quite short as compared
with the corresponding proof given in [5].

**Theorem 1.** \( \mathcal{L}_\alpha \)-Set is a category.

**Proof.** To begin with, let us show that \( \tilde{g}_m \circ \tilde{f}_m = (g \circ f)_m \), where \( \tilde{f}_m : \mathcal{L}_\alpha(X) \to \mathcal{L}_\alpha(Y) \) and
\( \tilde{g}_m : \mathcal{L}_\alpha(Y) \to \mathcal{L}_\alpha(Z) \) are given by applying
the minimal extension principle to mappings
\( f : X \to Y \) and \( g : Y \to Z \).

We will proceed by cases:

1. Assume that \( \tilde{g}_m(\tilde{f}_m(A))(z) = 0 \), then we have
two cases:
   (a) \( g^{-1}(z) = \varnothing \).
   In this case \( (g \circ f)^{-1}(z) = \varnothing \) and then
   \( (g \circ f)_m(A)(z) = 0 \).
   (b) \( \tilde{f}_m(A)(y) = 0 \) for all \( y \in g^{-1}(z) \).
   Here, for all \( y \in g^{-1}(z) \) we have either
   \( f^{-1}(y) = \varnothing \) or \( A(x) = 0 \) for all \( x \in f^{-1}(y) \).
   Assume that \( f^{-1}(y) = \varnothing \) for all \( y \in g^{-1}(z) \), then \( (g \circ f)^{-1}(z) = \varnothing \) and,
   therefore, \( (g \circ f)_m(A)(z) = 0 \). Otherwise,
   \( (g \circ f)^{-1}(z) \neq \varnothing \), but \( A(x) = 0 \) for all \( x \in (g \circ f)^{-1}(z) \) and we also have
   \( (g \circ f)_m(A)(z) = 0 \) in this case.
   Therefore, if \( \tilde{g}_m(\tilde{f}_m(A))(z) = 0 \) then
   \( (g \circ f)_m(A)(z) = 0 \).

2. Now, assume \( \tilde{g}_m(\tilde{f}_m(A))(z) > 0 \) and consider
   the following equalities:
   \( \tilde{g}_m(\tilde{f}_m(A))(z) = \frac{\sum \{ f_m(A)(y) \mid y \in g^{-1}(z), \tilde{f}_m(A)(y) > 0 \}}{\sum \{ f_m(A)(y) \mid y \in g^{-1}(z), \tilde{f}_m(A)(y) > 0 \}} \)
   \( = \frac{\sum \{ f_m(A)(y) \mid x \in f^{-1}(y), A(x) > 0 \}}{\sum \{ f_m(A)(y) \mid x \in f^{-1}(y), A(x) > 0 \}} \)
   \( = \frac{\sum \{ A(x) \mid x \in \bigcup_{y \in g^{-1}(z)} f^{-1}(y), A(x) > 0 \}}{\sum \{ A(x) \mid x \in \bigcup_{y \in g^{-1}(z)} (g \circ f)^{-1}(z), A(x) > 0 \}} \)
   \( = \frac{(g \circ f)_m(A)(z)}{(g \circ f)_m(A)(z)} \)
   The equality (*) holds because of the hypothesis \( \alpha > 0 \), for in this case if \( \tilde{f}_m(A)(y) = 0 \) then \( f^{-1}(y) = \varnothing \). Thus, we have that if
   \( \tilde{g}_m(\tilde{f}_m(A))(z) > 0 \), then \( \tilde{g}_m(\tilde{f}_m(A))(z) = \frac{(g \circ f)_m(A)(z)}{(g \circ f)_m(A)(z)} \).

The two cases altogether show that \( \tilde{g}_m \circ \tilde{f}_m = (g \circ f)_m \), and this fact allows us easily to obtain
the axioms of category. \( \square \)

**Remark 1.** It is important to note that if \( \alpha = 0 \) we
do not get a category. This is due to the fact
that if \( \tilde{f}_m(A)(y) = 0 \), then \( f^{-1}(y) \) is not
necessarily empty. A counterexample can be found in [5].

The definition of the \( \mathcal{L}_\alpha \) functor between \( \text{Set} \)
and \( \mathcal{L}_\alpha \)-Set is straightforward:

**Definition 5.** Let \( L \) be a completely distributive
lattice and \( \alpha \in L \), \( \alpha > 0 \). The covariant
\( \alpha \)-upper \( L \)-fuzzy power-set functor, \( \mathcal{L}_\alpha : \text{Set} \to \mathcal{L}_\alpha \)-Set, is obtained by defining \( \mathcal{L}_\alpha(X) \) as in De-
finition 3 and by defining \( \mathcal{L}_\alpha f = \tilde{f}_m \) for each
morphism \( f : X \to Y \) in \( \text{Set} \).
Now, it is easy to check that the above definition is really a functor.

