Systematic Construction of examples for cycling in the simplex method

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1) Linear programming problem

canonical form:
$$\max z = c^T x \qquad c, x \in \mathfrak{R}^n$$
$$\text{s.t.} \qquad Ax \le b \qquad A \in \mathfrak{R}^{m \times n}, b \in \mathfrak{R}^m, b \ge 0$$
$$x \ge 0$$

standard form:
$$\max z = c^T x$$
$$\text{s.t.} \qquad Ax + Iu = b \qquad I \in \mathfrak{R}^{m \times m}, u \in \mathfrak{R}^m$$
$$x, u \ge 0.$$

Fields of application:
Sales and production planning
Blending problems
Cutting stock problems
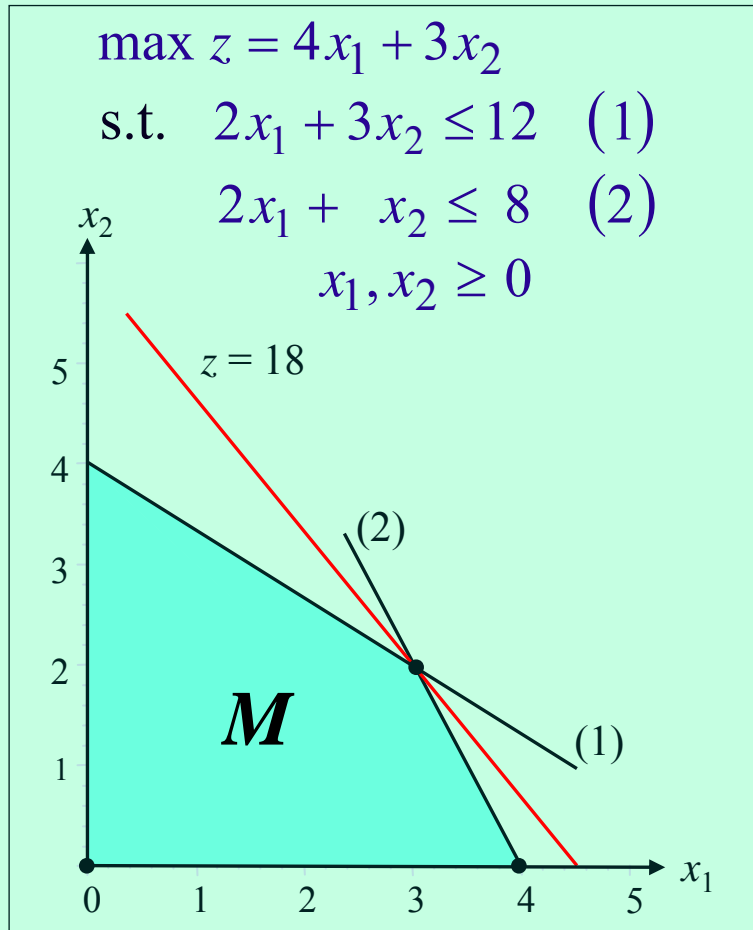Agriculture
Financial planning

Solution methods:
1951: Simplex method of Dantzig   (nonpolynomial)
1979: Ellipsoid method of Khachian   (polynomial)
1984: Interior point method of Karmakar   (polynomial)

2) Example for the simplex method and cycling

$$\max z = 4x_1 + 3x_2$$

s.t. $2x_1 + 3x_2 \leq 12$ (1)

$2x_1 + x_2 \leq 8$ (2)

$x_1, x_2 \geq 0$

BV	x_1	x_2	x_3	x_4	
x_3	2	3	1	0	12
x_4	2	1	0	1	8
Δz	-4	-3	0	0	0

x_3	0	2	1	-1	4
x_1	1	1/2	0	1/2	4
Δz	0	-1	0	2	16

x_2	0	1	1/2	-1/2	2
x_1	1	0	-1/4	3/4	3
Δz	0	0	1/2	3/2	18

Pivo selection rule of Dantzig:

Column v such that: $0 > \Delta z_v \leq \Delta z_j \forall j.$

Row μ such that: $\tilde{a}_{\mu,v} > 0 : \tilde{b}_\mu / \tilde{a}_{\mu,v} \leq \tilde{b}_i / \tilde{a}_{i,v} \quad \forall i \text{ with } \tilde{a}_{i,v} > 0.$

A tableau is called *degenerate*, if at least one value of the right-hand side is zero. When every tableau is nondegenerate, the simplex algorithm solves a linear problem in a finite number of iterations. Otherwise *cycling* may occur, i.e. after some iterations the algorithm returns to a previously generated tableau.