**Lemma 1.** The construction of \( \mathcal{L}_\alpha : \text{Set} \to \mathcal{L}_\alpha\text{-Set} \) above verifies the axioms of a functor.

**Proof.** We have already proved that \( \mathcal{L}_\alpha(f \circ g) = \mathcal{L}_\alpha(f) \circ \mathcal{L}_\alpha(g) \) in the proof of Theorem 1. Checking that \( \mathcal{L}_\alpha(1_X) = 1_{\mathcal{L}_\alpha X} \) is straightforward. □

The process above can be almost literally dualised, when considering the \( \alpha \)-lower sets and the maximal extension principle.

**Definition 6 (Category of \( L^\alpha \)-fuzzy sets).** Let \( L \) be a completely distributive lattice and \( \alpha \in L \). The category of \( L^\alpha \)-fuzzy sets, denoted \( L^\alpha\text{-Set} \), has as objects the class \( \{ L^\alpha(X) \mid X \in \text{Set} \} \). The set of morphisms between two objects \( L^\alpha(X) \) and \( L^\alpha(Y) \) is defined as

\[
\{ f_M : L^\alpha(X) \to L^\alpha(Y) \mid f : X \to Y \text{ is a mapping in Set} \}
\]

**Theorem 2.** \( L^\alpha\text{-Set} \) is a category.

**Proof.** The proof follows the steps of that of Theorem 1. □

**Remark 2.** Note that in this case no problem arises when \( \alpha = 1 \). Actually, the category \( L^1\text{-Set} \) is equal to \( \mathcal{L}\text{-Set} \).

**Definition 7.** Let \( L \) be a completely distributive lattice and \( \alpha \in L \). The 
\( \alpha \)-lower \( L \)-fuzzy power-set functor, \( L^\alpha : \text{Set} \to L^\alpha\text{-Set} \), is obtained by defining \( L^\alpha(X) \) as in Definition 3 and by \( L^\alpha f = f_M \) for a morphism \( f : X \to Y \) in \( \text{Set} \).

It is easy to check that the above definition is really a functor.

**On the Rationality of Extension Principles**

Regarding extension principles for fuzzy sets, it is important to check the rationality of the extension. In a categorical context, this amounts to showing that the extended categories are essentially the category of sets, or in more technical words, that the extended category is categorically equivalent to the category of sets and mappings.

To begin with, we recall the following characterisation of equivalent categories, see [8] for a proof of this result:

**Lemma 2.** Two categories \( C_1 \) and \( C_2 \) are equivalent if and only if there exists a functor \( \Phi : C_1 \to C_2 \) such that

1. For all pair of objects \( A, B \) in \( C_1 \), we have a bijection between \( \text{Hom}_{C_1}(A, B) \) and \( \text{Hom}_{C_2}(\Phi(A), \Phi(B)) \).
2. For all object \( A' \) in \( C_2 \), there exists an object \( A \) in \( C_1 \), such that \( \Phi(A) \) and \( A' \) are isomorphic objects in \( C_2 \).

**Theorem 3.** The categories \( \text{Set} \), \( \mathcal{L}_\alpha\text{-Set} \) and \( L^\alpha\text{-Set} \) are equivalent.

**Proof.** The definition of the category \( \mathcal{L}_\alpha\text{-Set} \) suggests to consider \( \Phi \) as the functor \( \mathcal{L}_\alpha \) in Definition 5. In addition, the hypotheses of Lemma 2 follow directly from the definition of \( \mathcal{L}_\alpha\text{-Set} \). Therefore \( \text{Set} \) and \( \mathcal{L}_\alpha\text{-Set} \) are equivalent categories.

The equivalence between \( \text{Set} \) and \( L^\alpha\text{-Set} \) is proved in a similar way. □

The rationality of these categories allows the definition of a structure of monad on the corresponding set functors, modulo the equivalence of categories, that is, the fact that \( \mathcal{L}_\alpha \) and \( L^\alpha \) are endofunctors arises the question whether they can be extended to monads.

**4 Monads**

A monad can be seen as the abstraction of the concept of adjoint functors and in a sense an abstraction of universal algebra. It is interesting to note that monads are useful not only in universal algebra, but it is also an important tool in topology when handling regularity, iteratedness and compactifications, and also in the study of toposes and related topics.