Cycling example:

$$\max 3x_1 + \frac{59}{20}x_2 - 50x_3 - \frac{2}{5}x_4$$

$$\frac{1}{40}x_1 + \frac{1}{400}x_2 + 3x_3 + 2x_4 \leq 0$$

$$\frac{1}{20}x_1 + \frac{9}{200}x_2 - \frac{1}{2}x_3 + \frac{2}{25}x_4 \leq 0$$

$$x_1, \ldots, x_4 \geq 0$$

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BV	x_1	x_2	x_3	x_4	x_5	x_6
x_5	$\frac{1}{40}$	$\frac{1}{400}$	3	2	1	0
x_6	$\frac{1}{20}$	$\frac{9}{200}$	$-\frac{1}{2}$	$\frac{2}{25}$	0	1
	-3	$-\frac{59}{20}$	50	$\frac{2}{5}$	0	0

BV	x_1	x_2	x_3	x_4	x_5	x_6
x_2	$\frac{130}{109}$	1	0	$\frac{992}{109}$	$\frac{400}{109}$	$\frac{2400}{109}$
x_3	$\frac{4}{545}$	0	1	$\frac{1796}{2725}$	$\frac{36}{109}$	$-\frac{2}{109}$
	$\frac{33}{218}$	0	0	$-\frac{622}{109}$	$-\frac{620}{109}$	$\frac{7180}{109}$

x_1	1	$\frac{1}{10}$	120	80	40	0
x_6	0	$\frac{1}{25}$	$-\frac{13}{2}$	$-\frac{98}{25}$	-2	1
	0	$-\frac{53}{20}$	410	$\frac{1202}{5}$	120	0

x_3	$-\frac{49}{620}$	$-\frac{449}{6200}$	1	0	$\frac{2}{31}$	$-\frac{50}{31}$
x_4	$\frac{65}{496}$	$\frac{109}{992}$	0	1	$\frac{25}{62}$	$\frac{75}{31}$
	$\frac{223}{248}$	$\frac{311}{496}$	0	0	$-\frac{105}{31}$	$\frac{2470}{31}$

x_1	1	0	$\frac{545}{4}$	$\frac{449}{5}$	45	$-\frac{5}{2}$
x_2	0	1	$-\frac{325}{2}$	-98	-50	25
	0	0	$-\frac{165}{8}$	$-\frac{193}{10}$	$-\frac{25}{2}$	$\frac{265}{4}$

x_4	$\frac{5}{8}$	$\frac{9}{16}$	$-\frac{25}{4}$	1	0	$\frac{25}{2}$
x_5	$-\frac{49}{40}$	$-\frac{449}{400}$	$\frac{31}{2}$	0	1	-25
	$-\frac{13}{4}$	$-\frac{127}{40}$	$\frac{105}{2}$	0	0	-5

The next iteration results in the initial tableau!

In the following we present conditions which are necessary and sufficient for cycling and use them to construct cycling examples.

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3) "History" of simplex cycling

During two years after the publication of Dantzig´s simplex method in 1951 it was an open problem whether simplex cycling is possible. In more than 50 years only a few cycling examples have been published.

Collection of Gass/Vinjamuri (2004):

1953 Hoffman

1955 Beale

1965 Yudin / Gol´shtein

1969 Marshall / Suurballe

1983 Chvátal

1984 Solow

1993 Nering / Tucker

1996 Sierksma

1997 Kuhn (in: Balinski / Tucker)

The construction of cycling examples was a kind of "**mental sport**", occasionally almost mystified. There did not exist any systematic construction procedure

Some citations:

Hoffman 1951: "Cycling is certainly *not completely understood*."

Beale 1955: "…linear programmers are still *intrigued by* cycling and seek an understanding of the basic reasons underlying its occurrence…"

Lee 1997: "None of those examples is as *mysterious* as Hoffman´s"

Guerrero-García/Santos-Palomo 2007: Submission of the paper: "On Hoffman´s *celebrated* cycling example"

4) Basic idea of the construction of cycling examples

Here we restrict ourselves to the easiest case with $n = 4$ and $m = 2$, i. e. the linear program has the form

$$\max z = c_1 x_1 + \ldots + c_4 x_4$$

$$\text{s.t.} \quad a_1 x_1 + \ldots + a_4 x_4 \leq 0$$

$$b_1 x_1 + \ldots + b_4 x_4 \leq 0$$

$$x_1, \ldots, x_4 \geq 0$$

Without loss of generality we may assume that the right-hand side zero. Let the "simplex cycle" be

$$C = (\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{1,6\}, \{1,2\}),$$

i. e. the first basis consists of columns 1 and 2, the second of columns 2 and 3, etc.

The tableaux must have the following properties (right-hand side eliminated):

BV	x_1	x_2	x_3	x_4	x_5	x_6
x_5	[+]	a_2	a_3	a_4	1	0
x_6	b_1	b_2	b_3	b_4	0	1
	[–]	$-c_2$	$-c_3$	$-c_4$	0	0

Basic indices

$\{5, 6\}$

BV	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1	□	□	□	□	0
x_6	0	[+]	□	□	□	1
	0	[–]	□	□	□	0

$\{1, 6\}$

BV	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1	0	[+]	□	□	□
x_2	0	1	□	□	□	□
	0	0	[–]	□	□	□

$\{1, 2\}$

$\vdots$ $\vdots$ $\vdots$

[+] : element must be **positive**
[–] : element must be the **most negative**

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In order to obtain necessary and sufficient conditions for cycling, every tableau must be expressed in terms of a_i, b_i, c_i.

The tableau T_{ij} associated with the basic variables x_i and x_j is of the form

$$T_{ij} = \frac{1}{D_{ij}} \begin{pmatrix} D_{1j} & D_{2j} & D_{3j} & D_{4j} & D_{5j} & D_{6j} \\ D_{i1} & D_{i2} & D_{i3} & D_{i4} & D_{i5} & D_{i6} \\ \overline{D}_{ij1} & \overline{D}_{ij2} & \overline{D}_{ij3} & \overline{D}_{ij4} & \overline{D}_{ij5} & \overline{D}_{ij6} \end{pmatrix}$$

where $\quad D_{ij} := \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}, \qquad \overline{D}_{ijv} := \begin{vmatrix} a_i & a_j & a_v \\ b_i & b_j & b_v \\ -c_i & -c_j & -c_v \end{vmatrix} \quad (1 \le i, j, v \le 6, i < j).$

For example, a pivot step from $T_{3,4}$ to $T_{4,5}$ is possíble, iff $T_{3,4}$ has the following properties:

$$\ldots x_5 \ldots$$
$$\ldots D_{5,4} / D_{3,4} > 0 \ldots$$
$$\ldots D_{3,5} / D_{3,4} \ldots$$
$$\ldots 0 > \overline{D}_{3,4,5} / D_{3,4} \le \overline{D}_{3,4,j} / D_{3,4} \ \forall j \ldots$$