**Definition 8.** Let \( C \) be a category. A monad (or triple, or algebraic theory) over \( C \) is written as \( \Phi = (\Phi, \eta, \mu) \), where \( \Phi : C \to C \) is a (covariant) functor, and \( \eta : \text{id}_C \to \Phi \) and \( \mu : \Phi \circ \Phi \to \Phi \) are natural transformations for which \( \mu \circ \Phi \eta = \eta \circ \mu \) and \( \mu \circ \Phi \mu = \mu \circ \mu \Phi \) hold.
In the particular case of the functor $\mathcal{L} = L_{id}$, it was proved in [7] that $(L_{id}, \eta, \mu)$ with $\eta_X : X \to L_{id}X$ and $\mu_X : L_{id}L_{id}X \to L_{id}X$ defined by

$$\eta_X(x)(x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu_X(A)(x) = \bigvee_{A \in L_{id}X} A(x) \land A(A).$$

is a monad.

The generalization of this result can be easily shifted to the case of $(\mathcal{L}^*, \eta^*, \mu^*)$. Also $(\mathcal{L}_\alpha, \eta_\alpha, \mu_\alpha)$ can be made a monad where $\eta_\alpha : X \to \mathcal{L}_\alpha X$ is defined by

$$\eta_\alpha X(x)(x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases}$$

and $\mu_\alpha X : \mathcal{L}_\alpha \mathcal{L}_\alpha X \to \mathcal{L}_\alpha X$ is defined by

$$\mu_\alpha X(A)(x) = \begin{cases} \bigwedge_{A \in I} A(x) \land A(A) & \text{if } I \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $I = \{ A \in \mathcal{L}_\alpha X \mid A(x) \land A(A) > 0 \}$.

**Lemma 3.** $\eta_\alpha : 1 \to \mathcal{L}_\alpha$ and $\mu_\alpha : \mathcal{L}_\alpha \mathcal{L}_\alpha \to \mathcal{L}_\alpha$ are natural transformations.

**Proof.** Straightforward. \qed

As stated previously, the reason why $(\mathcal{L}_\alpha, \eta_\alpha, \mu_\alpha)$ is a monad do not rely on duality; therefore, it is interesting to give a detailed proof of it.

**Theorem 4.** $(\mathcal{L}_\alpha, \eta_\alpha, \mu_\alpha)$ is a monad.

**Proof.** In the following we will drop the subscript $\alpha$ in $\mathcal{L}$, $\mu$ and $\eta$.

The left unit identity requires us to prove that $\mu_X(\eta_X(A))(x) = A(x)$, for all $x \in X$ and $A \in \mathcal{L}X$. Consider the set $I = \{ B \in \mathcal{L}X \mid B(x) \land \eta_X(A)(B) > 0 \}$, then:

$$\mu_X(\eta_X(A))(x) = \begin{cases} \bigwedge_{B \in I} B(x) \land \eta_X(A)(B) & \text{if } I \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

The equality (*) follows from the definition of $\eta_\alpha$ for $\eta_X(\alpha)(B) > 0$ if and only if $A = B$.

For the right unit identity we have to prove that $\mu_X(\mathcal{L}\eta_X(A))(x) = A(x)$, for all $x \in X$ and $A \in \mathcal{L}X$. Given $I = \{ B \in \mathcal{L}X \mid B(x) \land L\eta_X(A)(B) > 0 \}$ we have

$$\mu_X(\mathcal{L}\eta_X(A))(x) = \begin{cases} \bigwedge_{B \in I} B(x) \land \mathcal{L}\eta_X(A)(B) & \text{if } I \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mu_X(\mathcal{L}\eta_X(A))(x) > 0$ iff there exist $B \in \mathcal{L}X$ and $y \in \eta_X(B)$ with $B(x)$ and $A(y) > 0$; in this case:

$$\mu_X(\mathcal{L}\eta_X(A))(x) = \bigwedge_{B \in \mathcal{L}X} B(x) \land \mathcal{L}\eta_X(A)(B)$$

$$= \bigwedge_{B \in \mathcal{L}X} \left( B(x) \land \bigwedge_{y \in \eta_X(B)} A(y) \right)$$

$$= \bigwedge_{B \in \mathcal{L}X} B(x) \land A(y) = A(x)$$

For the last equality, notice that for each $B$ with $B(x) > 0$ and $y \in \eta_X(B)$, we have that $B(x) = \eta_X(y)(x) > 0$; thus, necessarily $y = x$ and $B(x) = 1$.

For the associativity of $\mu$, we have to prove that $\mu_X(\mathcal{L}\mu_X(\mathfrak{A}))(x) = \mu_X(\mu_X(\mathfrak{A}))(x)$ for all $x \in X$ and $\mathfrak{A} \in \mathcal{L}\mathcal{L}X$.