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After some simplifications we obtain the following conditions which are necessary and sufficient for the simplex algorithm to run through the cycle

$$C = \left(\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{1,6\}, \{1,2\}\right):$$

$$D_{1,2} > 0 > \overline{D}_{1,2,3} \le \overline{D}_{1,2,4}, \ \overline{D}_{1,2,5} \qquad (1)$$

$$D_{3,2} > 0 > \overline{D}_{3,2,4} \le \overline{D}_{3,2,5}, \ \overline{D}_{3,2,6} \qquad (2)$$

$$D_{3,4} > 0 > \overline{D}_{3,4,5} \le \overline{D}_{3,4,6}, \ \overline{D}_{3,4,1} \qquad (3)$$

$$D_{5,4} > 0 > \overline{D}_{5,4,6} \le \overline{D}_{5,4,1}, \ \overline{D}_{5,4,2} \qquad (4)$$

$$D_{5,6} > 0 > \overline{D}_{5,6,1} \le \overline{D}_{5,6,2}, \ \overline{D}_{5,6,3} \qquad (5)$$

$$D_{1,6} > 0 > \overline{D}_{1,6,2} \le \overline{D}_{1,6,3}, \ \overline{D}_{1,6,4} \qquad (6)$$
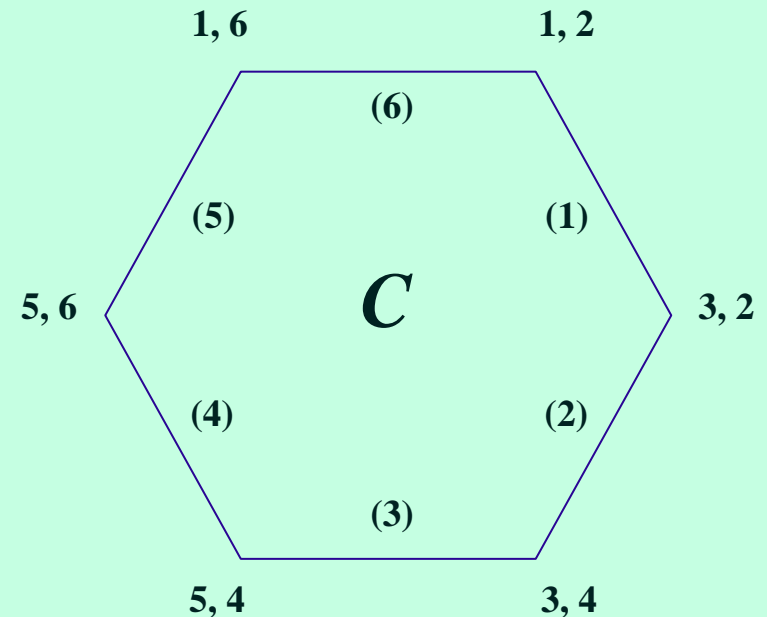
Note that the variables in the above system are the elements a_i, b_i e c_i of the initial tableau. For example, $\overline{D}_{3,4,5} < 0$ means

$$\begin{vmatrix} a_3 & a_4 & a_5 \\ b_3 & b_4 & b_5 \\ -c_3 & -c_4 & -c_5 \end{vmatrix} < 0$$

A specific solution of the above system is:

$$c_1 = 3, \quad c_2 = \frac{59}{20}, \quad c_3 = -50, \quad c_4 = -\frac{2}{5},$$

$$a_1 = \frac{1}{40}, \quad a_2 = \frac{1}{400}, \quad a_3 = 3, \quad a_4 = 2,$$

$$b_1 = \frac{1}{20}, \quad b_2 = \frac{9}{200}, \quad b_3 = -\frac{1}{2}, \quad b_4 = \frac{2}{25}$$

which corresponds to the illustrated cycling example.

Observation:
Determining a cycling example is equivalent to solving a *system of determinantal inequalities* of the above type! *Thus, nonlinear programming software can be applied to construct cycling examples!*

Crucial idea for the characterization of the cycling property:
Express all tableaux in terms of the initial tableau, using Cramer's Rule:

If the problem is

$\max z = c_1 x_1 + \ldots + c_4 x_4$
s.t.
$$a_1 x_1 + \ldots + a_4 x_4 \leq \mathbf{0}$$
$$b_1 x_1 + \ldots + b_4 x_4 \leq \mathbf{0}$$
$$x_1, \ldots, x_4 \geq \mathbf{0}$$

i.e. the initial tableau is

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 1 & 0 \\ b_1 & b_2 & b_3 & b_4 & 0 & 1 \\ -c_1 & -c_2 & -c_3 & -c_4 & 0 & 0 \end{pmatrix},$$

the tableau T_{ij}, having basic variables x_i and x_j can be expressed as

$$T_{ij} = \frac{1}{D_{ij}} \begin{pmatrix} D_{1j} & D_{2j} & D_{3j} & D_{4j} & D_{5j} & D_{6j} \\ D_{i1} & D_{i2} & D_{i3} & D_{i4} & D_{i5} & D_{i6} \\ \overline{D}_{ij1} & \overline{D}_{ij2} & \overline{D}_{ij3} & \overline{D}_{ij4} & \overline{D}_{ij5} & \overline{D}_{ij6} \end{pmatrix}$$

where $D_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$, $\overline{D}_{ij\nu} = \begin{vmatrix} a_i & a_j & a_\nu \\ b_i & b_j & b_\nu \\ -c_i & -c_j & -c_\nu \end{vmatrix}$.

For example,

$$T_{12} = \frac{1}{D_{12}} \begin{pmatrix} D_{12} & D_{22} & D_{32} & D_{42} & D_{52} & D_{62} \\ D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ \overline{D}_{121} & \overline{D}_{122} & \overline{D}_{123} & \overline{D}_{124} & \overline{D}_{125} & \overline{D}_{126} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \dfrac{545}{4} & \dfrac{449}{5} & 45 & -\dfrac{5}{2} \\ 0 & 1 & -\dfrac{325}{2} & -98 & -50 & 25 \\ 0 & 0 & -\dfrac{165}{8} & -\dfrac{193}{10} & -\dfrac{25}{2} & \dfrac{265}{4} \end{pmatrix}$$ (for the numerical example).

Thus, the sequence of tableaux is

$$T_{56} = \frac{1}{D_{56}} \begin{pmatrix} D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ \overline{D}_{561} & \overline{D}_{562} & \overline{D}_{563} & \overline{D}_{564} & \overline{D}_{565} & \overline{D}_{566} \end{pmatrix}$$

$$T_{16} = \frac{1}{D_{16}} \begin{pmatrix} D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \\ D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ \overline{D}_{161} & \overline{D}_{162} & \overline{D}_{163} & \overline{D}_{164} & \overline{D}_{165} & \overline{D}_{166} \end{pmatrix}$$

$$\vdots \qquad\qquad \vdots$$

$$T_{45} = \frac{1}{D_{45}} \begin{pmatrix} D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{65} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ \overline{D}_{451} & \overline{D}_{452} & \overline{D}_{453} & \overline{D}_{454} & \overline{D}_{455} & \overline{D}_{456} \end{pmatrix}$$

and we get the conditions for cycling

$$\frac{D_{16}}{D_{56}} > 0, \quad 0 > \frac{\overline{D}_{561}}{D_{56}} \leq \frac{\overline{D}_{56\nu}}{D_{56}} \text{ for } \nu = 2,...,6$$

$$\frac{D_{12}}{D_{16}} > 0, \quad 0 > \frac{\overline{D}_{162}}{D_{16}} \leq \frac{\overline{D}_{16\nu}}{D_{16}} \text{ for } \nu = 1,3,...,6$$

$$\vdots \qquad \qquad \vdots$$

$$\frac{D_{65}}{D_{45}} > 0, \quad 0 > \frac{\overline{D}_{456}}{D_{45}} \leq \frac{\overline{D}_{45\nu}}{D_{45}} \text{ for } \nu = 1,...,5$$

which can be simplified to

$$D_{16} > 0, \quad 0 > \overline{D}_{162} \leq \overline{D}_{163}, \ \overline{D}_{164}$$
$$D_{12} > 0, \quad 0 > \overline{D}_{123} \leq \overline{D}_{124}, \ \overline{D}_{125}$$
$$D_{32} > 0, \quad 0 > \overline{D}_{324} \leq \overline{D}_{325}, \ \overline{D}_{326}$$
$$D_{34} > 0, \quad 0 > \overline{D}_{345} \leq \overline{D}_{346}, \ \overline{D}_{341}$$
$$D_{54} > 0, \quad 0 > \overline{D}_{546} \leq \overline{D}_{541}, \ \overline{D}_{542}$$
$$D_{56} > 0, \quad 0 > \overline{D}_{561} \leq \overline{D}_{562}, \ \overline{D}_{563}$$