Consider the set $I = \{ A \in \mathcal{L}X \mid A(x) \land \mathcal{L}\mu_X(\mathfrak{A})(B) > 0 \}$.

$$\mu_X(\mathcal{L}\mu_X(\mathfrak{A}))(x) = \begin{cases} \bigwedge_{A \in I} A(x) \land \mathcal{L}\mu_X(\mathfrak{A})(A) & \text{if } I \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Therefore $\mu_X(\mathcal{L}\mu_X(\mathfrak{A}))(x) = 0$ if and only if the following condition holds: For all $A \in \mathcal{L}X$ such that $A(x) > 0$ and all $\mathfrak{A} \in \mathcal{M}(A)$ we have that $\mathfrak{A}(A) = 0$.

Assume that $\mu_X(\mathcal{L}\mu_X(\mathfrak{A}))(x) > 0$, then there is $A \in \mathcal{L}X$ such that
Thus, if \( \mu_X(A) = A \) we have \( \mu_X(A)(x) = A(x) > 0 \), therefore, by definition of \( \mu_X \), there exists \( B \) such that \( B(x) > 0 \) and \( A(B) > 0 \) and \( \mu_{\mathcal{L}X}(A)(B) > 0 \). As a result, the subscripts used in the following calculations of meets are all nonempty (we are going to use just the associativity of meets):

\[
\begin{align*}
\mu_X(\mathcal{L} \mu_X(A))(x) &= \bigwedge_{A \in \mathcal{L}X} A(x) \land \bigwedge_{A \in \mu_X^{-1}(A)} A(A) \\
&= \bigwedge_{A \in \mathcal{L}X, \mu_X(A)(x) > 0} \mu_X(A)(x) \land A(A) \\
&= \bigwedge_{A \in \mathcal{L}X, B \in \mathcal{L}X} \left( \bigwedge_{B(x) > 0} B(x) \land A(B) \right) \land A(A) \\
&= \bigwedge_{B \in \mathcal{L}X, \mu_{\mathcal{L}X}(A)(B) > 0} B(x) \land A(B) \land A(A) \\
&= \mu_X(\mathcal{L} \mu_X(A))(x)
\end{align*}
\]

Therefore, we have proved that, if \( \mu_X(\mathcal{L} \mu_X(A))(x) > 0 \), then \( \mu_X(\mathcal{L} \mu_X(A))(x) = \mu_X(\mathcal{L} \mu_X(A))(x) \).

To finish the proof, it suffices to prove that if \( \mu_X(\mu_{\mathcal{L}X}(A))(x) > 0 \), then \( \mu_X(\mathcal{L} \mu_X(A))(x) > 0 \).

Consider \( J = \{B \in \mathcal{L}X \mid B(x) > 0, \mu_{\mathcal{L}X}(A)(B) > 0\} \)

\[
\mu_X(\mu_{\mathcal{L}X}(A))(x) = \begin{cases} 
\bigwedge_{B \in J} B(x) \land \mu_{\mathcal{L}X}(A)(B) & \text{if } J \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

Thus, if \( \mu_X(\mu_{\mathcal{L}X}(A))(x) > 0 \), then there exists \( B \in \mathcal{L}X \) and \( A \in \mathcal{L}X \) such that, \( B(x) > 0 \), \( A(B) > 0 \), and \( \mathcal{A}(A) > 0 \). Now, if we consider \( A = \mu_X(A) \) then the following conditions are satisfied:

\[
- A(x) > 0 \\
- 0 < A(x) = \mu_X(A)(x), \text{ because } B(x) > 0, \text{ and } A(B) > 0 \\
- A \in \mu_X^{-1}(A) \text{ and } \mathcal{A}(A) > 0.
\]

Therefore we can conclude that

\[
\mu_X(\mathcal{L} \mu_X(A))(x) > 0
\]

as well. \( \square \)

5 Conclusions and further work

Previous constructions of fuzzy set categories have been generalised by considering \( L \)-fuzzy sets in which the values of the characteristic functions run on a completely distributive lattice. Later, \( L \)-fuzzy categories are defined using these \( L \)-fuzzy sets, and its rationality is proven in the categorical sense. Finally, the \( L \)-fuzzy functors given by the extension principles have been shown to have structure of monad.

Some interesting questions for further work on this basis are given by the idea of fuzzifying a “set of terms” as a “\( L \) fuzzy set of terms”, which naturally leads to the following question: Under which conditions the composition of these \( L \)-fuzzy powerset functors with the term monad turns out to be a monad? In [1] we have investigated conditions which guarantee the structure of monad of the composition of two monads; these conditions have been applied [2] to the composition of the \( L \)-fuzzy powerset monads and the term monad.

References