5) Cycling examples for diverse pivot rules

5a) Dantzig´s rule with the "Least-index tie-breaking rule":
(When several basic variables satisfy the criterion for leaving the basis, choose the variable with the least index):

$m = 2, n = 4:$ (5.1)

$$\max 14x_1 - 25x_2 + \frac{7}{20}x_3 - 20x_4$$

$$x_1 - 2x_2 - \frac{1}{10}x_3 + 5x_4 \leq 0$$

$$\frac{7}{10}x_1 - \frac{3}{10}x_2 - \frac{1}{100}x_3 + \frac{19}{50}x_4 \leq 0$$

$$x_1, \ldots, x_4 \geq 0$$

Solution: unbounded

$m = 2, n = 4:$ (5.2)

$$\max \frac{1}{2}x_1 + \frac{2}{5}x_2 - 5x_3 - \frac{1}{5}x_4$$

$$\frac{1}{40}x_1 - \frac{1}{100}x_2 + 3x_3 + 2x_4 \leq 0$$

$$\frac{1}{20}x_1 + \frac{1}{50}x_2 + \frac{1}{50}x_3 + \frac{2}{25}x_4 \leq 0$$

$$x_1, \ldots, x_4 \geq 0$$

Solution: $x_1 = \ldots x_4 = 0$

Example with *nonzero right-hand side*:

$$m = 4,\ n = 4:$$

$$\max 14x_1 - 25x_2 + \frac{7}{20}x_3 - 20x_4$$

$$x_1 - 2x_2 - \frac{1}{10}x_3 + 5x_4 \le 0$$

$$\frac{7}{10}x_1 - \frac{3}{10}x_2 - \frac{1}{100}x_3 + \frac{19}{50}x_4 \le 0 \qquad (5.3)$$

$$x_1 + x_2 + x_3 + x_4 \le 5$$

$$x_1 + 2x_2 + 3x_3 + x_4 \le 10$$

$$x_1, \ldots, x_4 \ge 0$$

Solution: $(x_1,\ x_2,\ x_3,\ x_4) = (10/211,\ 0,\ 700/211,\ 0)$

Example with *higher dimensions*:

$m = 2,\ n = 6:$

$C = \left(\{1,2\},\ \{2,3\},\ \{3,4\},\ \{4,5\},\ \{5,6\},\ \{6,7\},\ \{7,8\},\ \{1,8\},\ \{1,2\} \right)$

$$\max \frac{1}{100}x_1 - \frac{1}{100}x_2 + \frac{9}{1000}x_3 - \frac{3}{200}x_4 + \frac{1}{500}x_5 - \frac{3}{20}x_6$$

$$\frac{1}{20}x_1 - 100x_2 - \frac{2}{5}x_3 - 100x_4 - x_5 + 65x_6 \leq 0 \qquad (5.4)$$

$$\frac{9}{10}x_1 - x_2 + \frac{3}{5}x_3 - \frac{3}{2}x_4 - \frac{1}{100}x_5 + \frac{1}{100}x_6 \leq 0$$

$$x_1,\dots,x_6 \geq 0$$

Solution: unbounded

5b) Largest-coefficient tie-breaking rule

In practical calculations various problems arise if the pivot element is "*too small*" (ill-conditioned basis matrices). Therefore a common practical tie-braking rule for the leaving basic variable consists in selecting the largest (positive) element of the entering column as the pivot element.

Examples:

$$m = 2, \ n = 4 :$$

$$\max \frac{1}{100} x_1 + \frac{1}{1000} x_2 + \frac{1}{2000} x_3 - \frac{63}{100} x_4$$

$$20 x_1 + \frac{7}{100} x_2 - \frac{7}{100} x_3 + 100 x_4 \leq 0 \qquad (5.5)$$

$$-100 x_1 - \frac{3}{10} x_2 - \frac{1}{100} x_3 + \frac{1}{4} x_4 \leq 0$$

Solution: unbounded $\qquad\qquad\qquad\qquad x_1, \dots, x_4 \geq 0$

$m = 3,\ n = 7:$

$$\max \frac{1}{10}x_1 + \frac{1}{20}x_2 + \frac{2}{25}x_3 - \frac{1}{10}x_4 + \frac{1}{25}x_5 + \frac{2}{25}x_6 - \frac{33}{200}x_7$$

$$\frac{5}{2}x_1 - \frac{13}{10}x_2 + 5x_3 + \frac{8}{5}x_4 + \frac{1}{5}x_5 + \frac{3}{5}x_6 + \frac{3}{2}x_7 \le 0$$

$$\frac{12}{5}x_1 + \frac{2}{5}x_2 + 2x_3 - \frac{13}{20}x_4 + \frac{1}{25}x_5 + \frac{7}{10}x_6 - \frac{1}{2}x_7 \le 0$$

$$-25x_1 - \frac{8}{5}x_2 - 24x_3 - \frac{1}{5}x_4 - 2x_5 - 3x_6 + 5x_7 \le 0$$

$$x_1,\dots,x_7 \ge 0$$

(5.6)

Solution: unbounded

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5c) Steepest-edge column selection criterion

Most of the linear programming solvers offer the "*steepest-edge column selection*" criterion as an alternative for the most negative reduced cost rule. Here the entering variable is selected on the basis of the *most negative ratio of reduced cost to the length of the vector*, corresponding to a unit change in the nonbasic variable, i. e.

$$\tilde{a}_{m+1,\nu} < 0$$

$$\frac{\tilde{a}_{m+1,\nu}}{\sqrt{\tilde{a}_{1,\nu}^2 + \ldots + \tilde{a}_{m,\nu}^2 + 1}} \leq \frac{\tilde{a}_{m+1,j}}{\sqrt{\tilde{a}_{1,j}^2 + \ldots + \tilde{a}_{m,j}^2 + 1}} \qquad \text{for all } j = 1, \ldots, m+n$$

(the leaving basic variable is determined by the "*Largest-coefficient tie-breaking rule*").

$m = 3,\ n = 6:$

$$\max -36x_1 + \frac{3}{5}x_2 - 20x_3 - \frac{1}{4}x_4 + \frac{1}{20}x_5 + \frac{1}{20}x_6$$

$$2x_1 + \frac{1}{5}x_2 - 5x_3 - \frac{9}{10}x_4 + x_5 + \frac{23}{1000}x_6 \le 0$$

$$-41x_1 - \frac{6}{5}x_2 + 12x_3 + \frac{1}{5}x_4 - \frac{14}{5}x_5 - \frac{1}{500}x_6 \le 0 \qquad (5.7)$$

$$165000x_1 + 2600x_2 + 9600x_3 + 125x_4 - 100x_5 - 300x_6 \le 0$$

$$x_1, \ldots, x_6 \ge 0$$

Solution: unbounded

6) Cycling examples with permutation structure

The cycling example of Hoffman (1953) has the following form. After two pivot steps the tableau is a column permutation of the initial tableau:

BV	x_1	x_2	x_3	x_4	x_5	$\cdots$	x_{10}
Δz	0	0	$-c_3$	$-c_4$	$-c_5$	$\cdots$	$-c_{10}$
x_1	1	0	a_3	a_4	a_5	$\cdots$	a_{10}
x_2	0	1	b_3	b_4	b_5	$\cdots$	b_{10}
Δz	$\dfrac{c_3}{a_3}$	0	0	$-c_4 + \dfrac{c_3}{a_3}a_4$	$-c_5 + \dfrac{c_3}{a_3}a_5$	$\cdots$	$-c_{10} + \dfrac{c_3}{a_3}a_{10}$
x_3	$\dfrac{1}{a_3}$	0	1	$\dfrac{a_4}{a_3}$	$\dfrac{a_5}{a_3}$	$\cdots$	$\dfrac{a_{10}}{a_3}$
x_2	$-\dfrac{b_3}{a_3}$	1	0	$b_4 - \dfrac{b_3}{a_3}a_4$	$b_5 - \dfrac{b_3}{a_3}a_5$	$\cdots$	$b_{10} - \dfrac{b_3}{a_3}a_{10}$
Δz	$-c_9$	$-c_{10}$	0	0	$-c_3$	$\cdots$	$-c_8$
x_3	a_9	a_{10}	1	0	a_3	$\cdots$	a_8
x_4	b_9	b_{10}	0	1	b_3	$\cdots$	b_8

$\vdots$

Such permutation structure occurs iff the initial tableau has the following form, see Zörnig (2008, formula (2.7)):

x_1 x_2	x_3 x_4	x_5 x_6	$\dots$	x_{n+1} x_{n+2}
0	$-c_B^T$	$-c_B^T(I+B)$	$\dots$	$-c_B^T(I+B+B^2+\dots+B^{n/2-1})$
I	B	B^2	$\dots$	$B^{n/2}$

$$(6.1)$$

where the matrix $B \in \Re^{2\times 2}$ satisfies $B^{n/2+1} = I$, and the tableau satisfies some determinantal inequalities.

Observe that two pivot steps, substituting x_1 by x_3 and x_2 by x_4 correspond to a premultiplication of the tableau by

$$\begin{pmatrix} 1 & -c_B^T \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} 1 & c_B^T B^{-1} \\ 0 & B^{-1} \end{pmatrix}.$$

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Using the above theory and some matrix theory ($B^{n/2+1} = I$, i. e. B is involutary) one can construct cycling examples with permutation structure.

Example: In (6.1) choose n = 8 and

$$\begin{pmatrix} -c_B^T \\ B \end{pmatrix} = \begin{pmatrix} -c_3 & -c_4 \\ a_3 & a_4 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{4} & \dfrac{3}{10} \\ \dfrac{1}{10} & \dfrac{\sqrt{5}}{10} - \dfrac{53}{25} \\ \dfrac{1}{2} & \dfrac{\sqrt{5}}{2} - \dfrac{3}{5} \end{pmatrix}$$

7) Practical relevance/Concluding remarks

The large number of anticycling rules, published over the decades demonstrate the practical significance of cycling.

The occurrence of this phenomenon is not restricted to the original version of the simplex algorithm.

Almost all improvements and variants of the simplex method, as well as many of the simplex type algorithms in (nonlinear) mathematical programming involve the possibility of cycling or stalling, for example:

- steepest edge simplex algorithm
- primal-dual simplex algorithm
- exterior point simplex algorithm
- transportation problems
- network problems
- quadratic problems

- linear complementarity problems
- bottleneck programming
- piecewise-linear programming
- linearly constrained optimization
- integral simplex method for combinatorial optimization

Occurrence of cycling in practice:

(see Zörnig 2006: page 2248)

Discussion of cycling in the Internet:

- "**Brian´s Digest**",

   see http://www.worms.ms.unimelb.edu. au/digest.html

- "**Linear Programming FAQ´s**", see

   http://www-unix.mcs.anl.gov/otc/Guide/fac/linear-programming-fac.html

Types of cycling

It is indispensable to distinguish between classical cycling and computer cycling.

Classical cycling: arises when the problem data may be expressed as rational fractions and computations are performed without round-off errors, i.e. the data are always transformed from rational fractions to rational fractions.

Computer cycling: caused by round-off errors.

All examples above are of the first type!

They need not cause cycling when professional software is applied.

Practical use of the results

The results answer the classical question, under which conditions (classical) cycling may arise. The theory permits the construction of cycling examples with higher dimensions (all examples in the literature are small). The possibility of construction is only limited by the capacity of the software used to solve the systems of determinantal inequalities.

There is a large number of general linear programming test problems available, but only a few examples for classical cycling.

For example, the "TOMLAB OPERA Toolbox", developed at Mälardalen University in Sweden offers only three (!) cycling examples among their test problems.

A great collection of constructed cycling examples could be useful to evaluate the practical performance of (anticycling) procedures or new variants of simplex type methods